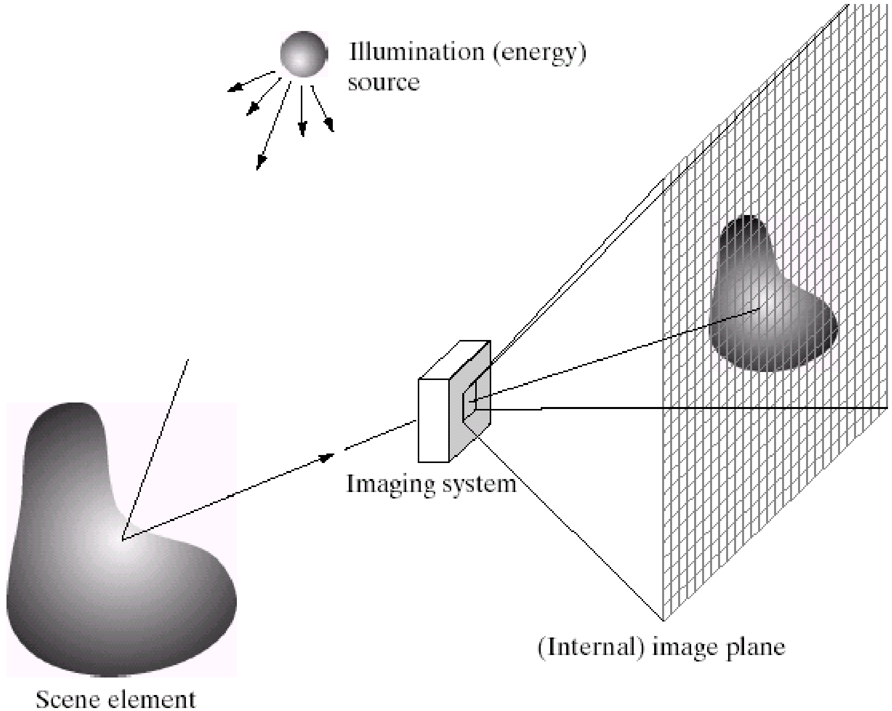
Sampling and Reconstruction

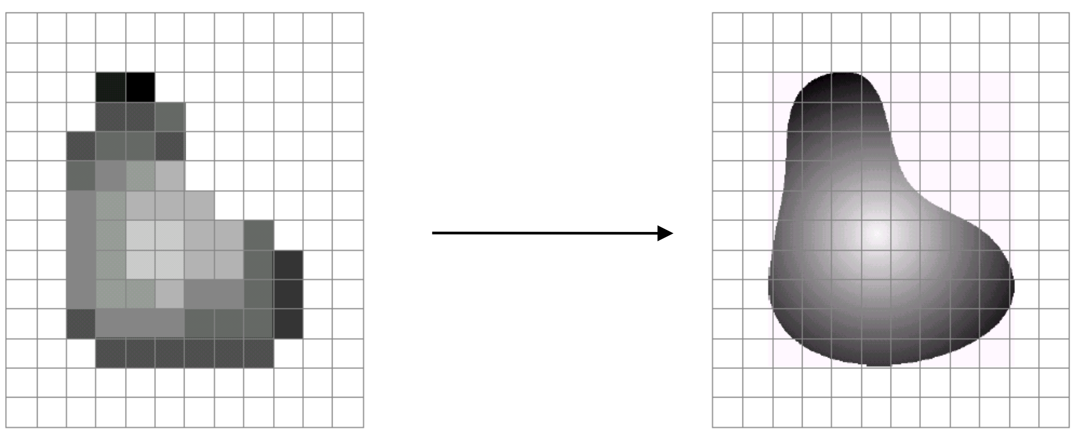
Salvador Dali, “Dali from the Back Painting Gala from the Back Eternalized by Six Virtual Corneas Provisionally Reflected by Six Real Mirrors”

CSC320: Introduction to Visual Computing
Michael Guerzhoy

Many slides from
Steve Marschner, Alexei Efros

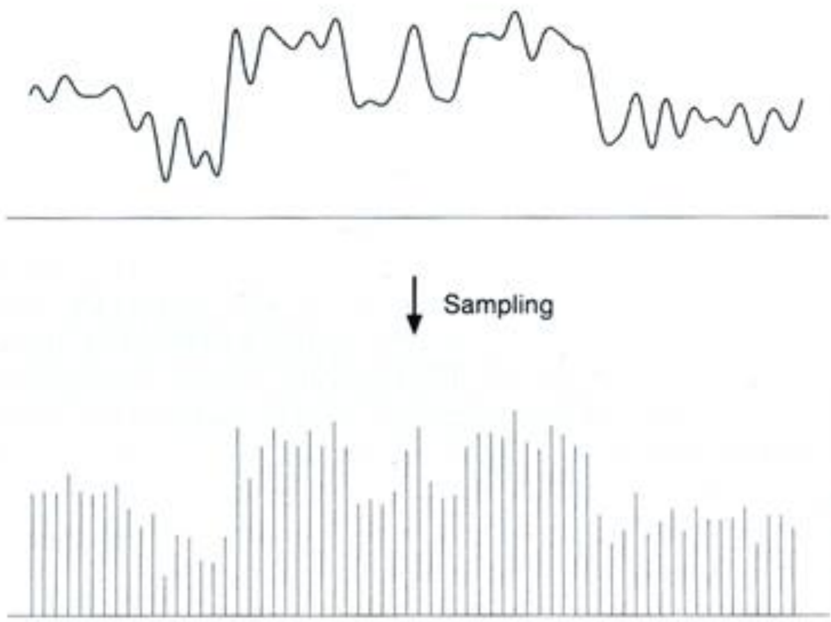
Sampling and Reconstruction





Sampled representations

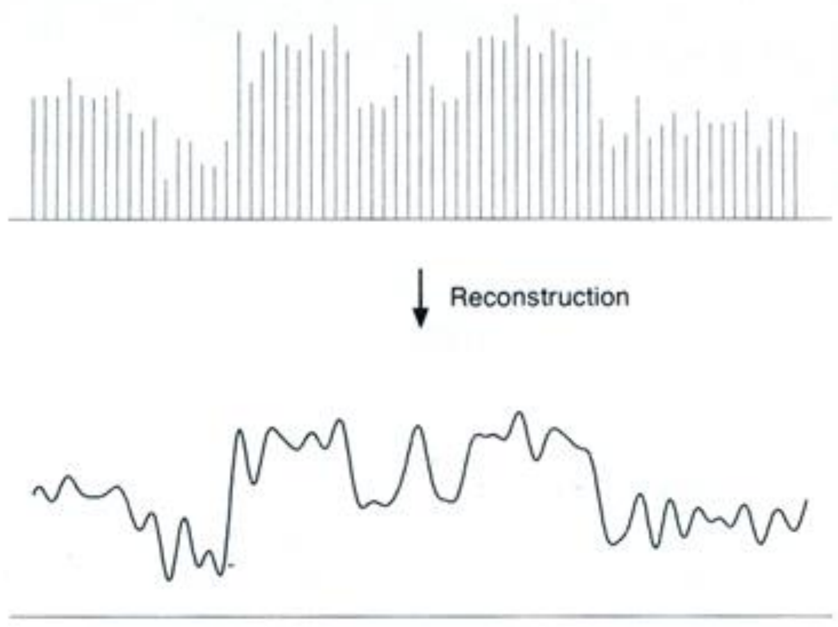
- How to store and compute with continuous functions?
- Common scheme for representation: samples
  - write down the function's values at many points



