CSC236 Summer 2004
Assignment 1 sample solution

8th June 2004

1. **Lattice logistics.**

   (a) **Claim 1(a)i**: Suppose $n \geq k > 0$ are natural numbers. Then there are $n - k + 1$ segments of length k with their end points on the lattice points of a length-n lattice.

   Proof: Place the length-n lattice on a coordinate system so that the coordinates of lattice points are natural numbers h, with $0 \leq h \leq n$. A segment of length k is uniquely determined by its coordinate, h', that is nearest the origin, and if this segment begins and ends on lattice points, you must have $0 \leq h' \leq n-k$. There are $n - k + 1$ integers in the range $(0, n - k)$, so there are $n - k + 1$ segments of length k that begin and end on the lattice points of a length-n lattice. QED.

   Claim 1(a)ii: There are $\sum_{i=1}^{n} i$ segments of positive length with end points on the lattice points of a length-n lattice.

   Proof: A segment of positive length with end points on a length-n lattice has length k, where $1 \leq k \leq n$, so by Claim 1(a)i there are $n - k + 1$ such segments for each $1 \leq k \leq n$, or

   $$\sum_{k=1}^{n}(n-k-1) = \sum_{n-k+1=1}^{n}(n-k+1) = \sum_{i=1}^{n} i \qquad \text{(where } i = n-k+1\text{)},$$

   segments in all. QED.

   Claim 1(a)iii: The sum $\sum_{i=1}^{n} i = n(n+1)/2$. (So, by Claim 1(a)ii, there are $n(n+1)/2$ segments of positive length with end points on a length-n lattice).

   Proof: In the course notes (pages 22 and 23) there is a proof that $\sum_{i=1}^{n} i = n(n+1)/2$. Since the term $i = 0$ doesn't change the sum, this is the same as the sum $\sum_{i=1}^{n} i$. QED.

   (b) **Claim 1(b)i**: Suppose $n \geq k > 0$ are natural numbers. Then there are $(n - k + 1)^2$ squares of area k^2 with their corners on lattice points of an $n \times n$ two-dimensional lattice.

   Proof: Place the $n \times n$ lattice on a coordinate system so that the lattice points have coordinates (h, i), where $0 \leq h, i \leq n$. Each square of area k^2 with its corners on lattice points is uniquely determined by the coordinates (h', i') nearest the origin. A square with sides of length k fits on an $n \times n$ lattice if and only if these coordinates satisfy $0 \leq h' \leq n-k$ and $0 \leq i' \leq n-k$. There are $n-k+1$ integers in the range $(0, n-k)$, so there are $(n - k + 1)^2$ such squares in all. QED.

   Claim 1(b)ii: There are $\sum_{i=1}^{n} i^2$ squares of positive area with their corners on the lattice points of an $n \times n$ lattice.

   Proof: A square of positive area with its corners on the lattice points of an $n \times n$ lattice has sides of length k, where $1 \leq k \leq n$. By Claim 1(b)i there are $n - k + 1$ such squares for each $1 \leq k \leq n$, so there are

   $$\sum_{i=1}^{n}(n-k+1)^2 = \sum_{n-k+1=1}^{n}(n-k+1)^2 = \sum_{i=1}^{n} i^2 \qquad \text{(where } i = n-k+1\text{)},$$

   squares in all. QED.

Claim 1(b)iii: $P(n)$: "The sum $\sum_{i=0}^{n} i^2 = n(n+1)(2n+1)/6$" for all $n \in \mathbb{N}$. (By Claim 1(b)ii, this means there are $n(n+1)(2n+1)/2$ squares with their corners on the lattice points of an $n \times n$ lattice, since the term $i = 0$ doesn't change the sum).

Proof (induction on n): $P(0)$ states that 0 equals 0, which is certainly true, so the base case holds.

