CSC165, Summer 2014
Assignment 3
Solutions

1. (a) (4 pts.) Prove or disprove that, for any universal set U and predicates P and Q,

$$[\exists x \in U, P(x) \wedge Q(x)] \Rightarrow [\exists x \in U, P(x)) \wedge (\exists x \in U, Q(x))]$$

Solution: The statement is **true**.

Proof:

Assume $\exists x \in U, P(x) \wedge Q(x)$

Let $x_0 \in U$ be such that $P(x_0) \wedge Q(x_0)$ # naming it x_0

Then $P(x_0)$ # $P(x_0) \wedge Q(x_0) \Rightarrow P(x_0)$ since $R \wedge S \Rightarrow R$

Then $\exists x \in U, P(x)$ # x_0 is such an x

Then $Q(x_0)$ # $P(x_0) \wedge Q(x_0) \Rightarrow Q(x_o)$ since $R \wedge S \Rightarrow S$

Then $\exists x \in U, Q(x)$ # x_0 is such an x

Then $(\exists x \in U, P(x)) \wedge (\exists x \in U, Q(x))$ # conjunction of two true statements above

Then $[\exists x \in U, P(x) \wedge Q(x)] \Rightarrow [(\exists x \in U, P(x)) \wedge (\exists x \in U, Q(x))]$ # introduce implication

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(b) (4 pts.) Prove or disprove that, for any universal set U and predicates P and Q,

$$[\exists x \in U, P(x)) \wedge (\exists x \in U, Q(x))] \Rightarrow [\exists x \in U, P(x) \wedge Q(x)]$$

Solution:

The statement is **false**. We prove its negation

$$\neg[[\exists x \in U, P(x)) \wedge (\exists x \in U, Q(x))] \Rightarrow [\exists x \in U, P(x) \wedge Q(x)]].$$

Proof:

Let $U = \{1, 10\}$, $P(x) : x < 5$, $Q(x) : x > 7$

Then $P(1)$ # $1 < 5$

Then $\exists x \in U, P(x)$ # $1 \in U$ is such an x

Then $Q(10)$ # $10 < 7$

Then $\exists x \in U, Q(x)$ # $10 \in U$ is such an x

Then $[\exists x \in U, P(x)) \wedge (\exists x \in U, Q(x))]$ # conjunction of two true statements

Then $\neg[P(1) \wedge Q(1)]$ # $Q(1)$ is false

Then $\neg[P(10) \wedge Q(10)]$ # $P(10)$ is false

Then $\forall x \in U, \neg[P(x) \wedge Q(x)]$ # we enumerated all the cases for $x \in U$

Then $\neg[\exists x \in U, P(x) \wedge Q(x)]$ # quantifier negation

Then $[[\exists x \in U, P(x)] \wedge [\exists x \in U, Q(x)]] \wedge \neg[\exists x \in U, P(x) \wedge Q(x)]$ # conjunction

Then $\neg[[\exists x \in U, P(x)) \wedge (\exists x \in U, Q(x))] \Rightarrow [\exists x \in U, P(x) \wedge Q(x)]]$ # implication negation

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(c) (4 pts.) Prove or disprove that, for any universal set U and predicate P

$$[\exists x \in U, P(x)] \Rightarrow [\forall x \in U, P(x)]$$

Solution:
The statement is **false**. We prove its negation

$$\neg[[\exists x \in U, P(x)] \Rightarrow [\forall x \in U, P(x)]].$$

Proof:
Let $U = \{1, 10\}$, $P(x) : x < 5$
Then $P(1)$ # $1 < 5$
Then $\exists x \in U, P(x)$ # $1 \in U$ is such an x

Then $\neg P(10)$ # $10 \geqslant 7$
Then $\exists x \in U, \neg P(x)$ # $10 \in U$ is such an x
Then $\neg[\forall x \in U, P(x)]$ # negation of universal

Then $[\exists x \in U, P(x)] \wedge [\neg[\forall x \in U, P(x)]]$ # conjunction

Then $\neg[\exists x \in U, Q(x)] \Rightarrow [\exists x \in U, P(x) \wedge Q(x)]$ # implication negation
QED

(d) (4 pts.) Prove or disprove that, for any universal set U and predicate P

$$[\forall x \in U, P(x)] \Rightarrow [\exists x \in U, P(x)]$$

Solution:
The statement is **false**. We prove its negation

$$\neg[[\forall x \in U, P(x)] \Rightarrow [\exists x \in U, P(x)]].$$

Proof:
Let $U = \varnothing$, $P(x) = true$
Then $\forall x \in U, P(x)$ # elements in the empty set have all properties