[FvDFH fig.14.14b / Wolberg]

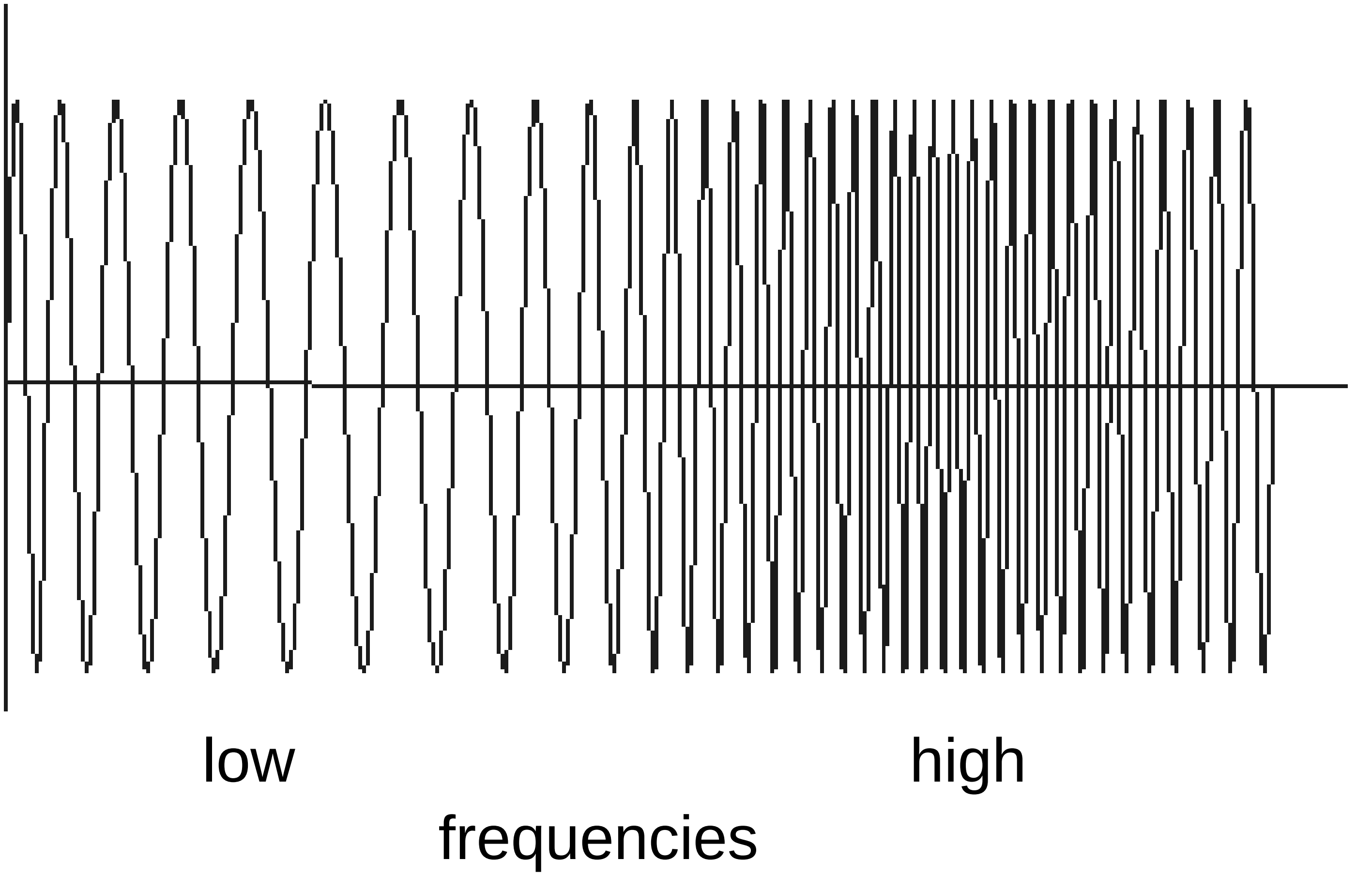
Reconstruction

- Making samples back into a continuous function
  - for output (need realizable method)
  - for analysis or processing (need mathematical method)
  - amounts to “guessing” what the function did in between



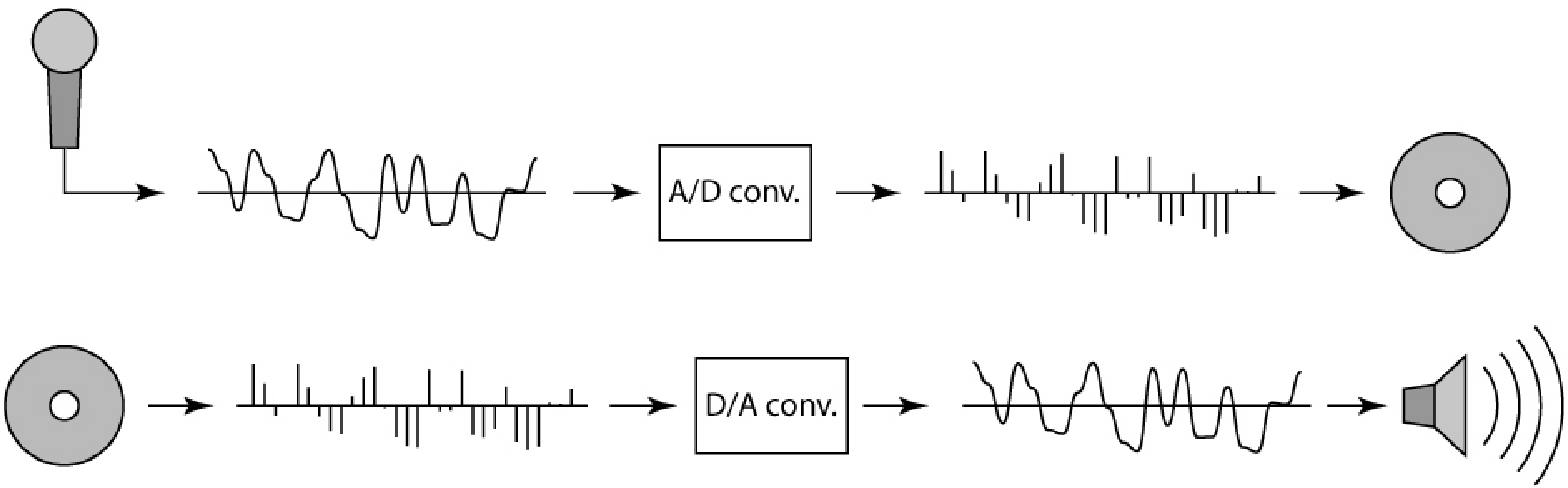
[FvDFH fig.14.14b / Wolberg]