Induction step: I want to show that $P(n) \Rightarrow P(n+1)$, so I assume $P(n)$ for an arbitrary $n \in \mathbb{N}$. Now the sum $\sum_{i=0}^{n} i^2$ can be broken into two terms, and I can use the inductive hypothesis on the first term

$$\begin{aligned}
\sum_{i=0}^{n+1} i^2 &= \left(\sum_{i=0}^{n} i^2\right) + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \text{(by inductive hypothesis)} \\
&= (n+1)\left(\frac{n(2n+1)+6(n+1)}{6}\right) = (n+1)\left(\frac{2n^2+7n+6}{6}\right) \\
&= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)([n+1]+1)(2[n+1]+1)}{6},
\end{aligned}$$

which is what $P(n+1)$ claims. So $P(n) \Rightarrow P(n+1)$.

I conclude that $P(n)$ is true for any $n \in \mathbb{N}$. QED.

2. **More lattices.**

(a) **Claim 2(a)i:** Suppose $n \geq k > 0$ are natural numbers. Then there are $(n-k+1)^3$ cubes of size k^3 with corners on the lattice points of an $n \times n \times 1$ three-dimensional lattice.

Proof: Place an $n \times n \times n$ lattice in a coordinate system so that the coordinates of lattice points are natural numbers (h, i, j), with $0 \leq h, i, j \leq n$. A cube of size k^3 is uniquely determined by the coordinates of its corner (h', i', j') nearest the origin. Since the cube must fit inside the larger lattice, (h', i', j') are the coordinates nearest the origin of a cube of size k^3 if and only if

$$0 \leq h' \leq n-k \qquad 0 \leq i' \leq n-k \qquad 0 \leq j' \leq n-k.$$

There are $n-k+1$ natural numbers in the range $(0, n-k)$, so there are $(n-k+1)^3$ possibilities for the coordinates of the corner nearest the origin of a cube of size k^3 with its corners on the lattice points of an $n \times n \times n$ lattice. So there are $(n-k+1)^3$ such cubes. QED.

Claim 2(a)ii: An $n \times n \times n$ lattice contains $\sum_{i=1}^{n} i^3$ cubes of positive volume with corners on lattice points.

Proof: A cube of positive volume with corner points on the lattice points of an $n \times n \times n$ lattice has sides of length k, where $1 \leq k \leq n$. By Claim 2(a)i, there are $(n-k+1)^3$ such cubes for each k in this range, so there are

$$\sum_{k=1}^{n}(n-k+1)^3 = \sum_{n-k+1=1}^{n}(n-k+1)^3 = \sum_{i=1}^{n} i^3 \qquad \text{(where } i = n-k+1\text{)},$$

cubes in all. QED.

Claim 2(a)iii: $P(n)$: "The sum $\sum_{i=0}^{n} i^3 = [n(n+1)/2]^2$" for all $n \in \mathbb{N}$. (By Claim 2(a)ii this means that there are $[n(n+1)/2]^2$ cubes with their corners on lattice points on an $n \times n \times n$ lattice, since the term $i = 0$ doesn't change the sum).

Proof (induction on n): $P(0)$ states that 0 equals 0, which is certainly true, so the base case holds.

Induction step: I want to show that $P(n) \Rightarrow P(n+1)$, so I assume $P(n)$ for some arbitrary $n \in \mathbb{N}$. Now I can break up the sum $\sum_{i=0}^{n+1} i^3$ into two parts, and use the inductive hypothesis on the first part

$$\begin{aligned}
\sum_{i=0}^{n+1} i^3 &= \left(\sum_{i=0}^{n} i^3\right) + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= (n+1)^2\left(\frac{n^2+4(n+1)}{4}\right) = (n+1)^2\left(\frac{n^2+4n+4}{4}\right) = \left(\frac{(n+1)(n+2)}{2}\right)^2,
\end{aligned}$$

where the right-hand sum is what $P(n+1)$ claims, so $P(n) \Rightarrow P(n+1)$.
I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

(b) **Claim 2(b)i**: There are $[n(n+1)/2]^2$ rectangle with positive volume with corners on lattice points of an $n \times n$ lattice.