Then $\neg\exists x \in U, P(x)$ # U is empty

Then $[\forall x \in U, P(x)] \wedge [\neg\exists x \in U, P(x)]$ # conjunction

Then $\neg[[\forall x \in U, P(x)] \Rightarrow [\exists x \in U, P(x)]]$ # implication negation

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2. For this question, you will prove that the roots of $x^2 + 6x + 8.5$ are irrational. In other words, that

$$\forall x \in \mathbb{R}, x^2 + 6x + 8.5 = 0 \Rightarrow x \notin \mathbb{Q}.$$

You may use, without proof, the quadratic formula. In other words, you may use the fact that for all real a, b, and c,

$$\forall x \in \mathbb{R}, (ax^2 + bx + c = 0) \wedge (b^2 - 4ac \geq 0) \Rightarrow (x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}) \vee (x = \frac{-b + \sqrt{b^2 - 4ac}}{2a})$$

Please write complete proofs in each subquestion, without referring back to earlier subquestions.

(a) (4 pts.) Prove that $\forall x \in \mathbb{R}, x \in \mathbb{Q} \Rightarrow (x+1) \in \mathbb{Q}$. *Hint: go back to the definition of* $\mathbb{Q}$, *and show that if* x *satisfies that definition, then so does* $(x+1)$.

Solution:

Assume $x \in \mathbb{R}$
  Assume $x \in \mathbb{Q}$
    Then $x = \frac{p}{q}$, for $p \in \mathbb{Z}, q \in \mathbb{Z}^*$ # definition of rational numbers
    Then $x + 1 = \frac{p}{q} + \frac{q}{q} = \frac{p+q}{q}$ # algebra
    $p + q \in \mathbb{Z}$ # integers are closed under addition
    Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}, x + 1 = \frac{p'}{q'}$ # $p+q$ and q are such p' and q'
    Then $x + 1 \in \mathbb{Q}$ # definition of $\mathbb{Q}$
  Then $x \in \mathbb{Q} \Rightarrow (x+1) \in \mathbb{Q}$ # introduce implication
Then $\forall x \in \mathbb{R}, x \in \mathbb{Q} \Rightarrow (x+1) \in \mathbb{Q}$ # introduce universal
QED

(b) (4 pts.) Prove that $\forall x \in \mathbb{R}, x \notin \mathbb{Q} \Rightarrow (x+1) \notin \mathbb{Q}$. *Hint: can you prove the contrapositive of this statement?*

Solution:

We prove the contrapositive first.
Assume $x \in \mathbb{R}$
  Assume $(x+1) \in \mathbb{Q}$
    Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}^*, x + 1 = \frac{p'}{q'}$, # definition of $\mathbb{Q}$
    Let p and q be such that $x + 1 = \frac{p}{q}$ # instantiation
    Then $x = \frac{p}{q} - \frac{q}{q} = \frac{p-q}{q}$ # algebra
    $p - q \in \mathbb{Z}$ # integers are closed under subtraction
    Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}, x = \frac{p'}{q'}$ # $p-q$ and q are such p' and q'
    Then $x \in \mathbb{Q}$ # by definition of $\mathbb{Q}$
  Then $(x+1) \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$ # introduce implication
Then $\forall x \in \mathbb{R}, (x+1) \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$ # introduce universal
Then $\forall x \in \mathbb{R}, x \notin \mathbb{Q} \Rightarrow (x+1) \notin \mathbb{Q}$ # contrapositive

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(c) (16 pts.) Prove that $\forall x \in \mathbb{R}, x^2 + 6x + 8.5 = 0 \Rightarrow x \notin \mathbb{Q}$. *Hint: you may want to use results that are similar to 2a and 2b.*

Solution:

We first prove that adding a whole number to an irrational number results in an irrational number, and that dividing an irrational number by a whole number results in an irrational number.

Adding a whole number to an irrational number results in an irrational number:

Assume $x \in \mathbb{R}$

  Assume $(x + n) \in \mathbb{Q}, n \in \mathbb{Z}$

    Then $x + n = \frac{p}{q}$, for $p \in \mathbb{Z}, q \in \mathbb{Z}^*$ # definition of $\mathbb{Q}$

    Then $x = \frac{p}{q} - \frac{nq}{q} = \frac{p-nq}{q}$ # algebra

    $p - nq \in \mathbb{Z}$ # integers are closed under subtraction and multiplication

    Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}, x = \frac{p'}{q'}$ # $p - nq$ and q are such p' and q'

    Then $x \in \mathbb{Q}$ # by definition of $\mathbb{Q}$

  Then $(x + n) \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$ # introduce implication

Then $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x + n) \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$ # introduce universal

Then $\boldsymbol{\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, x \notin \mathbb{Q} \Rightarrow (x + n) \notin \mathbb{Q}}$ # contrapositive