1D Example: Audio



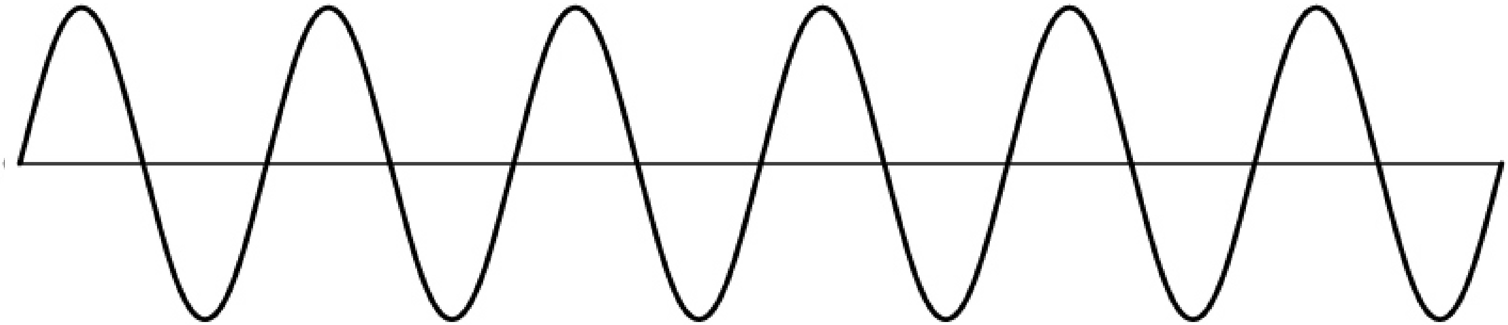
Sampling in digital audio

- Recording: sound to analog to samples to disc
- Playback: disc to samples to analog to sound again
  - how can we be sure we are filling in the gaps correctly?



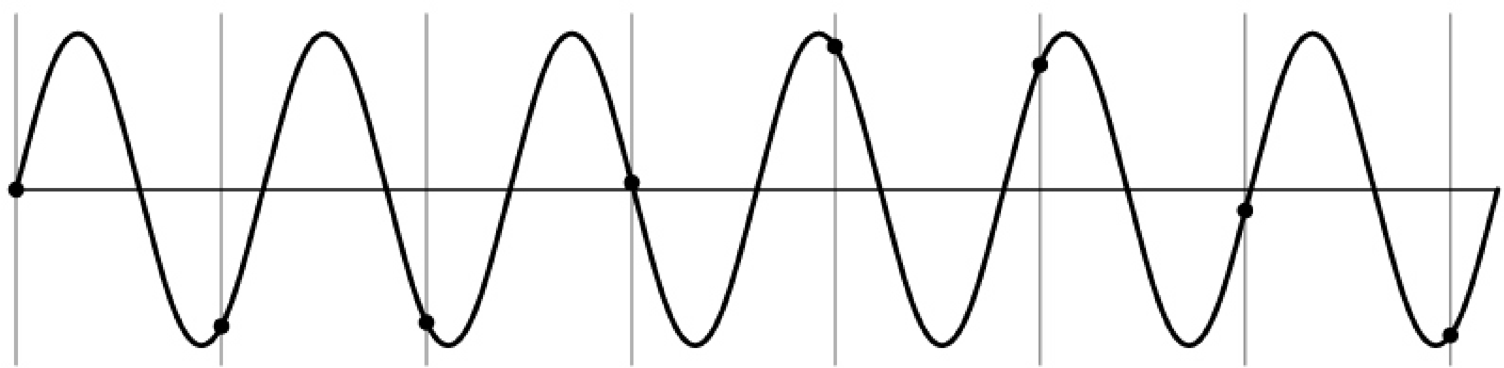
Sampling and Reconstruction

- Simple example: a sign wave



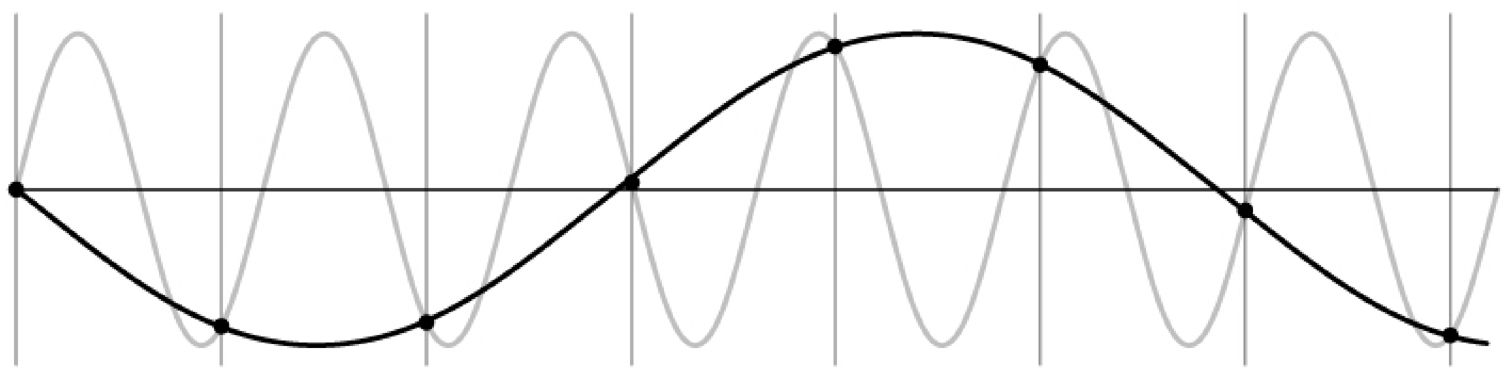
Undersampling

- What if we "missed" things between the samples?
- Simple example: undersampling a sine wave
  - unsurprising result: information is lost



