1. Consider the following statement:

   If m and n are odd integers, then mn is an odd integer.

   (a) Express the statement using logical notation.

   $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, [(m \text{ is odd} \wedge n \text{ is odd}) \Rightarrow (mn \text{ is odd})]$
   Alternate: $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, [(\exists k \in \mathbb{Z}, m = 2k+1) \wedge (\exists k \in \mathbb{Z}, n = 2k+1) \Rightarrow (\exists k \in \mathbb{Z}, mn = 2k+1)]$

   (b) This statement can be proven using a direct proof. Write a detailed proof *structure* for the statement. **Don't write a complete proof** — for now, focus on the proof structure only and leave out *all* of the "middle" of the argument.

   Assume $m, n \in \mathbb{Z}$. # m and n are arbitrary elements of $\mathbb{Z}$
   Assume $(m \text{ is odd} \wedge n \text{ is odd})$. # the antecedent
   ⋮
   Then mn is odd. # definition of odd
   Then $(m \text{ is odd} \wedge n \text{ is odd}) \Rightarrow (mn \text{ is odd})$. # introduce $\Rightarrow$
   Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, [(m \text{ is odd} \wedge n \text{ is odd}) \Rightarrow (mn \text{ is odd})]$ # introduce $\forall$

   (c) Now, complete the proof of the statement.

   Assume $m, n \in \mathbb{Z}$. # m and n are arbitrary elements of $\mathbb{Z}$
   Assume $(m \text{ is odd} \wedge n \text{ is odd})$. # the antecedent
   Then $(\exists k \in \mathbb{Z}, m = 2k+1)$ and $(\exists k \in \mathbb{Z}, n = 2k+1)$. # definition of odd
   Let $i \in \mathbb{Z}$ be such that $m = 2i+1$. # label the quotient $m/2$ by i
   Let $j \in \mathbb{Z}$ be such that $n = 2j+1$. # label the quotient $n/2$ by j
   Then $mn = (2i+1)(2j+1)$ # substitution
   $= 4ij + 2i + 2j + 1$ # algebraic manipulation
   $= 2(2ij + i + j) + 1$
   Let $p = 2ij + i + j$.
   Then $p \in \mathbb{Z}$. # since $\mathbb{Z}$ closed under $+$, $\times$.
   Then $mn = 2p + 1$. # substitution
   Then $\exists k \in \mathbb{Z}, mn = 2k+1$.
   Then mn is odd. # definition of odd
   Then $(m \text{ is odd} \wedge n \text{ is odd}) \Rightarrow (mn \text{ is odd})$. # introduce $\Rightarrow$
   Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, [(m \text{ is odd} \wedge n \text{ is odd}) \Rightarrow (mn \text{ is odd})]$ # introduce $\forall$

2. Consider the following statement:

   If m and n are integers with mn odd, then m and n are odd.

(a) Express the statement using logical notation.

$\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, [(mn \text{ is odd}) \Rightarrow (m \text{ is odd} \wedge n \text{ is odd})]$
Alternate: $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, [(\exists k \in \mathbb{Z}, mn = 2k+1) \Rightarrow (\exists k \in \mathbb{Z}, m = 2k+1) \wedge (\exists k \in \mathbb{Z}, n = 2k+1)]$

(b) This statement can be proven using an **indirect** proof. Write a detailed proof *structure* for the statement. **Don't write a complete proof** — for now, focus on the proof structure only and leave out *all* of the "middle" of the argument.

Assume $m, n \in \mathbb{Z}$. # m and n are arbitrary elements of $\mathbb{Z}$
  Assume (m is even $\vee$ n is even). # the negation of the consequent
    [*Since at least one of m or n is even, let us label one of the even numbers as m and make no assumption about n. The number n could be odd or even. (This argument is often labelled "Without loss of generality, assume m is even." or "WLOG assume m is even.")*]
    WLOG, assume m is even.
      ⋮
      Then mn is even.
    Then (m is even) $\Rightarrow$ (mn is even). # introduce $\Rightarrow$
  Then (m is even $\vee$ n is even) $\Rightarrow$ (mn is even). # introduce disjuction in antecedent
  Then (mn is odd) $\Rightarrow$ (m is odd $\wedge$ n is odd). # apply contrapositive
Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, [(mn \text{ is odd}) \Rightarrow (m \text{ is odd} \wedge n \text{ is odd})]$. # introduce $\forall$

(c) Now, complete the proof of the statement.

Assume $m, n \in \mathbb{Z}$. # m and n are arbitrary elements of $\mathbb{Z}$
  Assume (m is even $\vee$ n is even). # the negation of the consequent
    WLOG, assume m is even.
      Then $\exists k \in \mathbb{Z}, m = 2k$. # definition of even
      Let $i \in \mathbb{Z}$ be such that $m = 2i$. # label the quotient $m/2$ by i
      Then $mn = 2in$ # substitution
      $= 2(in)$ # associativity
      Let $p = in$.
      Then $p \in \mathbb{Z}$. # since $\mathbb{Z}$ closed under $\times$.
      Then $mn = 2p$. # substitution
      Then $\exists k \in \mathbb{Z}, mn = 2k$.
      Then mn is even. # definition of even
    Then (m is even) $\Rightarrow$ (mn is even). # introduce $\Rightarrow$
  Then (m is even $\vee$ n is even) $\Rightarrow$ (mn is even). # introduce disjuction in antecedent
  Then (mn is odd) $\Rightarrow$ (m is odd $\wedge$ n is odd). # apply contrapositive
Then $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, [(mn \text{ is odd}) \Rightarrow (m \text{ is odd} \wedge n \text{ is odd})]$. # introduce $\forall$