Proof: A rectangle with corners on lattice points of an $n \times n$ lattice is uniquely determined by the its projection onto the bottom and left side of the lattice. These projections are segments of positive length on two length-n lattices, so (by Claim 1(a)iii) there are $n(n+1)/2$ choices for the projection onto the bottom, and $n(n+1)/2$ choices for the projection onto the left side, for $[n(n+1)/2]^2$ combinations of these projections. QED.

3. **Slicing lines and planes.**

(a) **Claim 3(a)**: $P(n)$: "The maximum number of segments (including rays and entire lines) a line is divided into by n points is $n+1$ segments" is true for all $n \in \mathbb{N}$.

Proof (induction on n): $P(0)$ states that 0 points divide a line into one segment, which is clearly true (the segment is the line itself). So the base case holds.
Induction step: I want to show that $P(n) \rightarrow P(n+1)$, so I assume $P(n)$ for some arbitrary $n \in \mathbb{N}$. Given $n+1$ points $p_1, \ldots, p_{n+1}$, I arrange them from left to right along a line. This divides the line into $n+1$ pieces with left-most points $p_1, \ldots, p_{n+1}$ (respectively), plus a piece with right-most point p_1, for $n+2$ pieces in all. This is the maximum number of pieces possible, since if you could divide a line into more than $n+2$ pieces with $n+1$ points, then by removing one point you would divide it into $n+1$ pieces (the pieces adjacent to the removed point become a single piece), and this contradicts $P(n)$. Therefore, $P(n) \Rightarrow P(n+1)$.
I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

(b) **Claim 3(b)i**: $P(n)$: "The maximum number of parts (regions) that n lines can divide the plane into is $n(n+1)/2+1$" is true for all $n \in \mathbb{N}$.

Proof (induction on n): $P(0)$ states that the maximum number of regions that 0 lines can divide the plane into is 1, which is clearly true since 0 lines divide the plane into a single region (the plane itself), so the base case holds.
Induction step: I want to show that $P(n) \Rightarrow P(n+1)$, so I assume that $P(n)$ is true for some arbitrary $n \in \mathbb{N}$. Suppose you have $n+1$ lines $l_1, \ldots, l_{n+1}$. An upper bound on the number of new regions contributed by line l_{n+1} is given by considering how many segments l_{n+1} is divided into by the previous n lines. Each segment that l_{n+1} is divided into by its intersections with the (at most) previous n lines lies in a different existing region of the plane, and divides that existing region of the plane into two parts (corresponding to the half-planes on each side of l_{n+1}), and by 3(a) there are no more than $n+1$ such segments.
By arranging the lines in general position (every pair intersect, no three intersect) you guarantee that l_{n+1} encounters the n other lines, and thus contributes $n+1$ new regions. By the induction hypothesis, there are $[n(n+1)/2]+1$ regions created by the previous n lines, plus a further $n+1$ regions, for a total of$[(n+1)(n+2)/2]+1$ regions, which is what $P(n+1)$ asserts. Thus, $P(n) \Rightarrow P(n+1)$.
I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

4. **Claim 4(a)i**: $P(n)$: "n planes in **general position** (every pair of planes intersect in a single line, no three planes intersect in a single line, every triple of planes intersect in a single point, no four planes intersect in a single point) divide three-dimensional space into $\sum_{i=0}^{n-1}([i(i+1)/2]+1) + 1$ parts" is true for all $n \in \mathbb{N}$.

Proof (induction on n): $P(0)$ states that 0 planes divide three-dimensional space into one part (the sum $\sum_{i=0}^{n-1}([i(i+1)/2]+1)$ is empty), which is clearly true (the single part is the three-dimensional space itself). Thus the base case holds.

Induction step: I want to show that $P(n) \Rightarrow P(n+1)$, so I assume that $P(n)$ is true for some arbitrary $n \in \mathbb{N}$. Now consider $n+1$ planes $\pi_1, \ldots, \pi_{n+1}$.

An upper bound on the number of new regions contributed by π_{n+1} is obtained by considering its intersections with the previous n planes. Each two-dimensional region that the previous n planes divide π_{n+1} into lies in a different existing region of space created by the previous n planes. Each two-dimensional region that the (at most) previous n planes divide π_{n+1} into divides the existing region into two parts (corresponding to the half-spaces on either side of π_{n+1}), so (by Claim 3(b)i) plane π_{n+1} contributes a maximum of $n(n+1)/2+1$ new regions.