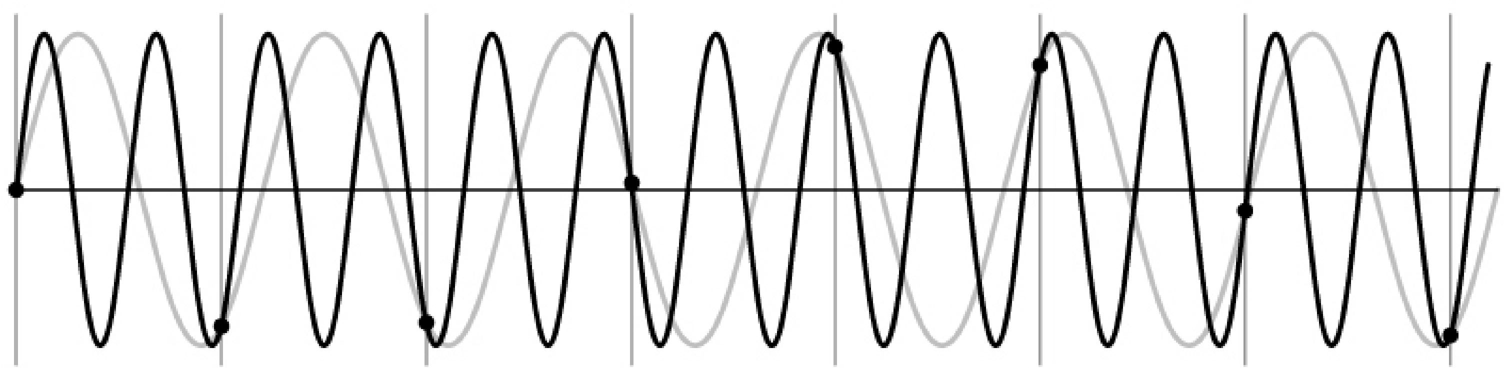
Undersampling

- What if we “missed” things between the samples?
- Simple example: undersampling a sine wave
  - unsurprising result: information is lost
  - surprising result: indistinguishable from lower frequency



Undersampling

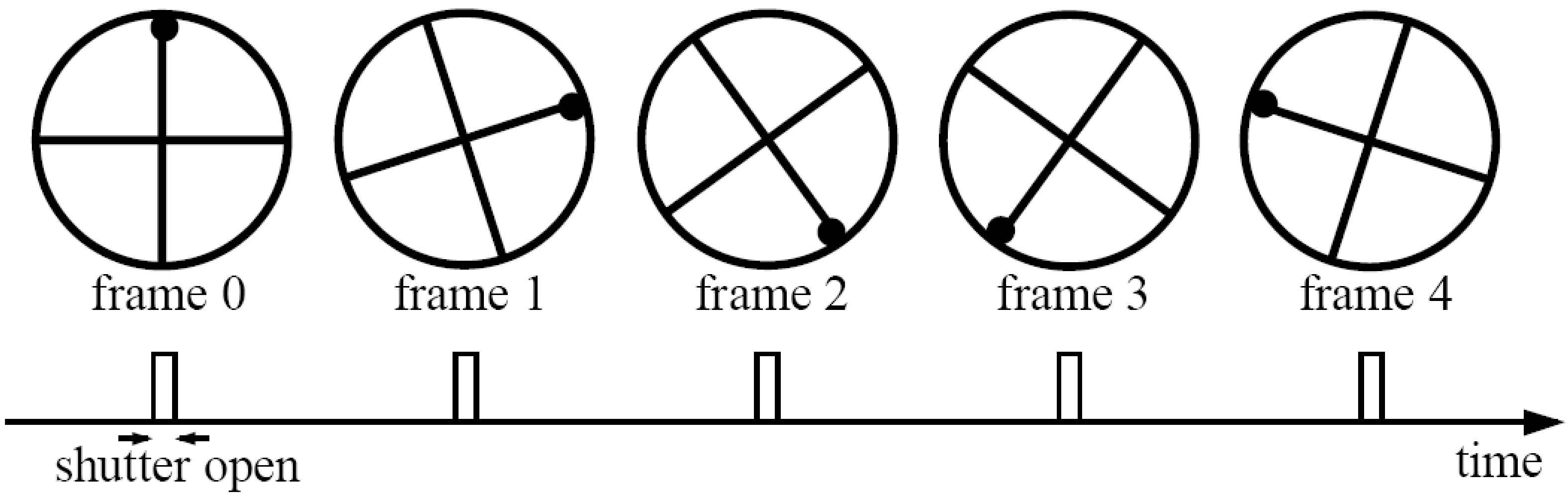
- What if we "missed" things between the samples?
- Simple example: undersampling a sine wave
  - unsurprising result: information is lost
  - surprising result: indistinguishable from lower frequency
  - also, was always indistinguishable from higher frequencies
  - *aliasing*: signals "traveling in disguise" as other frequencies



Aliasing in video

Imagine a spoked wheel moving to the right (rotating clockwise). Mark wheel with dot so we can see what's happening.

If camera shutter is only open for a fraction of a frame time (frame time = 1/30 sec. for video, 1/24 sec. for film):



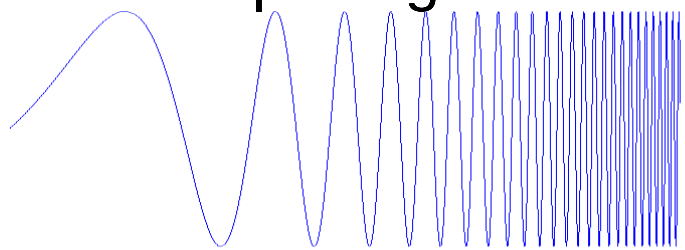
Without dot, wheel appears to be rotating slowly backwards! (counterclockwise)

Slide by Steve Seitz

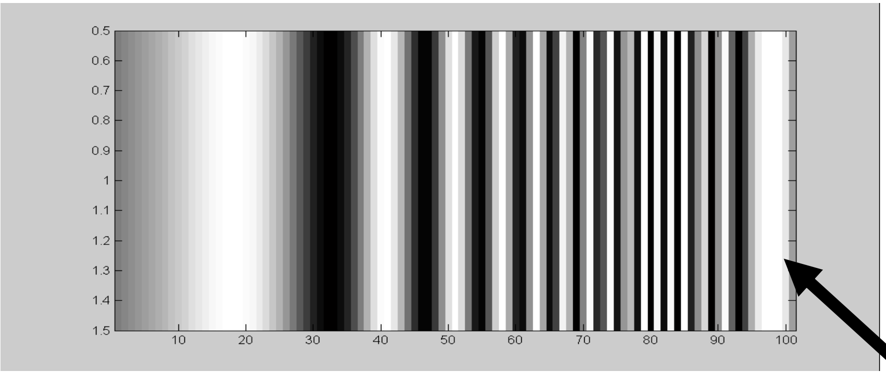
Aliasing in images

What's happening?