Multiplying an irrational number by a whole number results in an irrational number:

Assume $x \in \mathbb{R}$

  Assume $(x/n) \in \mathbb{Q}, n \in \mathbb{Z}$

    Then $x/n = \frac{p}{q}$, for $p \in \mathbb{Z}, q \in \mathbb{Z}^*$ # definition of $\mathbb{Q}$

    Then $x = \frac{np}{q}$ # algebra

    $np \in \mathbb{Z}$ # integers are closed under multiplication

    Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}, x = \frac{p'}{q'}$ # np and q are such p' and q'

    Then $x \in \mathbb{Q}$ # by definition of $\mathbb{Q}$

  Then $(x/n) \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$ # introduce implication

Then $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x/n) \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$ # introduce universal

Then $\boldsymbol{\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, x \notin \mathbb{Q} \Rightarrow (x/n) \notin \mathbb{Q}}$ # contrapositive

We now prove the main result.

Assume $x \in \mathbb{R}$

  Assume $x^2 + 6x + 8.5 = 0$

    Then $6^2 - 4(1)(8.5) = 2 \geq 0$ # algebra

    Then $\left(x = \frac{-6-\sqrt{2}}{2}\right) \vee \left(x = \frac{-6+\sqrt{2}}{2}\right)$ # using the quadratic formula

    Case 1: $x = \frac{-6-\sqrt{2}}{2}$

    $-\sqrt{2} \notin Q$ # Dividing $\sqrt{2} \notin \mathbb{Q}$ by $-1 \in \mathbb{Z}$

    $-6 - \sqrt{2} \notin Q$ # Adding $\sqrt{2} \notin \mathbb{Q}$ and $-6 \in \mathbb{Z}$

    Then $\frac{-6-\sqrt{2}}{2} \notin \mathbb{Q}$ # dividing $(-6 - \sqrt{2}) \notin \mathbb{Q}$ by $2 \in \mathbb{Z}$

    Then $x \notin \mathbb{Q}$ # substituting x

    Case 2: $x = \frac{-6+\sqrt{2}}{2}$

    $-6 + \sqrt{2} \notin Q$ # Adding $\sqrt{2} \notin \mathbb{Q}$ and $-6 \in \mathbb{Z}$

    Then $\frac{-6-\sqrt{2}}{2} \notin \mathbb{Q}$ # dividing $(-6 - \sqrt{2}) \notin \mathbb{Q}$ by $2 \in \mathbb{Z}$

    Then $x \notin \mathbb{Q}$ # substituting x

    Then $x \notin \mathbb{Q}$ # this holds in both of the two cases

  Then $x^2 + 6x + 8.5 = 0 \Rightarrow x \notin \mathbb{Q}$ # introduce implication

Then $x \in \mathbb{R}, x^2 + 6x + 8.5 = 0 \Rightarrow x \notin \mathbb{Q}$ # introduce universal

Here's an alternative proof (due to E.W., a student in the class):

Assume $x \in \mathbb{R}$

Assume $x^2 + 6x + 8.5 = 0$

Then $6^2 - 4(1)(8.5) = 2 \geq 0$ # algebra

Then $\left(x = \frac{-6-\sqrt{2}}{2}\right) \vee \left(x = \frac{-6+\sqrt{2}}{2}\right)$ # using the quadratic formula

Case 1: $x = \frac{-6-\sqrt{2}}{2}$

Assume $x \in \mathbb{Q}$

Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}, x = p/q$ # definition of $\mathbb{Q}$

Let $x = p/q, p \in \mathbb{Z}, q \in \mathbb{Z}$ # pick such p and q

Then $p/q = \frac{-6-\sqrt{2}}{2}$ # substitution

Then $\sqrt{2} = \frac{-6q-2p}{q}$ # algebra

Then $\exists p'', \exists q'', \sqrt{2} = p''/q''$ # $-6q - 2p$ and q are such p'' and q'', integers closed under addition/multiplication

Then $\sqrt{2} \in \mathbb{Q}$ # definition of $\mathbb{Q}$

Contradiction! # $\sqrt{2} \notin \mathbb{Q}$

Then $x \notin \mathbb{Q}$ # assuming $x \in \mathbb{Q}$ leads to a contradiction

Case 2: $x = \frac{-6+\sqrt{2}}{2}$

Assume $x \in \mathbb{Q}$

Then $\exists p' \in \mathbb{Z}, \exists q' \in \mathbb{Z}, x = p/q$ # definition of $\mathbb{Q}$

Let $x = p/q, p \in \mathbb{Z}, q \in \mathbb{Z}$ # pick such p and q

Then $p/q = \frac{-6+\sqrt{2}}{2}$ # substitution

Then $\sqrt{2} = \frac{6q+2p}{q}$ # algebra

Then $\exists p'', \exists q'', \sqrt{2} = p''/q''$ # $6q+2p$ and q are such p'' and q'', integers closed under addition/multiplication

