CSC165, Summer 2014
Assignment 4
Weight: 8%
Due Jul. 4th, 2:00 p.m.

You are allowed to use (i.e., put in your justifications), without proof, the following facts:

- $\forall x \in \mathbb{R}, x^2 \geqslant 0$
- For a, b, c real, $a > 0$, $[b^2 - 4ac < 0] \Rightarrow [\forall x, ax^2 + bx + c >= 0]$
- $f(n) = 1/n$ is a decreasing function for $n > 0$. In other words,

$$\forall n_1 \in \mathbb{N}, \forall n_2 \in \mathbb{N}, n_2 > n_1 \Rightarrow 1/n_2 < 1/n_1$$

.

1. Prove or disprove **using a detailed structured proof, justifying every step**:

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, [x(x+1) = y(y+1)] \Leftrightarrow [x = y]$$

Solution: The statement is **false**.

Proof:

Let $x_0 = -1$, $y_0 = 0$
Then $x_0 \in \mathbb{Z}$, $y_0 \in \mathbb{Z}$
Then $x_0(x_0+1) = 0$ # $x_0 + 1 = 0$
Then $y_0(y_0+1) = 0$ # $y_0 = 0$
Then $x_0(x_0+1) = y_0(y_0+1)$ # both equal 0
Then also $\neg[x_0 = y_0]$ # $-1 \neq 0$
Then $[x(x+1) = y(y+1)] \wedge \neg[x = y]$ # conjunction of two true statements
Then $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, [x(x+1) = y(y+1)] \wedge \neg[x = y]$ # x_0, y_0 are such x, y
Then $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \neg([x(x+1) = y(y+1)] \Rightarrow [x = y])$ # implication negation
Then $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, [\neg([x(x+1) = y(y+1)] \Rightarrow [x = y])] \vee [\neg([x(x+1) = y(y+1)] \Leftarrow [x = y])]$
$\#$ True $\vee$ R is True for any R
Then $\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \neg([x(x+1) = y(y+1)] \Leftrightarrow [x = y])$ # equivalence negation
Then $\exists x \in \mathbb{Z}, \neg[\forall y \in \mathbb{Z}, [x(x+1) = y(y+1)] \Leftrightarrow [x = y]]$ # quantifier negation
Then $\neg[\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, [x(x+1) = y(y+1)] \Leftrightarrow [x = y]]$ # quantifier negation

Note: less detailed arguments that still end with the same conclusion may be fine.

2. Prove or disprove **using a detailed structured proof, justifying every step**:

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [(x^2 y < 0) \wedge (x \neq 0)] \Leftrightarrow [x^{100} y < 0]$$

Solution: The statement is **true**.

Proof:

Assume $x \in \mathbb{R}, y \in \mathbb{R}$

Then $x^{49} \in \mathbb{R}$ # the reals are closed under multiplication and $x^{49} = x * x * x * ... * x$

Also $x^{49} \neq 0$ # $\forall x \in \mathbb{R}, x = 0 \Leftrightarrow x^n = 0$

Then $(x^{49})^2 > 0$ # $\forall a \in \mathbb{R}, a \neq 0 \Rightarrow a^2 > 0$

Then $x^{98} > 0$ # algebra

Assume $(x^2 y < 0) \wedge (x \neq 0)$

Then $x^{98}(x^2 y < 0)$ # $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, [a > 0, b < 0] \Rightarrow ab < 0$

Then $x^{100} y < 0$ # algebra/associativity of multiplication

Then $[(x^2 y < 0) \wedge (x \neq 0)] \Rightarrow [x^{100} y < 0]$ # assuming the antecedent leads to the consequent

Assume $x^{100} y < 0$

Then $x^{98}(x^2 y) < 0$ # algebra/associativity of multiplication

Also $1/x^{98} > 0$ # $\forall a \in \mathbb{R}, a > 0 \Rightarrow 1/a > 0$ Then $(1/x^{98})x^{98}(x^2 y) < (1/x^{98})0$ # multiply both sides by the same value

Then $x^2 y < 0$ # algebra

Then $[x^{100} y < 0] \Rightarrow [(x^2 y < 0) \wedge (x \neq 0)]$ # assuming the antecedent leads to the consequent

Then $[(x^2 y < 0) \wedge (x \neq 0)] \Leftrightarrow [x^{100} y < 0]$ # equivalence

Then $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [(x^2 y < 0) \wedge (x \neq 0)] \Leftrightarrow [x^{100} y < 0]$ # introduce universal

3. As of Jun.17, I predict that Brazil, Germany, Italy, and Argentina will make it to the semi-finals of the FIFA World Cup. Suppose you have a group of 25 footballers from those four countries. Prove that out of those 25, there is a group of at least 6 footballers who come from the same country. *It is not necessary to use the detailed structured proof format, but you still must justify every step.*

Solution: The statement is **true**.

Proof:

Let N_c be the number of footballers in the group who come from country c $\in$ {Brazil, Italy, Argentina, Germany}

Let N be the total number of footballers in the group

Assume there is no group of 6 footballers who come from the same country and $N = 25$

Then $\forall c \in$ Brazil, Italy, Argentina, Germany, $N_c \leqslant 5$ # By assumption, the natural number N_c is smaller than 6

Then $N = N_{Brazil} + N_{Italy} + N_{Argentina} + N_{Germany} \leqslant 5+5+5+5 = 20$ # Only footballers from the 4 countries are in the group + substitution

But it is not the case that $25 \leqslant 20$ # arithmetic, we assumed that $N = 25$

Contradiction!

