Learning with Maximum Likelihood

René Magritte, “La reproduction interdite” (1937)

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Likelihood: Bernoulli Variables

- Suppose a fair coin is tossed n times, independently
  - $Y \sim Bernoulli(\theta)$
- The likelihood (discrete case) is the probability of observing the dataset when the parameters are θ
  - $P(Y_i = 1|\theta) = \theta$
  - $P(Y_i = 1|\theta) = \theta$
  - $P(Y_i = y_i|\theta) = \theta^{y_i}(1-\theta)^{1-y_i}$

  - $P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_m = y_m|\theta) = \prod_{i=1}^{m} P(Y_i = y_i|\theta)$
    - Because of independence of cases

Maximum likelihood: Bernoulli

- Suppose we observe the data $Y^{(1)} = y^{(1)}, Y^{(2)} = y^{(2)}, \ldots, Y^{(m)} = y^{(m)}$ (m i.i.d. Bernoulli variables), and would like to know what θ is
- One possibility: find the θ that maximizes the likelihood function
  - What value of θ makes the data set that we are actually observing (i.e., the training set) the most plausible?
- $P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_m = y_m|\theta)$ is maximized at $\theta = \frac{1}{m}\sum_{i=1}^{m} y_i$

Likelihood: Gaussian Noise

- Assume each data point is generated using some process.
  - E.g., $y^{(i)} = \theta^T x^{(i)} + \epsilon^{(i)},\ \epsilon^{(i)} \sim N(0, \sigma^2)$
- We can now compute the likelihood of single datapoint
  - I.e., the probability of the point for a set θ.
  - E.g., $P(\mathrm{y}^{(i)} | \theta, x^{(i)}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(y^{(i)} - \theta^T x^{(i)}\right)^2}{2\sigma^2}\right)$ We can then compute the likelihood for the entire training set $\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(m)}, y^{(m)})\}$ (assuming each point is independent)
  - E.g., $P(y | \theta, x) = \Pi_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(y^{(i)} - \theta^T x^{(i)}\right)^2}{2\sigma^2}\right)$

Maximum Likelihood

- $\mathrm{P(data}|\theta) = P(y|\theta, x) = \Pi_1^m \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{\left(y^{(i)}-\theta^T x^{(i)}\right)^2}{2\sigma^2})$
- $\log P(data|\theta) = \sum -\frac{\left(y^{(i)}-\theta^T x^{(i)}\right)^2}{2\sigma^2} + 2m/\log(2\pi\sigma^2)$

  is maximized for a value of θ for which

  $\sum_{i=1}^m \left(y^{(i)} - \theta^T x^{(i)}\right)^2$ is minimized
- Note: x is fixed