By arranging the $n+1$ planes in general position (every pair of planes intersects in a line, no three planes intersect in a line, every triple of planes intersect in a point, no four planes intersect in a point) you guarantee that π_{n+1} intersects all n of the previous planes in n lines in general position, and contributes the maximum number of new regions. By the inductive hypothesis the first n planes divide three-dimensional space into $(\sum_{i=0}^{n-1}[i(i+1)/2]+1)$ parts, and π_{n+1} adds $[n(n+1)/2]+1$ regions. Adding the new regions to the old gives us $(\sum_{i=0}^{n}[n(n+1)/2]+1)$ $+\ 1$, which is what $P(n+1)$ asserts. Therefore $P(n) \Rightarrow P(n+1)$.

I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

Claim 4(a)ii: $P(n)$: "The expression $(\sum_{i=0}^{n-1}[i(i+1)/2]+1)+1$ equals $[(n-1)n(n+1)/6]+n+1$" for all $n \in \mathbb{N}$. (Thus, by Claim 4(a)i, n planes in general position slice three-dimensional space into $[(n-1)n(n+1)/6]+n+1$ regions).

Proof (induction on n): $P(0)$ states that 1 equals 1, since the summation is empty. This is clearly true, so the base case holds.

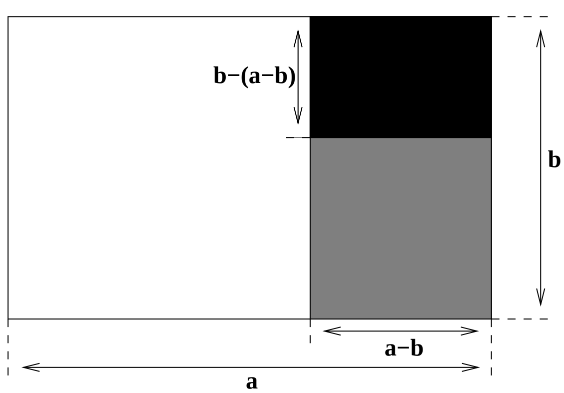
Induction step: I want to show that $P(n) \Rightarrow P(n+1)$, so I assume that $P(n)$ (the induction hypothesis, or IH) is true for some arbitrary $n \in \mathbb{N}$. Now the sum $(\sum_{i=0}^{n+1-1}[i(i+1)/2]+1)+1$ can be broken into two terms, and the induction hypothesis applied to the first term

$$\begin{aligned}
\left(\sum_{i=0}^{n+1-1}[i(i+1)/2]+1\right)+1 &= \left(\sum_{i=0}^{n-1}[i(i+1)/2]+1\right)+1+([n(n+1)/2]+1) \\
&= \frac{(n-1)n(n+1)}{6}+n+1+([n(n+1)/2]+1) \qquad \text{(by IH)} \\
&= \frac{3n(n+1)+(n-1)n(n+1)}{6}+(n+1)+1 \\
&= \frac{n(n+1)(n+2)}{6}+(n+1)+1,
\end{aligned}$$

which is what $P(n+1)$ asserts. Thus, $P(n) \Rightarrow P(n=1)$.

I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

5. **Remark**: Although the algebra that follows may seem daunting and unmotivated, the idea that $a/b = b/(a-b)$ implies that $b/(a-b) = (a-b)/(b-[a-b])$ is clear if you sketch the corresponding rectangles (and insist on a "pleasing" ratio). Then the job is to translate something that is geometrically clear into algebra.



Claim: There are no integers a, and b such that $a/b = b/(a-b)$.

