Prove or disprove each of the following statements. Write detailed proof structures and justify your work.

1. For all real numbers r, s, if r and s are both positive, then $\sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$.

FIRST, *write the statement symbolically:*

$$\forall r \in \mathbb{R}, \forall s \in \mathbb{R}, r > 0 \wedge s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$$

SECOND, *try a direct proof:*

Assume $r \in \mathbb{R}$ *and* $s \in \mathbb{R}$
 Assume $r > 0$ *and* $s > 0$
  Then, $\sqrt{r} + \sqrt{s} = \ldots$ NO OBVIOUS WAY TO CONTINUE.

NEXT, *try an indirect proof:*

Assume $r \in \mathbb{R}$ *and* $s \in \mathbb{R}$.
 Assume $\sqrt{r} + \sqrt{s} = \sqrt{r+s}$.
  Then, $(\sqrt{r} + \sqrt{s})^2 = (\sqrt{r+s})^2$. # *square both sides*
  Then, $(\sqrt{r})^2 + 2\sqrt{r}\sqrt{s} + (\sqrt{s})^2 = r + s$. # *expand both sides*
  Then, $2\sqrt{rs} = 0$. # *subtract* $r + s$ *from both sides*
  Then, $rs = 0$. # *divide by 2 and square both sides*
  Then, $r = 0 \vee s = 0$.
  # NOW, DO A SUB-PROOF BY CASES.
  Assume $r = 0$.
   Then, $r \not> 0$.
   Then, $r \not> 0 \vee s \not> 0$.
   Then, $\neg(r > 0 \wedge s > 0)$.
  Assume $s = 0$.
   Then, $s \not> 0$.
   Then, $r \not> 0 \vee s \not> 0$.
   Then, $\neg(r > 0 \wedge s > 0)$.
  In either case, $\neg(r > 0 \wedge s > 0)$.
 Then, $\sqrt{r} + \sqrt{s} = \sqrt{r+s} \Rightarrow \neg(r > 0 \wedge s > 0)$
 Then, $r > 0 \wedge s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$.
Then, $\forall r \in \mathbb{R}, \forall s \in \mathbb{R}, r > 0 \wedge s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$.

2. For all real numbers x and y, $x^4 + x^3y - xy^3 - y^4 = 0$ exactly when $x = \pm y$.

FIRST, *write the statement symbolically (be careful to handle that "±" correctly):*

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^4 + x^3y - xy^3 - y^4 = 0 \Leftrightarrow (x = y \vee x = -y)$$

SECOND, *start the proof structure for the universal quantifiers:*

Assume $x \in \mathbb{R}$ *and* $y \in \mathbb{R}$.
  # TO PROVE AN EQUIVALENCE, WE PROVE THE IMPLICATION IN EACH DIRECTION.
  First assume $x^4 + x^3y - xy^3 - y^4 = 0$.
    Then, $x^3(x + y) - y^3(x + y) = 0$.   # *factor out the expression*
    Then, $(x^3 - y^3)(x + y) = 0$.   # *factor out the expression*
    Then, $x^3 - y^3 = 0 \vee x + y = 0$.   # $ab = 0 \Leftrightarrow a = 0 \vee b = 0$
    # NOW, DO A SUB-PROOF BY CASES.
    Assume $x^3 - y^3 = 0$
      Then, $x^3 = y^3$ # *add* y^3 *to both sides*
      Then, $x = y$ # *take cube roots on both sides, cube root is one-to-one so we can do it*
      Then, $x = y \vee x = -y$ # *introduce* $\vee$
    Assume $x + y = 0$
      Then, $x = -y$ # *subtract* y *from both sides*
      Then, $x = y \vee x = -y$ # *introduce* $\vee$
    In either case, $x = y \vee x = -y$.
  Then, $x^4 + x^3y - xy^3 - y^4 = 0 \Rightarrow x = \pm y$.
  Next assume $x = \pm y$.
    Then, $x = y \vee x = -y$.   # *expand "±"*
    # NOW, DO A SUB-PROOF BY CASES.
    Assume $x = y$.
      Then, $x^3 = y^3$.   # *cube both sides*
      Then, $x^3 - y^3 = 0$.   # *subtract* y^3 *from both sides*
      Then, $(x^3 - y^3)(x + y) = 0$.   # *multiply both sides by* $(x + y)$
      Then, $x^4 + x^3y - xy^3 - y^4 = 0$.   # *expand*
    Assume $x = -y$.
      Then, $x + y = 0$.   # *add* y *to both sides*
      Then, $(x^3 - y^3)(x + y) = 0$.   # *multiply both sides by* $(x^3 - y^3)$
      Then, $x^4 + x^3y - xy^3 - y^4 = 0$.   # *expand*
    In both cases, $x^4 + x^3y - xy^3 - y^4 = 0$.
  Then, $x = \pm y \Rightarrow x^4 + x^3y - xy^3 - y^4 = 0$.
  Then, $x^4 + x^3y - xy^3 - y^4 = 0 \Leftrightarrow x = \pm y$.   # *introduce* $\Leftrightarrow$
Then, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^4 + x^3y - xy^3 - y^4 = 0 \Leftrightarrow (x = y \vee x = -y)$.

Notice how the detailed proof structure makes it easy to keep track of assumptions, and cases and sub-cases, and to know exactly when we are done.