Input signal:

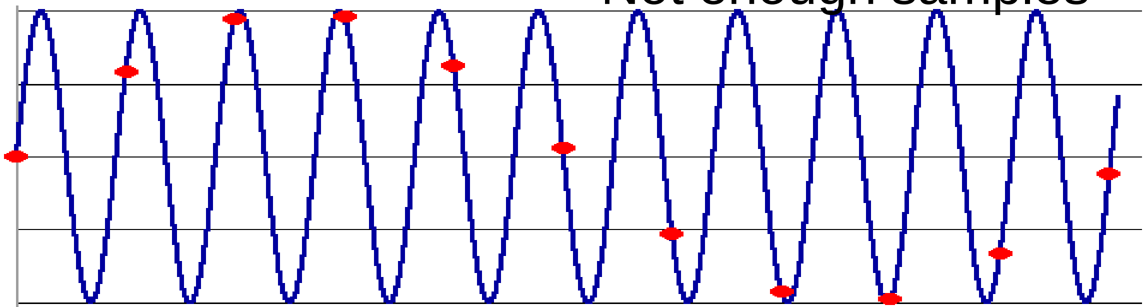


Plot as image:



x = 0:.05:5; imagesc(sin((2.^x).*x))

Alias!
Not enough samples



Antialiasing

What can we do about aliasing?

Sample more often

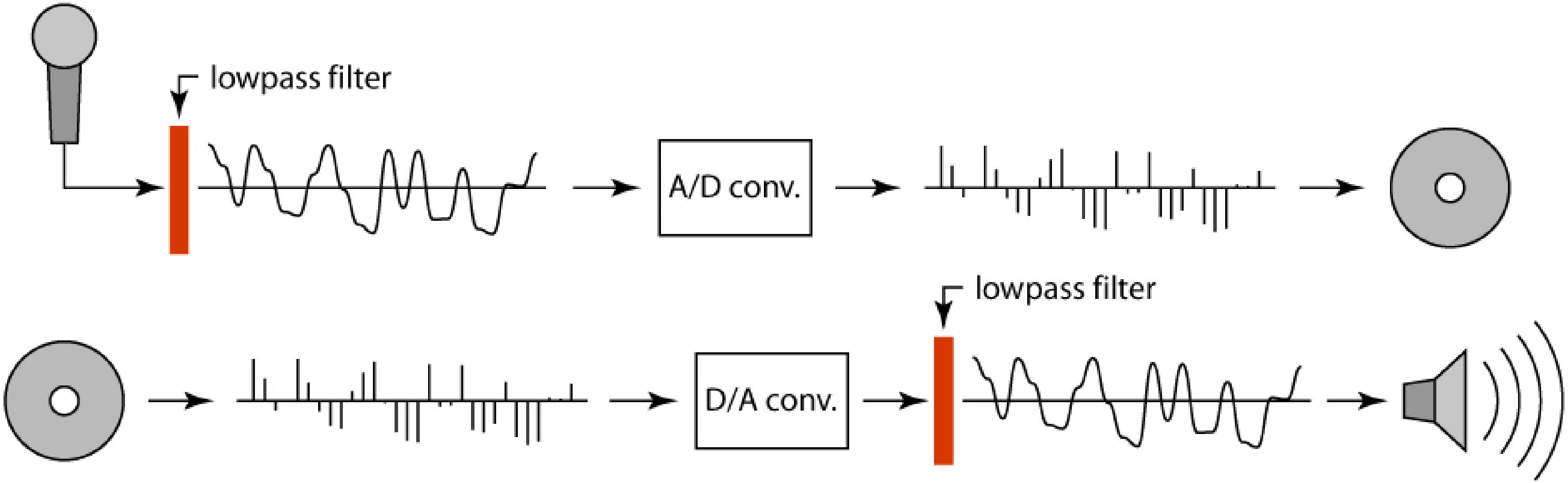
- Join the Mega-Pixel craze of the photo industry
- But this can’t go on forever

Make the signal less “wiggly”

- Get rid of some high frequencies
- Will loose information
- But it’s better than aliasing

Preventing aliasing

- Introduce lowpass filters:
  - remove high frequencies leaving only safe, low frequencies
  - choose lowest frequency in reconstruction (disambiguate)

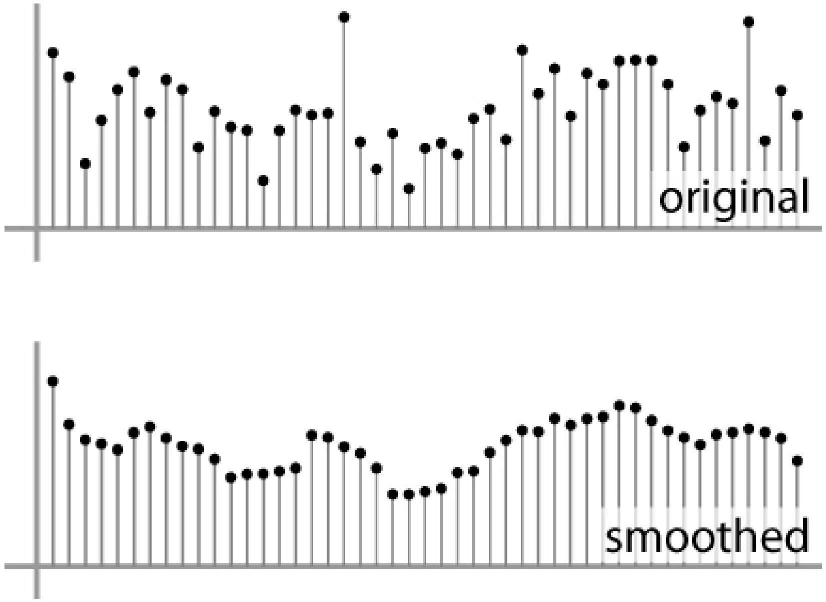


Linear filtering: a key idea

- Transformations on signals; e.g.:
  - bass/treble controls on stereo
  - blurring/sharpening operations in image editing
  - smoothing/noise reduction in tracking
- Key properties
  - linearity: filter(f + g) = filter(f) + filter(g)
  - shift invariance: behavior invariant to shifting the input
    - delaying an audio signal
    - sliding an image around
- Can be modeled mathematically by *convolution*