Then there is a group of at least 6 footballers who come from the same country # assuming otherwise leads to a contradiction

4. Prove or disprove the following equivalence. If you decide it is true and so need to prove it, do not use a truth table. Justify every step in your derivation using the equivalences in Section 2.17 of the notes.

$$[(\neg Q \wedge P) \vee \neg Q] \Leftrightarrow [(\neg Q \wedge \neg R) \vee (\neg Q \wedge R)]$$

Solution: The statement is **true**.

Proof:

$$\begin{aligned}
(\neg Q \wedge P) \vee \neg Q &\Leftrightarrow (\neg Q \wedge P) \vee (\neg Q \wedge (P \vee \neg P)) \quad \# \textit{ identity laws} \\
&\Leftrightarrow (\neg Q \wedge P) \vee (\neg Q \wedge P) \vee (\neg Q \wedge \neg P) \quad \# \textit{ distributive laws} \\
&\Leftrightarrow (\neg Q \wedge P) \vee (\neg Q \wedge \neg P) \quad \# \textit{ idempotency laws} \\
&\Leftrightarrow (\neg Q \wedge (P \vee \neg P)) \quad \# \textit{ distributive laws} \\
&\Leftrightarrow \neg Q \quad \# \textit{ identity laws} \\
&\Leftrightarrow \neg Q \wedge (\neg R \vee R) \quad \# \textit{ identiy laws} \\
&\Leftrightarrow (\neg Q \wedge \neg R) \vee (\neg Q \wedge R) \quad \# \textit{ distributive laws}
\end{aligned}$$

5. Let R^+ be the set of positive real numbers. Prove or disprove **using a detailed structured proof, justifying every step**:

$$\exists \epsilon \in \mathbb{R}^+, \forall B \in \mathbb{N}, \forall n \in \mathbb{N}, [n > B] \Rightarrow \left[\left|\frac{2n^2 + 15n}{n^2} - 2\right| < \epsilon\right]$$

Solution: The statement is **true**.

Proof:

Let $\epsilon_0 = 16$

Assume $B \in \mathbb{N}, n \in \mathbb{N}$

Assume $n > B$

Then $B \geq 0$ # The smallest natural number is zero

Then $n \geq 1$ # $n > B \geqslant 0$

Then $|\frac{2n^2+15n}{n^2} - 2| = |\frac{15}{n}| = \frac{15}{n} \leq 15$ # algebra, $\forall n_1 \in \mathbb{N}, \forall n_2 \in \mathbb{N}, n_2 > n_1 \Rightarrow 1/n_2 < 1/n_1$ applied to $n_1 = 1$

Then $|\frac{2n^2+15n}{n^2} - 2| < \epsilon_0$ # $16 > 15$

Then $[n > B] \Rightarrow |\frac{2n^2+15n}{n^2} - 2| < \epsilon_0$ # assuming the antecedent implies the consequent

Then $\forall B \in \mathbb{N}, \forall n \in \mathbb{N}, [n > B] \Rightarrow \left[\left|\frac{2n^2+15n}{n^2} - 2\right| < \epsilon\right]$ # introduce universal

Then $\exists \epsilon \in R^+, \forall B \in \mathbb{N}, \forall n \in \mathbb{N}, [n > B] \Rightarrow \left[\left|\frac{2n^2+15n}{n^2} - 2\right| < \epsilon\right]$ # $\epsilon_0 = 16$ is such an ϵ

6. Assume that $D \subset \mathbb{N}$ and $D \neq \varnothing$. Prove or disprove **using a detailed structured proof, justifying every step**:

$$[\forall x \in D, \exists y \in \mathbb{N}, y < x] \Leftrightarrow [0 \notin D]$$

Solution: The statement is **true**.

Proof:

Assume $0 \in D$

  Let $x_0 = 0$

  Then $x_0 \in D$

  Then $\forall y \in \mathbb{N}, y \geq x_0$ # 0 is the smallest natural number

  Then $\exists x \in D, \forall y \in \mathbb{N}, y \geq x$ # x_0 is such an x

  Then $\neg(\forall x \in D, \exists y \in \mathbb{N}, y < x)$ # quantifier negation

Then $0 \in D \Rightarrow \neg(\forall x \in D, \exists y \in \mathbb{N}, y < x$ # assuming the antecedent leads to the consequent

Then $\forall x \in D, \exists y \in \mathbb{N}, y < x \Rightarrow 0 \notin D$ # contrapositive

Assume $0 \notin D$

  Assume $x \in D$

    Then $x \neq 0$ # 0 $\not inD$ but $x \in D$ Let $y_0 = 0$

    Then $y_0 \in \mathbb{N}$ # definition of naturals

    Then $\forall x \in D, y_0 < x$ # the minimum natural number is 0 and $0 \notin D$ so $x \neq 0$

    Then $\exists y \in \mathbb{N}, y < x$ # y_0 is such a y

  Then $\forall x \in D, \exists y \in \mathbb{N}, y < x$ # introduce universal

Then $[0 \notin D] \Rightarrow [\forall x \in D, \exists y \in \mathbb{N}, y < x]$ # assuming the antecedent implies the consequent

Then $\forall x \in D, \exists y \in \mathbb{N}, y < x \Leftrightarrow 0 \notin D$ # equivalence