Proof (contradiction): Suppose there are integers a and b such that $a/b = b/(a-b)$. Without loss of generality we can assume that a and b are natural numbers, since

$$\frac{a}{b} = \frac{b}{a-b} \Rightarrow \left|\frac{a}{b}\right| = \left|\frac{b}{a-b}\right| \Rightarrow \frac{|a|}{|b|} = \frac{|b|}{|a-b|},$$

and the equality on the right involves only non-negative integers, in other words, natural numbers. Now construct a set $A = \{n \in \mathbf{N} : \exists n' \in \mathbf{N}, \text{ such that } n'/n = n/(n'-n)\}$. By our assumption, $b \in A$ (just set $n = b$, and $n' = a$), so A is a non-empty subset of the natural numbers, and (by the principle of well-ordering) has a least element b^*, with a corresponding natural number a^* such that $a^*/b^* = b^*/(a^* - b^*)$. Since a^*/b^* is defined, $b^* \neq 0$, which, in turn, implies that $a^* \neq 0$. Cross-multiplying and dividing by non-zero a gives

$$\frac{a^*}{b^*} = \frac{b^*}{a^* - b^*} \Rightarrow a^* - b^* = \frac{(b^*)^2}{a^*}.$$

Since a^* and b^* are integers, so is $a^* - b^*$. Furthermore, since a^* and b^* are positive, so is $(b^*)^2/a^* = a^* - b^*$, so $a^* - b^*$ is a positive natural number. Now consider the pleasing ratio

$$\begin{aligned}\frac{b^*}{a^* - b^*} &= \frac{a^*}{b^*} = \frac{a^* - b^*}{(b^*/a^*)(a^* - b^*)} = \frac{a^* - b^*}{b^* - [(b^*)^2/a^*]} \\ &= \frac{a^* - b^*}{b^* - (a^* - b^*)}.\end{aligned}$$

So $a^* - b^*$ and b^* are natural numbers with the property that $b^*/(a^* - b^*) = (a^* - b^*)/[b^* - (a^* - b^*)]$, which means that $a^* - b^* \in A$. Also, $a^* - b^*$ is less than b^*, since by cross multiplying we showed that $b^* - (a^* - b^*)$ is the product of positive factors (b^*/a^*) and $(a^* - b^*)$. But this contradicts b^* being the smallest member of A, so the assumption that there are integers a and b such that $a/b = b/(a-b)$ is false. QED.

Since I have shown that there is no rational number (ratio of integers) that satisfies $a/b = b/(a-b)$, the next thing to attempt is a solution that uses irrational numbers. Cross multiplying the equation and then solving the quadratic equation yields:

$$a^2 - ab - b^2 = 0 \Longrightarrow a = b\frac{1 \pm \sqrt{5}}{2}.$$

Since I'd like a rectangle with sides of positive length, I could normalize b to have length 1, and so a would have length $(1 + \sqrt{5})/2$.

6. **Approximate arithmetic.**

   (a) **Claim 6(a)i** $P(n)$: "If the binary expansion of natural number m contains $2k$ ones and no zeroes, then m is divisible by 3" is true for all $k \in \mathbb{N} - \{0\}$.

   Proof (induction on k): $P(1)$ states that the binary expansion equivalent to $\sum_{i=0}^{2\cdot 1-1} 2^i = 2 + 1$ is divisible by 3, which is clearly true, so the base case holds.

   Induction step: I wish to show that $P(k) \Rightarrow P(k+1)$, so I assume that $P(k)$ is true for some arbitrary $k \in \mathbb{N} - \{0\}$. Now the natural number with the binary expansion equivalent to $\sum_{i=0}^{2(k+1)-1} 2^i$ can be broken into two parts, and

$$\begin{aligned}\left(\sum_{i=0}^{2(k+1)-1} 2^i\right) \bmod 3 &= \left(\left[\sum_{i=2}^{2(k+1)-1} 2^i\right] \bmod 3 + \left[2^1 + 2^0\right] \bmod 3\right) \bmod 3 \qquad \text{(by Fact 1)} \\ &= \left(\left[2^2 \times \sum_{i=0}^{2k-1} 2^i\right] \bmod 3 + 0\right) \bmod 3 \\ &= ([1 \times 0] \bmod 3) \bmod 3 \qquad \text{(by IH and Fact 2)} \\ &= 0 \bmod 3.\end{aligned}$$

   This is exactly what $P(k+1)$ asserts, so $P(k) \Rightarrow P(k+1)$.
   I conclude that $P(k)$ is true for all $k \in \mathbb{N} - \{0\}$.

