CSC165, Summer 2014
Assignment 5
Solutions

The goal of this assignment is for you to keep practicing writing proofs. Our goal this semester is for you to learn to write proofs by the end of the course, and the only way to learn to write proofs is through practice. **In your proofs, justify each step**. If you are asked to prove or disprove a claim, first determine whether the claim is true, and then prove that it is true or prove that it is false (i.e., that its negation is true), depending on which is correct.

You may work in groups of no more than two students, and **you should submit a TEX file named `a5.tex` and a PDF file named `a5.pdf` that was produced by compiling your `a4.tex`** and that contains the answers to the questions below. You should also submit your Python code in `a5.py` These files should be submitted using MarkUs.

For this assignment, you will **not** receive 20% of the marks for leaving questions blank or writing "I cannot answer this."

1. Prove that

$$\forall a \in \mathbb{R}, \forall n \in \mathbb{N}, [0 < a < 1] \Rightarrow a^n \leqslant 1$$

using mathematical induction. Justify every step, and use the detailed structured proof format (you can follow the format used in the induction handout on the website.)

Solution:

First, we define the predicate: $P(k) := [\forall a \in \mathbb{R}, [0 < a < 1] \Rightarrow a^k \leqslant 1]$

Base case:

Assume $a \in \mathbb{R}$
  Assume $0 < a < 1$
    Then $a^0 = 1 \leqslant 1$ # algebra
  Then $[0 < a < 1] \Rightarrow a^0 \leqslant 1$ # introduce implication
Then $\forall a \in \mathbb{R}, [0 < a < 1] \Rightarrow a^0 \leqslant 1$ # introduce universal
Then $P(0)$ # *substitution*

Induction step:

Assume $k \in \mathbb{N}$
Assume $P(k)$ is true
Then $\forall a \in \mathbb{R}, [0 < a < 1] \Rightarrow a^k \leqslant 1$ # substitution
Assume $a \in \mathbb{R}$
Assume $0 < a < 1$
Then $a^k \leqslant 1$ # implication
Then $a * a^k \leqslant a * 1$ # $a > 0$
Then $a^{k+1} \leqslant a < 1$ # $a < 1$
Then $a^{k+1} < 1$ # transitivity
Then $0 < a < 1 \Rightarrow a^{k+1} < 1$ # introduce implication
Then $\forall a \in \mathbb{R}, [0 < a < 1 \Rightarrow a^{k+1} \leqslant 1]$ # introduce universal
Then $P(k+1)$ # substitution
Then $P(k) \Rightarrow P(k+1)$ # introduce implication
Then $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$ # introduce universal

We now conclude:
$P(0)$ # proven above
Also, $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$ # proven above
Then $\forall n \in \mathbb{N}, P(n)$ # principle of simple induction
Then $\forall n \in \mathbb{N}, \forall a \in \mathbb{R}, [0 < a < 1] \Rightarrow a^n \leqslant 1$ # substitution
Then $\forall a \in \mathbb{R}, \forall n \in \mathbb{N}, [0 < a < 1] \Rightarrow a^n \leqslant 1$ # universal quantifiers commute

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Note: this proof is somewhat fussier than what should get perfect marks, but note that it is necessary to make sure that a and n are defined wherever they are used.

2. Prove that

$$\forall n \in \mathbb{N}, [n > 2 \Rightarrow n! < n^n]$$

using mathematical induction. Justify every step, and use the detailed structured proof format (you can follow the format used in the induction handout on the website).

Solution:

We prove a helpful lemma: $\forall k \in \mathbb{N}, k^k < (k+1)^k$:
Assume $k \in \mathbb{N}$
Then $0 < \frac{k}{k+1} \leqslant 1$ # $k > 0, k+1 > 1, k < k+1$
Then $\frac{k}{k+1}^k \leqslant 1$ # Question 1
Then, $\frac{k^k}{(k+1)^k} \leqslant 1$ # algebra
Then, $k^k \leqslant (k+1)^k$ # Multiply both sides by $(k+1)^k$
Then $\forall k \in \mathbb{N}, k^k \leqslant (k+1)^k$ # introduce universal