Then $\sqrt{2} \in \mathbb{Q}$ # definition of $\mathbb{Q}$

Contradiction! # $\sqrt{2} \notin \mathbb{Q}$

Then $x \notin \mathbb{Q}$ # assuming $x \in \mathbb{Q}$ leads to a contradiction

Then $x \notin \mathbb{Q}$ # this holds in both of the two cases

Then $x^2 + 6x + 8.5 = 0 \Rightarrow x \notin \mathbb{Q}$ # introduce implication

Then $x \in \mathbb{R}, x^2 + 6x + 8.5 = 0 \Rightarrow x \notin \mathbb{Q}$ # introduce universal

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3. An "interpretation" for a logical statement consists of a domain D (any non-empty set of elements) and a meaning for each predicate symbol. For example, $D = \{1, 2\}$ and $P(x)$: "$x > 0$" is an interpretation for the statement $\forall x \in D, P(x)$ (in this case, one that happens to make the statement true). For each statement below, provide one interpretation under which the statement is true and another interpretation under which the statement is false— if either case is not possible, explain why clearly and concisely. You may reuse examples if you wish.

   (a) (4 pts.) $\exists x \in D, \forall y \in D, P(x, y) \Rightarrow P(y, x)$
   Solution:
   To make the statement true, let $D = \{1\}$ and define $P(x, y) : x = y$. Then, $P(x, y)$ is true for any choice of $x, y \in D$.

   To make the statement false, we have to find an interpretation for which $\forall x \in D, \exists y \in D, P(x, y) \wedge \neg P(y, x)$. Let $D = \mathbb{N}$ and $P(x, y) : x < y$. Then, this statement amounts to saying "for every natural number x, you can always find another natural number that is larger than x," which is false.

   (b) (4 pts.) $[\forall x \in D, \forall y \in D, P(x, y) \Rightarrow P(y, x)] \wedge [\forall x \in D, \forall y \in D, \neg P(x, y)]$
   Solution:
   Let $D = \{1\}$ and let $P(x, y) : x \neq y$. Then, this interpretation makes the statement true. The first part of the conjunction is vacuously true and clearly, $\neg P(1, 1)$.

   To make the statement false, let $D = \{1, 2\}$ and $P(x, y) : x < y$. Then, $P(1, 2)$ is true, but $P(2, 1)$ is false. This makes the first part of the conjunction false, which makes the entire conjunction false.

   (c) (4 pts.) $[\exists x \in D, Q(x)] \Rightarrow [\forall x \in D, P(x)]$
   Solution:
   For an interpretation that makes the statement true, let $D = \{1\}$, $Q(x) : x = x$ and $P(x) : x = 1$. To make the statement false, we need to find an interpretation for which $[\exists x \in D, Q(x)] \wedge [\exists x \in D, \neg P(x)]$ is true. Let $D = \{1\}$, $Q(x) : x = 1$, and $P(x) : x = 2$.

4. For each equivalence below, either provide a derivation from one side of the equivalence to the other (justify each step of your derivation with a brief explanation— for example, by naming one of the equivalences (See Tutorial 4), or show that the equivalence does not hold (warning: you cannot use a derivation to show non-equivalence— instead, think carefully about what an equivalence means, and how you can disprove it).

   (a) (4 pts.) $\neg Q \vee (P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q) \wedge \neg Q$
   Solution:
   The equivalence does not hold. To see this, consider when $Q = F$ and $P = T$. On the LHS, we get T, but on the RHS, we get F.

   (b) (4 pts.) $((P \vee Q) \Rightarrow R) \Leftrightarrow (P \Rightarrow (Q \vee R))$
   Solution:
   The equivalence does not hold. To see this, consider when $P = F$, $Q = T$, and $R = F$. Then, the LHS is F, but the RHS is T.

5. (4 pts.) Use a truth table to prove that $(P \Leftrightarrow Q) \Leftrightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$.

Solution:

The truth table is as follows (omitting $\neg P$ and $\neg Q$):

P	Q	$P \Leftrightarrow Q$	$P \wedge Q$	$\neg P \wedge \neg Q$	$((P \wedge Q) \vee (\neg P \wedge \neg Q))$	$(P \Leftrightarrow Q) \Leftrightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$
F	F	T	F	T	T	T
F	T	F	F	F	F	T
T	F	F	F	F	F	T
T	T	T	T	F	T	T

There are two boolean variables in the equivalence, and hence there must be $2 * 2 = 4$ different rows (since each of P and Q holds two possible values. The equivalence is true in all cases.

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