CSC236, Summer 2004, Assignment 2

Due: Thursday, June 24th, 10 pm

Instructions

Please work on all questions. Turn in the outline and structure of a proof, even if you cannot provide every step of the proof, and we will try to assign part marks. However, if there is any question you cannot see how to even begin, leave it blank you you will receive 20% of the marks for that question.

When you turn in your neat, finely-crafted, final version, indicate the question you feel you did the best work on by writing a circled numeral "1" next to it. We will mark the question you have placed a circled "1" beside (and possibly other questions, as well).

Be sure to give full credit to any sources you consult (other than course notes, TAs, and the instructor) in preparing this problem set. If you try to pass off somebody else's work as your own for credit, you are committing an academic offense, and that can entail serious consequences. Any ideas that you do not attribute to someone else are assumed to be the ideas of the author(s) listed below, and will be evaluated for grading.

Write your name(s) and student number(s) (maximum of two names and two student numbers) in the space below, and staple this page to the front of your assignment.

Name ______________________________

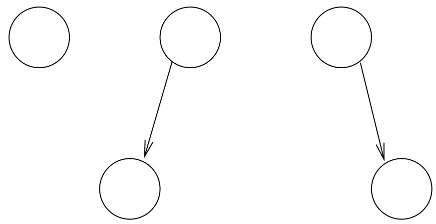
Student # ______________________________

Name ______________________________

Student # ______________________________

1. Two unlabelled binary trees with zero nodes are considered the same. Two unlabelled binary trees with one node each are considered the same. Two unlabelled binary trees with more than one node each are considered the same if the left subtrees of their root nodes are the same, and the right subtrees of their root nodes are the same.

   The drawing below suggests there is one unlabelled binary tree with zero nodes, one unlabelled binary tree with one node and two unlabelled binary trees with two nodes. Derive a formula (it doesn't need to be in closed form) for the number of unlabelled binary trees there are with n nodes, for all $n \in \mathbb{N}$. Prove that your formula is correct.



2. The FIBONACCI FUNCTION, $F(n)$ is defined in Example 3.2 (page 80) of the Course Notes. Verify each of the following claims for $n \in \{0, 1, 2\}$, and then prove that they are true for all $n \in \mathbb{N}$.

   (a) $\sum_{i=0}^{n} F(i) = F(n+2) - 1$.

   (b) $\sum_{i=0}^{n} F(2i) = F(2n+1) - 1$.

   (c) $\sum_{i=0}^{n} F(2i+1) = F(2n+2)$.

3. (a) On pages 88 and 89 a function $T(n)$ is defined to express the maximum number of steps required for a generic divide-and-conquer algorithm. Consider the case where n is a natural power of b, that is $n = b^k$ for some $k \in \mathbb{N}$, so

$$T(n) = \begin{cases} c, & n = 1 \\ aT\left(\frac{n}{b}\right) + dn^l, & n > 1. \end{cases}.$$

   Prove that for any $0 \leq i \leq k$, you have

$$T(n) = a^i T\left(\frac{n}{b^i}\right) + dn^l \sum_{j=0}^{i-1} \left(\frac{a}{b^l}\right)^j.$$

   This proof makes the informal reasoning near the bottom of page 89 precise. You may assume that c is a positive real number, that d, and l are non-negative real numbers, and that a and b are positive natural numbers.

   (b) Use simple induction to prove the principle of function definition by recursion (page 80, equation 3.5). To prove that function f exists, you must prove that $f(n)$ is defined for every $n \in \mathbb{N}$. To prove that $f(n)$ is unique, you must prove that any two functions that satisfy the definition also agree on their value for every $n \in \mathbb{N}$.

4. Here is a recursive definition of a function that takes a pair of natural numbers as arguments.

$$\forall m \in \mathbb{N}, 0 \leq n \leq m \qquad PT(m,n) = \begin{cases} 1, & n = 0 \\ 1, & n = m \\ PT(m-1, n-1) + PT(m-1, n), & 0 < n < m \\ \text{undefined}, & \text{otherwise} \end{cases}$$

   Verify the following for $m \in \{0, 1, 2, 3\}$, and then prove the following for all $m \in \mathbb{N}$:

   (a) $\sum_{n=0}^{m} PT(m,n) = 2^m$.

   (b) If $0 \leq n \leq m$, then $PT(m,n) = PT(m, m-n)$