Claim 6(a)ii: $P(k)$: "The natural number with the binary expansion equivalent to $\sum_{i=0}^{2k} 2^i$ is equal to 1 mod 3" is true for all $k \in \mathbb{N}$.

Proof (induction on k): $P(0)$ states that $2^0 = 1$ is equal to 1 mod 3, which is certainly true, so the base case holds.

Induction step: I want to show that $P(k) \Rightarrow P(k+1)$, so I assume that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$. Now the natural number with binary expansion equivalent to $\sum_{i=0}^{2(k+1)} 2^i$ can be broken into two parts, and

$$\begin{aligned}\left(\sum_{i=0}^{2(k+1)} 2^i\right) \bmod 3 &= \left(\left[\sum_{i=2}^{2(k+1)} 2^i\right] \bmod 3 + \left[2^1 + 2^0\right] \bmod 3\right) \bmod 3 \qquad \text{(by Fact 1)} \\ &= \left(\left[2^2 \sum_{i=0}^{2k} 2^i\right] \bmod 3 + 0\right) \bmod 3 = ([1 \times 1] \bmod 3) \bmod 3 \qquad \text{(by IH and Fact 2)} \\ &= 1.\end{aligned}$$

This is exactly what $P(k+1)$ asserts, so $P(k) \Rightarrow P(k+1)$.

I conclude that $P(k)$ is true for all $k \in \mathbb{N}$. QED.

(b) **Claim 6(b)i**: Q(k): "If the ternary expansion of natural number m has $2k$ ones and no zeroes, then m is divisible by 4" is true for every natural number k greater than 1.

Proof (induction on k): $Q(1)$ states that if the ternary expansion of m is $(11)_3$, then m is divisible by 4. Well, $(11)_3$ equals 4 in decimal notation, which is certainly divisible by 4, so the base case is true.

Induction step: I want to show that for any natural number k, $P(k) \Rightarrow P(k+1)$, so I assume $P(k)$. This means that number m with ternary expansion consisting of $2k$ ones and no zeroes (which equals $\sum_{i=0}^{2k-1} 3^i$) is divisible by 4, or $m \bmod 4 = 0$. By Fact 2 and Fact 3, $(9m+4) \bmod 4 = (1 \times 0 + 0) \bmod 4 = 0$, so

$$9\left(\sum_{i=0}^{2k-1} 3^i\right) + 4 = \left(\sum_{i=0}^{2k-1} 3^{i+2}\right) + 3^1 + 3^0 = \left(\sum_{i=2}^{2k+1} 3^i\right) + 3^1 + 3^0 = \sum_{i=0}^{2(k+1)-1} 3^i.$$

So, the sum on the right is divisible by 4 and equals the ternary expansion of the natural number m' that has $2(k+1)$ ones and no zeroes. So $P(k) \Rightarrow P(k+1)$.

I conclude that $P(k)$ is true for every natural number greater than 1. QED.

Claim 6(b)ii: $R(k)$: "If the ternary expansion of natural number m has $2k+1$ ones, and no zeroes, then $m \bmod 4 = 1$ " is true for every natural number k.

Proof (induction on k): $R(0)$ states that $(1)_3 \bmod 4 = 1 \bmod 4$. This is certainly true, since $(1)_3$ equals $(1)_{10}$, which equals $4 \times 0 + 1$. So the base case is true.