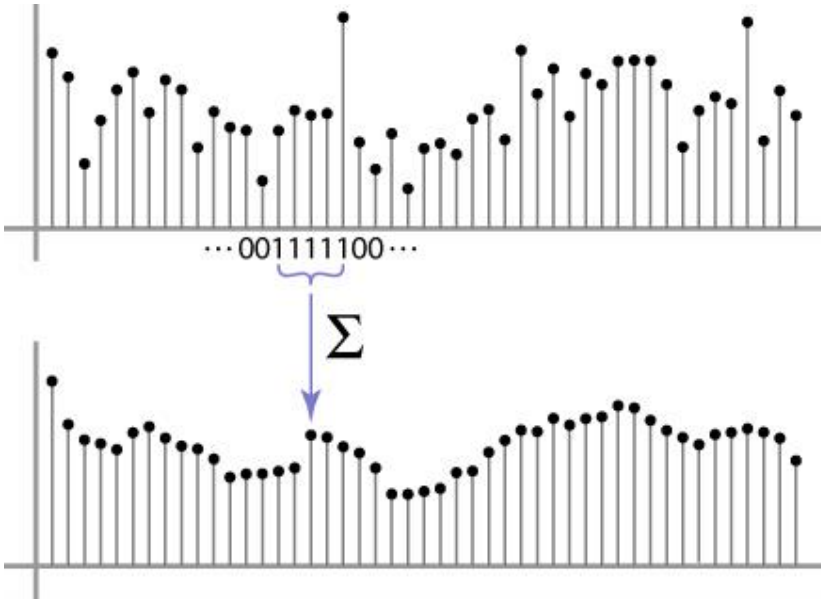
Moving Average

- basic idea: define a new function by averaging over a sliding window
- a simple example to start off: smoothing



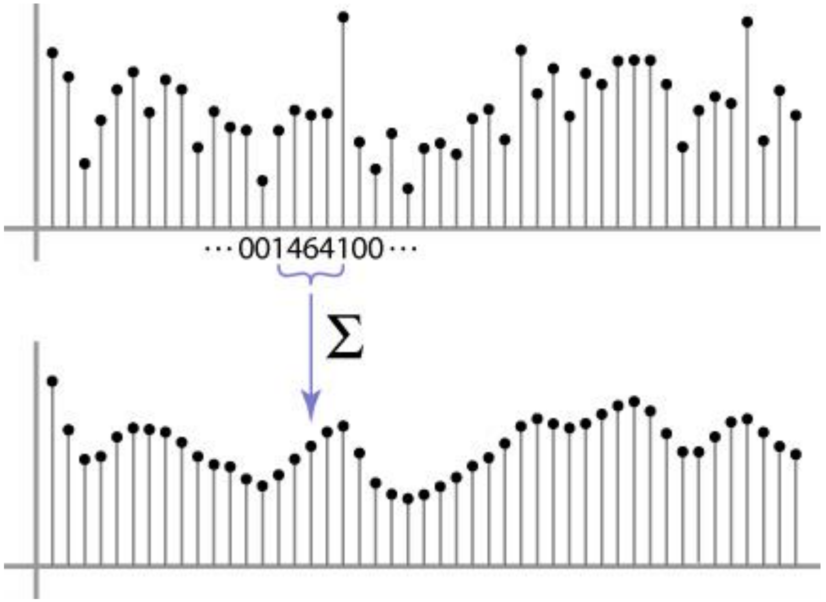
Weighted Moving Average

- Can add weights to our moving average
- *Weights* […, 0, 1, 1, 1, 1, 1, 0, …] / 5



Weighted Moving Average

- bell curve (gaussian-like) weights [..., 1, 4, 6, 4, 1, ...]



Moving Average In 2D

What are the weights H?

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$F[x, y]$

$H[u, v]$

Slide by Steve Seitz

Cross-correlation filtering

- Let's write this down as an equation. Assume the averaging window is (2k+1)x(2k+1):

$$G[i,j] = \frac{1}{(2k+1)^2} \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[i+u, j+v]$$

  - We can generalize this idea by allowing different weights for different neighboring pixels:

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v] F[i+u, j+v]$$

  - This is called a **cross-correlation** operation and written:

$$G = H \otimes F$$

  - H is called the "filter," "kernel," or "mask."

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Gaussian filtering

A Gaussian kernel gives less weight to pixels further from the center of the window

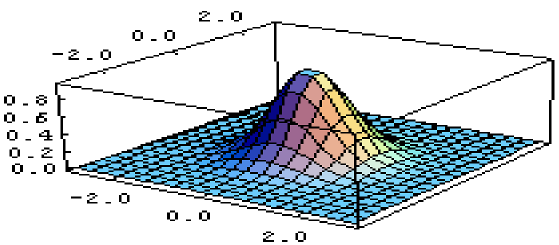
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$F[x, y]$

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$H[u, v]$

$$h(u, v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{\sigma^2}}$$



This kernel is an approximation of a Gaussian function:

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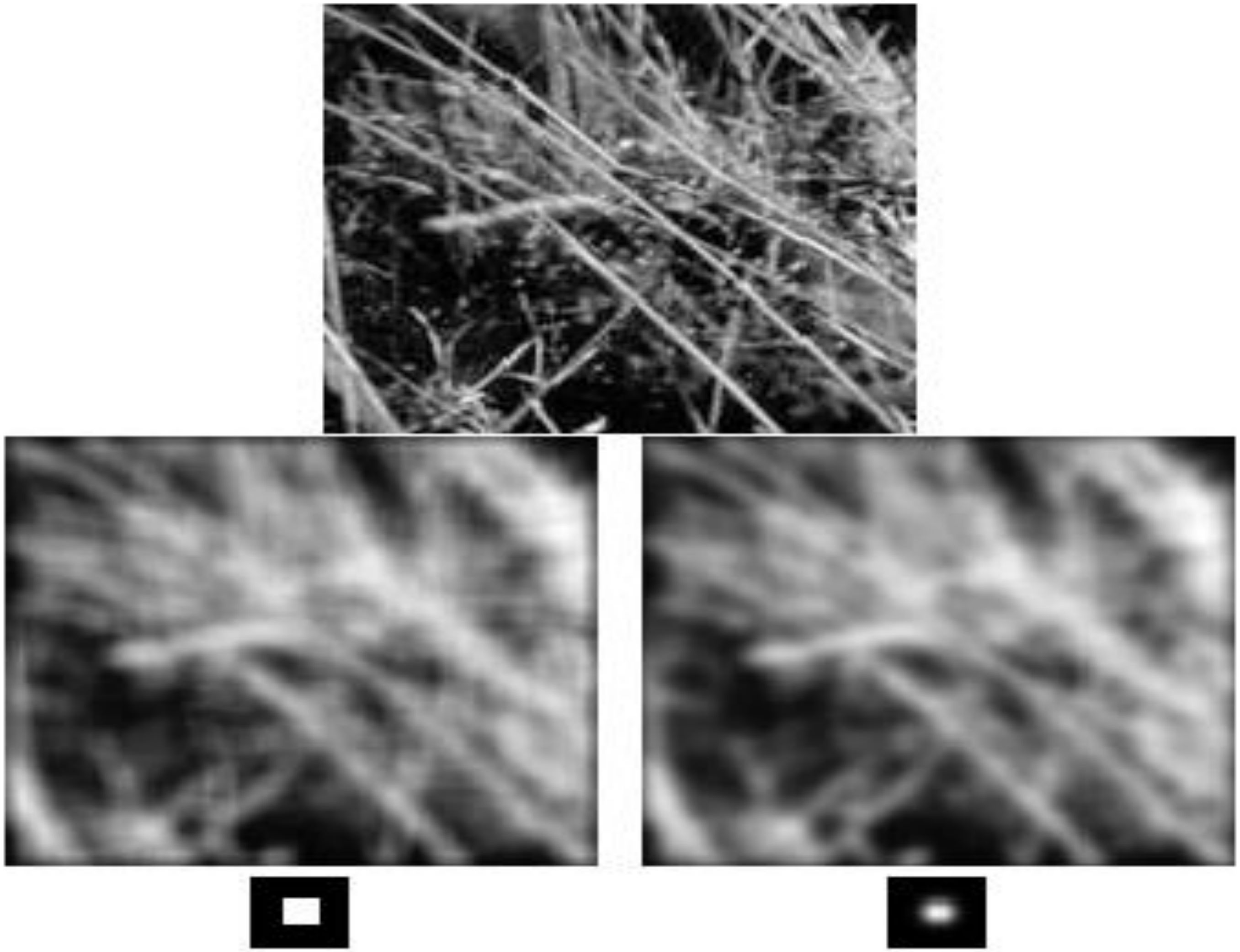
Mean vs. Gaussian filtering



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Convolution

cross-correlation: $G = H \otimes F$

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v] F[i+u, j+v]$$

A **convolution** operation is a cross-correlation where the filter is flipped both horizontally and vertically before being applied to the image:

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v] F[i-u, j-v]$$

It is written:

$$G = H \star F$$

Suppose H is a Gaussian or mean kernel. How does convolution differ from cross-correlation?

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Convolution is nice!

- Notation: $b = c \star a$
- Convolution is a multiplication-like operation
  - commutative $a \star b = b \star a$
  - associative $a \star (b \star c) = (a \star b) \star c$
  - distributes over addition $a \star (b + c) = a \star b + a \star c$
  - scalars factor out $\alpha a \star b = a \star \alpha b = \alpha(a \star b)$
  - identity: unit impulse *e* = […, 0, 0, 1, 0, 0, …]

    $a \star e = a$
- Conceptually no distinction between filter and signal
- Usefulness of associativity
  - often apply several filters one after another: $(((a * b_1) * b_2) * b_3)$
  - this is equivalent to applying one filter: a * $(b_1 * b_2 * b_3)$

Practice with linear filters

Original

0	0	0
0	1	0
0	0	0

?

Source: D. Lowe

Practice with linear filters

Original

0	0	0
0	1	0
0	0	0

Filtered
(no change)

Source: D. Lowe

Practice with linear filters

Original

0	0	0
0	0	1
0	0	0

?