Now, define the predicate: $P(k) := k! < k^k$

Base case:

$3! = 1 * 2 * 3 = 6$ # algebra
Also, $3^3 = 3 * 3 * 3 = 27$ # algebra
Then $3! < 3^3$ # $6 < 27$
Then $P(3)$ # substitution

Induction step:

Assume $k \in \mathbb{N}$

Assume $P(k)$ is true

Then $k! < k^k$ # substitution Then $(k+1) * k! < (k+1) * k^k$ # Multiply both sides by $k+1 > 0$

Then $(k+1)! < (k+1) * k^k < (k+1) * (k+1)^k = (k+1)^{(k+1)}$ # algebra, the lemma above

Then $(k+1)! < (k+1)^{(k+1)}$ # transitivity

Then $P(k+1)$ # substitution

Then $P(k) \Rightarrow P(k+1)$ # implication

Then $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$ # introduce universal

(Note that the induction starts at 3, but the induction step can start at 0 as usual, since False implies True and False implies False are both true.)

We can now conclude:

$P(3)$ is true # proven above

$\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$ # proven above

Then $\forall n \in \{3, 4, 5, 6, ...\}, P(n)$ # principle of simple induction

Then $\forall n \in \mathbb{N}, n > 2 \Rightarrow P(n)$ # $[(n \in \mathbb{N}) \wedge (n > 2)] \Leftrightarrow [n \in \{3, 4, 5, 6, ...\}]$

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3. (a) Write Python code to determine the how many integers between 0 and n (inclusive) are expressible as the sum of **squares** of two (possibly equal) positive natural numbers in at least two different ways. For example, $50 = 5^2 + 5^2 = 7^1 + 1^1$ is expressible as the sum of squares of two positive natural numbers in at least two different ways. On the other hand, the only way to express 2 as a sum of squares of positive natural number is $2 = 1^2 + 1^2$, and 3 is not expressible as a sum of squares of two positive natural numbers at all. Submit the code as `a5.py`, and submit the relevant parts of the output which shows your answer for $n = 100$ (this could be just one line), explaining clearly what it means, as part of your answer to this question. Justify your answer briefly (a complete formal proof is not required.)

   Solution:

   The output for `n = 100` is:

   `There are 3 numbers expressible as a sum of two squares in two different ways between 1 and 100.`

   The script works by checking whether each number between 1 and n is expressible as a sum of squares in at least two different ways. For each k, we consider all the possibilities $i^2 + (k-i)^2$ for $i < (k-i)$ for $0 < i \leqslant \lceil \sqrt{n/2} \rceil$, which covers all the possibilities.

   (b) What is a tight upper bound on the number of comparison operations (i.e., ==, <, <=, >, >=) that are executed when running your algorithm for a given `n`? Ignore the comparison operations that are performed when running functions from the `math` module.//

   Solution:

   The loop in `is_two_sum_sq` runs for $\sqrt{n/2}$ iterations at 2 comparisons each when checking whether a solution has been found, and there are at most 2 times that `found` is compared to 2, so the upper bound there is $2\sqrt{n/2} + 2$. Checking whether $\{1, 2, 3, ..., n\}$ are each expressible as a sum of two squares in two different ways takes

$$(2\sqrt{1/2} + 2) + (2\sqrt{2/2} + 2) + (2\sqrt{3/2} + 2) + ... + (2\sqrt{n/2} + 2) = 2n + (2/\sqrt{2}) \sum_{i=1}^{n} \sqrt{i}$$

   This is a tight upper bound, and would be sufficient as an answer. (Using a formula from http://mathforum.org/library/drmath/view/65309.html, you can get that the asymptotic upper bound is $\mathcal{O}(n^{3/2})$ in this case.)

The analysis would be more straightforward if I didn't try to minimize the number of iterations in line 14. If I had used n iterations instead of $\sqrt{n/2}$ iterations in `is_two_sum_sq`, I would be performing at most $2k+2$ comparisons for each k, and so in total the upper bound would be

$$\sum_{i=1}^{n}[2i+2] = n(n+1) + 2n = n^2 + 3n$$

4. Prove that
$$\exists k \in \mathbb{N}, \forall n \in \mathbb{N}, [n > k] \Rightarrow [1000n^2 + 10 \leqslant n^4].$$

Hints: you can divide both sides by n^2 and preserve the inequality, since n^2 is always positive. Reminder: $n^a/n^b = n^{a-b}$. You can then figure out (in your rough work) what value of k you need. Note that $10/n^2 < 1$ if $n \geqslant 4$. Justify every step, and use the detailed structured proof format.