Induction step: I want to show that for any natural number k, $R(k) \Rightarrow R(k+1)$, so I begin by assuming $R(k)$. This means that the natural number whose ternary expansion consists of $2k+1$ ones equals 1 mod 4, in other words $\left(\sum_{i=0}^{2k} 3^i\right) \bmod 4 = 1$. Since $9 \bmod 4 = 1$ and $4 \bmod 4 = 0$, using Fact 1 and Fact 2 this means that $[9\left(\sum_{i=0}^{2k} 3^i\right) + 4] \bmod 4 = [(1 \times 1 \bmod 4) + 0] \bmod 4 = 1$, so

$$9\left(\sum_{i=0}^{2k} 3^i\right) + 4 = \left(\sum_{i=0}^{2k} 3^{i+2}\right) + 3^0 + 3^1 = \left(\sum_{i=2}^{2k+2} 3^i\right) + 3^1 + 3^2 = \sum_{i=0}^{2(k+1)} 3^i,$$

equals 1 mod 4. The sum on the right equals the ternary expansion consisting of $2(k+1)+1$ ones, so $R(k+1)$ is true. This shows that $R(k) \Rightarrow R(k+1)$.

I conclude that $R(k)$ is true for all $k \in \mathbf{N}$. QED.

Claim 6(b)iii: If m is a natural number with a ternary expansion consisting of n twos and no zeroes, then m is divisible by 4 if n is even, and m mod 4 equals 2 if n is odd.

Proof (direct): Suppose m is a natural number with a ternary expansion consisting of n twos and no zeroes. Then

$m = \sum_{i=0}^{n-1}(2 \times 3^i)$. If n is even, then for some integer k, $n = 2k$, so by Claim 6(b)i and Fact 2

$$m \bmod 4 = \left(2\sum_{i=0}^{2k-1} 3^i\right) \bmod 4 = (2 \times 0) \bmod 4 = 0,$$

so m is divisible by 4, as claimed. If n is odd, then for some integer k, $n = 2k+1$, so by Claim 6(b)ii and Fact 2

$$m \bmod 4 = \left(2\sum_{i=0}^{2k} 3^i\right) \bmod 4 = (2 \times 1) \bmod 4 = 2,$$

so $m \bmod 4$ equals 2, as claimed. So the claims for odd and even n hold. QED.

7. **More approximate arithmetic.**

(a) **Claim 7(a)i**: $P(n)$: "$10^n \bmod 9$ equals 1" for all $n \in \mathbf{N}$.

Proof (induction on n): $P(0)$ states that $10^0 \bmod 9$ equals 1, in other words $1 \bmod 9 = 1$, which is clearly true since $1 = 9 \times 0 + 1$. So the base case is true.

Induction step: I want to show that for any natural number n, $P(n) \Rightarrow P(n+1)$, so I assume $P(n)$ is true. This means that $10^n \bmod 9 = 1$, so (by Fact 2 and the fact that $10 \bmod 9 = 1$) we have $(10 \times 10^n) \bmod 9 = (1 \times 1) \bmod 9 = 1$. Thus $P(n) \Rightarrow P(n+1)$.

I conclude that $P(n)$ is true for all $n \in \mathbf{N}$.

Claim 7(a)ii: $P(n)$: "Natural number $(\sum_{i=0}^{n} d_i \times 10^i) \bmod 9$ equals $(\sum_{i=0}^{n} d_i) \bmod 9$" is true for every $n \in \mathbf{N}$.

Proof (induction on n): $P(0)$ states that any one-digit number $\left(\sum_{i=0}^{0} d_i \times 10^i\right) \bmod 9 = d_i \bmod 9$, which is clearly true since the sum on the left has just one term, $d_i \times 1 = d_i$.

Induction step: I want to show that $P(n) \Rightarrow P(n+1)$, so I assume $P(n)$ is true for some arbitrary $n \in \mathbb{N}$. This means that I can break up $(\sum_{i=0}^{n+1} d_i \times 10^i) \bmod 9$ into two terms, and use the induction hypothesis (IH) on the first term

$$\begin{aligned}
\left(\sum_{i=0}^{n+1} d_i 10^i\right) \bmod 9 &= \left(\left[\sum_{i=0}^{n} d_i 10^i\right] \bmod 9 + \left[10^{n+1} d_{n+1}\right] \bmod 9\right) \bmod 9 \\
\text{(by IH, Claim 7(a)i, and Fact 2)} &= \left(\left[\sum_{i=0}^{n} d_i\right] \bmod 9 + [1 \times d_{n+1} \bmod 9] \bmod 9\right) \bmod 9 \\
\text{(apply } \bmod 9 \text{ twice)} &= \left(\left[\sum_{i=0}^{n} d_i\right] \bmod 9 + d_{n+1} \bmod 9\right) \bmod 9 \\
\text{(by Fact 1)} &= \left(\sum_{i=0}^{n+1} d_i\right) \bmod 9,
\end{aligned}$$

which is what $P(n+1)$ asserts. So $P(n) \Rightarrow P(n+1)$.