Source: D. Lowe

Practice with linear filters

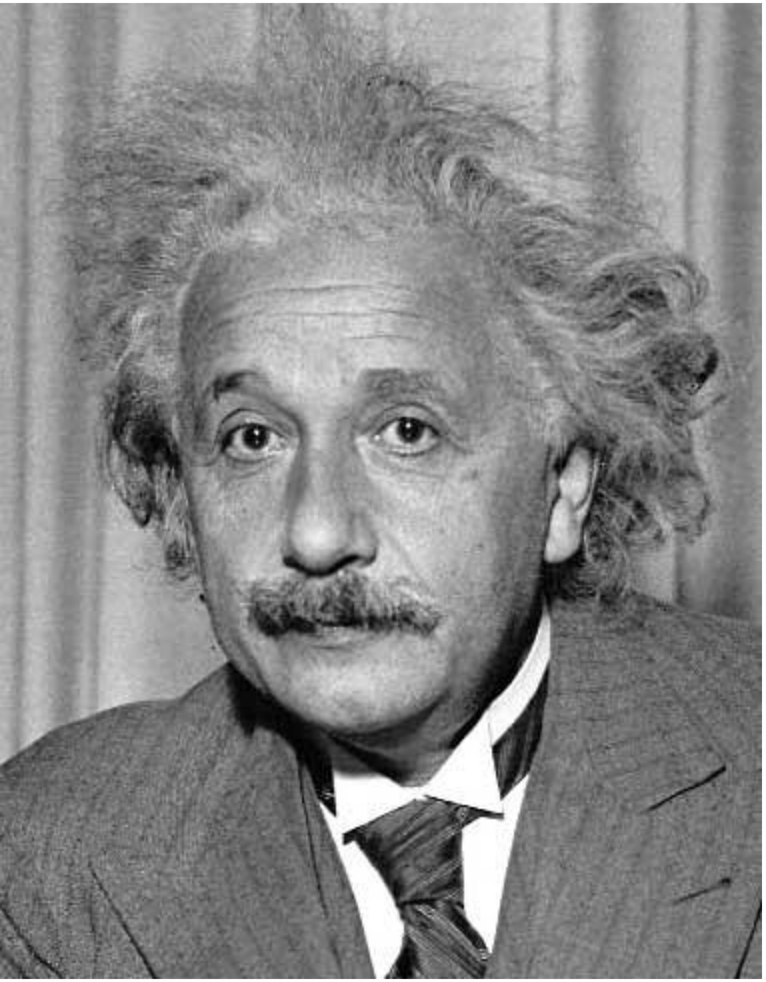
Original

0	0	0
0	0	1
0	0	0

Shifted left
By 1 pixel

Source: D. Lowe

Other filters



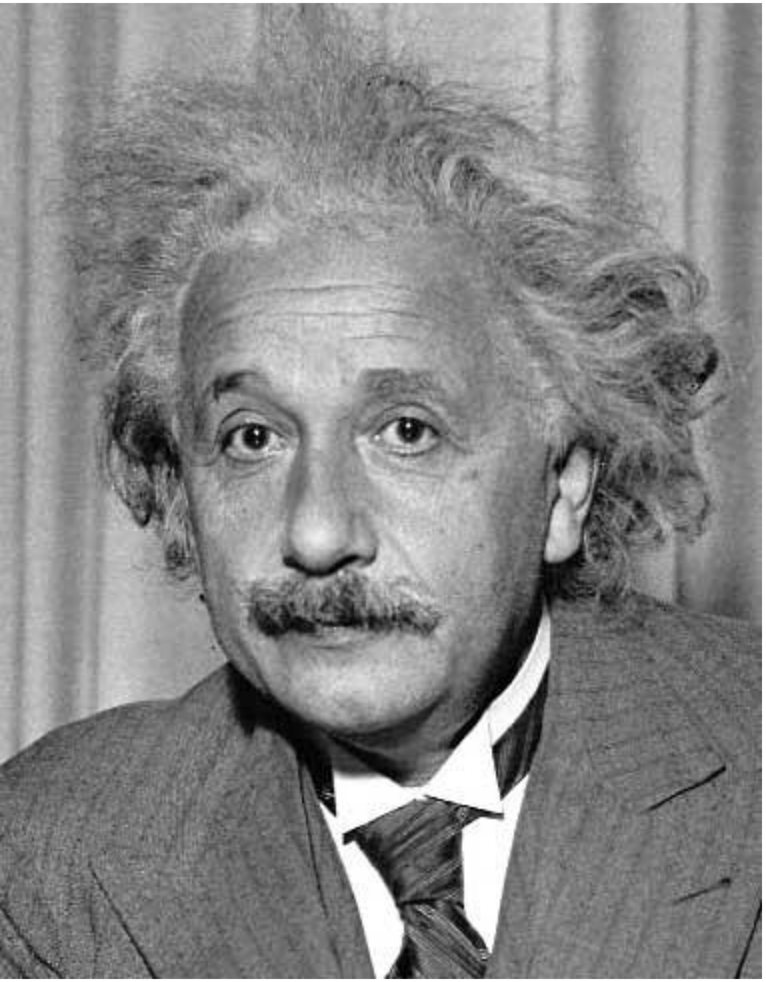
1	0	-1
2	0	-2
1	0	-1

Sobel

Vertical Edge
(absolute value)

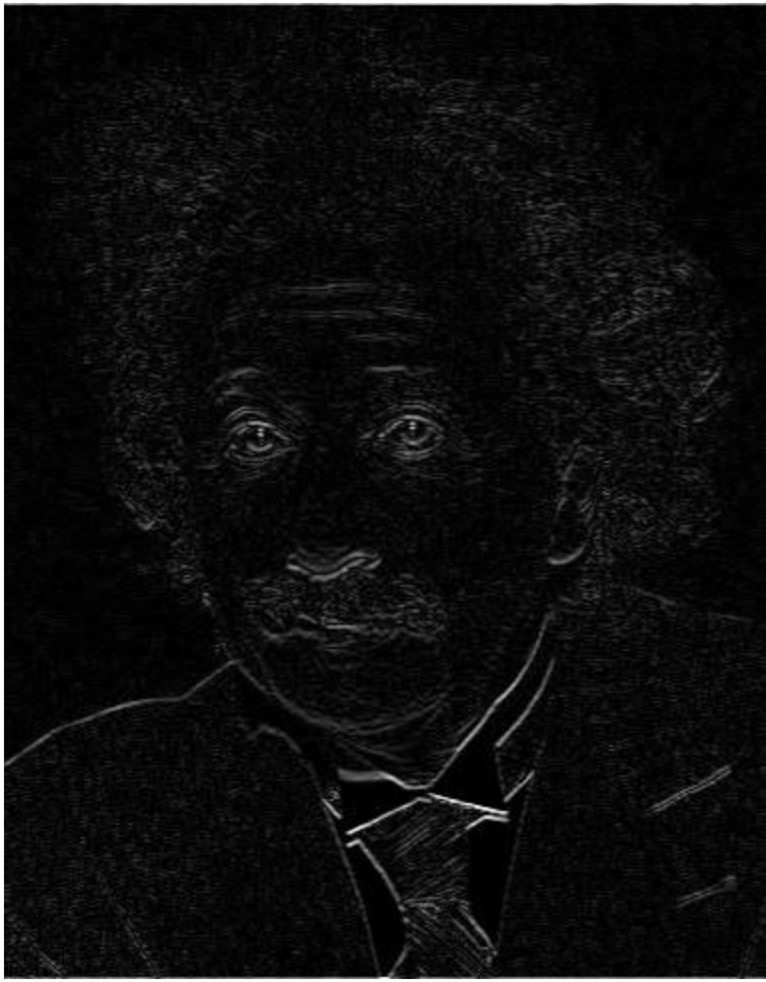
Q?

Other filters



1	2	1
0	0	0
-1	-2	-1

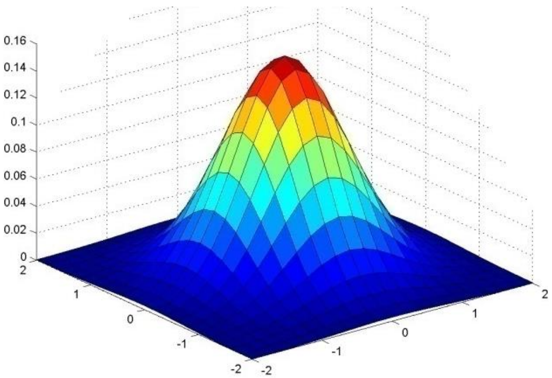
Sobel

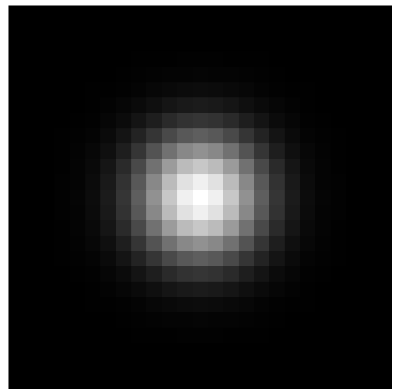


Horizontal Edge
(absolute value)

Important filter: Gaussian

Weight contributions of neighboring pixels by nearness





0.003	0.013	0.022	0.013	0.003
0.013	0.059	0.097	0.059	0.013
0.022	0.097	0.159	0.097	0.022
0.013	0.059	0.097	0.059	0.013
0.003	0.013	0.022	0.013	0.003

5 x 5, σ = 1

$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

Slide credit: Christopher Rasmussen