Solution:

Let k = 32
Then $k \in \mathbb{N}$ # $32 \in \mathbb{N}$
Assume $n \in \mathbb{N}$
  Assume $n > k$
    Then $10/n^2 < 1$ # $n^2 > 32^2 = 1024 > 10$
    Then $1000 + 10/n^2 < 1001 < 1023 < 32^2 < n^2$ # $10/n^2 < 1, n > 32$
    Then $1000 + 10/n^2 < n^2$ # transitivity of $<$
    Then $1000n^2 + 10 < n^4$ # multiply both sides by n^2
  Then $[n > k] \Rightarrow 1000n^2 + 10 < n^4$ # introduce implication
Then $\forall n \in \mathbb{N}, [n > k] \Rightarrow 1000n^2 + 10 < n^4$ # introduce universal

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5. Let $\mathbb{R}^+$ be the set of positive real numbers and $\mathbb{N}^+$ be the set of positive natural numbers. Prove that

$$\forall a \in \mathbb{R}^+, \forall b \in \mathbb{R}^+, \forall p_1 \in \mathbb{N}^+, \forall p_2 \in \mathbb{N}^+, [p_1 \leqslant p_2] \Rightarrow an^{p_1} \in \mathcal{O}(bn^{p_2}.)$$

You may not use, without proof, any properties of big-Oh, other than its definition. Justify every step, and use the detailed structured proof format.

Solution:

Assume $p_1 \in \mathbb{N}^+, p_2 \in \mathbb{N}^+, a \in \mathbb{R}^+, b \in \mathbb{R}^+$
  Assume $p_1 \leqslant p_2$
    Let $c = \frac{a}{b}$ # $b \neq 0$ since $b \in \mathbb{R}^+$
    Then $c \in \mathbb{R}^+$ # both a and b are $\in \mathbb{R}^+$ and it is closed under division
    Let $B = 1$
    Then $B \in \mathbb{N}$ # $1 \in \mathbb{N}$
    Assume $n \in \mathbb{N}$
      Assume $n \geqslant B$
        Then $n^{p_2 - p_1} \geqslant 1$ # $p_2 - p_1 \geqslant 0, n > 1 > 0$
        Then $n^{p_2} \geqslant n^{p_1}$ # multiply both sides by $n^{p_1} > 0$ (since $n > 0$)
        Then $n^{p_1} \leqslant n^{p_2}$ # algebra
        Then $an^{p_1} \leqslant an^{p_2}$ # multiply both sides by $a > 0$
        Then $an^{p_1} \leqslant \frac{a}{b} bn^{p_2}$ # algebra
        Then $an^{p_1} \leqslant c(bn^{p_2})$ # substitution
      Then $n \geqslant B \Rightarrow an^{p_1} \leqslant c(bn^{p_2})$ # introduce implication
    Then $\forall n \in \mathbb{N}, n \geqslant B \Rightarrow an^{p_1} \leqslant c(bn^{p_2})$ # introduce universal
    Then $\exists B \in \mathbb{N}, \exists c \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow an^{p_1} \leqslant c(bn^{p_2})$ # introduce existential, B and c provided above
    Then $an^{p_1} \in \mathcal{O}(bn^{p_2})$ # definition of big-Oh
  Then $[p_1 \leqslant p_2] \Rightarrow [an^{p_1} \in \mathcal{O}(bn^{p_2})]$ # introduce implication
Then $\forall a \in \mathbb{R}^+, \forall b \in \mathbb{R}^+, \forall p_1 \in \mathbb{N}^+, \forall p_2 \in \mathbb{N}^+, [p_1 \leqslant p_2] \Rightarrow an^{p_1} \in \mathcal{O}(bn^{p_2})$ # introduce universal

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