I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

(b) **Claim 7(b)i**: $P(n)$: "$10^n \bmod 11 = (-1)^n \bmod 11$" is true for all $n \in \mathbb{N}$.

Proof (induction on n): $P(0)$ states that $1 \bmod 11$ equals $1 \bmod 11$, which is clearly true. Thus the base case holds.

Induction step: I want to show that $P(n) \Rightarrow P(n+1)$, so I assume that $P(n)$ is true for some arbitrary $n \in \mathbb{N}$. I can now break $10^{n+1} \bmod 11$ into two factors, and I can use the induction hypothesis (IH) on the first factor

$$\begin{aligned}
10^{n+1} \bmod 11 &= (10^n \times 10) \bmod 11 = (10^n \bmod 11 \times 10 \bmod 11) \bmod 11 \quad \text{(by Fact 2)} \\
&= (10^n \bmod 11 \times [11 \bmod 11 - 1 \bmod 11] \bmod 11) \bmod 11 \quad \text{(by Fact 1)} \\
&= (10^n \bmod 11 \times [-1 \bmod 11] \bmod 11) \bmod 11 \quad \text{(since } 11 \bmod 11 = 0) \\
&= ([-1]^n \bmod 11 \times [-1] \bmod 11) \bmod 11 \quad \text{(IH, } \bmod 11 \text{ twice)} \\
&= ([-1]^{n+1}) \bmod 11 \quad \text{(by Fact 2)},
\end{aligned}$$

which is what $P(n+1)$ asserts, so $P(n) \Rightarrow P(n+1)$.
I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.

Claim 7(b)ii: $P(n)$: "Natural number $\left(\sum_{i=0}^{n} d_i 10^i\right)$ mod 11 equals $\left(\sum_{i=0}^{n}[-1]^i d_i\right)$ mod 11" is true for all $n \in \mathbb{N}$.

Proof (induction on n): $P(0)$ states that d_i mod 11 equals d_i mod 11, which is clearly true, so the base case holds.
Induction step: I want to show that $P(n) \Rightarrow P(n+1)$, so I assume that $P(n)$ is true for some arbitrary $n \in \mathbb{N}$. Now I can break up $\left(\sum_{i=0}^{n+1} d_i 10^i\right)$ mod 11 into two terms, and use the inductive hypothesis (IH) on the first term

$$\begin{aligned}
\left(\sum_{i=0}^{n+1} d_i 10^i\right) \bmod 11 &= \left(\left[\sum_{i=0}^{n} d_i 10^i\right] + d_{n+1}10^{n+1}\right) \bmod 11 \\
\text{(by Fact 1)} &= \left(\left[\sum_{i=0}^{n} d_i 10^i\right] \bmod 11 + [d_{n+1}10^{n+1}] \bmod 11\right) \bmod 11 \\
&= \left(\left[\sum_{i=0}^{n} d_i 10^i\right] \bmod 11 + [d_{n+1} \bmod 11 \times (-1)^{n+1} \bmod 11] \bmod 11\right) \bmod 11 \\
\text{(by IH and Fact 2)} &= \left(\left[\sum_{i=0}^{n} d_i[-1]^i\right] \bmod 11 + [d_{n+1}(-1)^{n+1}] \bmod 11\right) \bmod 11 \\
\text{(by Fact 1)} &= \left(\sum_{i=0}^{n+1} d_i[-1]^i\right) \bmod 11,
\end{aligned}$$

which is what $P(n+1)$ asserts, so $P(n) \Rightarrow P(n+1)$.
I conclude that $P(n)$ is true for all $n \in \mathbb{N}$. QED.