

Convergence of Langevin Monte Carlo in Chi-Squared and Rényi Divergence

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Abstract

We study sampling from a target distribution $\nu_* = e^{-f}$ using the unadjusted Langevin Monte Carlo (LMC) algorithm when the potential f satisfies a strong dissipativity condition and it is first-order smooth with a Lipschitz gradient. We prove that, initialized with a Gaussian random vector that has sufficiently small variance, iterating the LMC algorithm for $\tilde{O}(\lambda^2 d \epsilon^{-1})$ steps is sufficient to reach ϵ -neighborhood of the target in both Chi-squared and Rényi divergence, where λ is the logarithmic Sobolev constant of ν_* . Our results do not require warm-start to deal with the exponential dimension dependency in Chi-squared divergence at initialization. In particular, for strongly convex and first-order smooth potentials, we show that the LMC algorithm achieves the rate estimate $\tilde{O}(d \epsilon^{-1})$ which improves the previously known rates in both of these metrics, under the same assumptions. Translating this rate to other metrics, our results also recover the state-of-the-art rate estimates in KL divergence, total variation and 2-Wasserstein distance in the same setup. Finally, as we rely on the logarithmic Sobolev inequality, our framework covers a range of non-convex potentials that are first-order smooth and exhibit strong convexity outside of a compact region.

1 Introduction

We consider sampling from a target distribution $\nu_* = e^{-f}$ using the Langevin Monte Carlo (LMC)

$$x_{k+1} = x_k - \eta \nabla f(x_k) + \sqrt{2\eta} W_k, \quad (1.1)$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is the potential function, W_k is a d -dimensional isotropic Gaussian random vector independent from $\{x_l\}_{l \leq k}$, and η is the step size. This algorithm is the Euler discretization of the following stochastic differential equation (SDE)

$$dz_t = -\nabla f(z_t) dt + \sqrt{2} dB_t, \quad (1.2)$$

where B_t denotes the d -dimensional Brownian motion. The solution of the above SDE is referred to as the first-order Langevin diffusion, and the convergence behavior of the LMC algorithm (1.1) is intimately related to the properties of the diffusion process (1.2). Intuitively, fast mixing of LMC (1.1) is inherited from the Langevin diffusion (1.2) since the Euler discretization scheme with

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a sufficiently small step size ensures that the Markov chain generated by the LMC iterations tracks its continuous counterpart. Therefore, to ensure that the LMC algorithm converges, one typically starts from conditions that imply the fast convergence of the diffusion process, the Langevin dynamics given in (1.2).

Denoting the density of the Langevin diffusion z_t with π_t , the following Fokker-Planck equation describes the evolution of the continuous dynamics (1.2) [Ris96]

$$\frac{\partial \pi_t(x)}{\partial t} = \nabla \cdot (\nabla f(x) \pi_t(x)) + \Delta \pi_t(x) = \nabla \cdot \left(\pi_t(x) \nabla \log \frac{\pi_t(x)}{\nu_*(x)} \right). \quad (1.3)$$

Convergence to the equilibrium of the above equation has been studied extensively under various assumptions and distance measures. Defining Chi-squared, Rényi and Kullback–Leibler (KL) divergence measures between two probability distributions ρ and ν in \mathbb{R}^d , respectively as

$$\begin{aligned} \chi^2(\rho|\nu) &= -1 + \int \left(\frac{\rho(x)}{\nu(x)} \right)^2 \nu(x) dx \quad \text{and} \quad \text{KL}(\rho|\nu) = \int \log \left(\frac{\rho(x)}{\nu(x)} \right) \rho(x) dx, \\ R_\alpha(\rho|\nu) &= \frac{1}{\alpha - 1} \log \int \left(\frac{\rho(x)}{\nu(x)} \right)^\alpha \nu(x) dx \quad \text{for } \alpha > 1, \end{aligned} \quad (1.4)$$

the logarithmic Sobolev inequality (LSI) is a particularly useful condition on the target ν_* , which implies the exponential convergence of (1.3) in both Chi-squared and KL divergence. A probability density ν satisfies LSI if the following holds

$$\forall \rho, \quad \text{KL}(\rho|\nu) \leq \frac{\lambda}{2} \int \|\nabla \log \frac{\rho(x)}{\nu(x)}\|^2 \rho(x) dx. \quad (\text{LSI})$$

LSI is known to hold for strongly log-concave distributions [BÉ85] in which case the constant λ^{-1} is equal to the strong convexity constant of the potential. This condition is also robust against finite perturbations [HS87] which allows one to deal with non-convex potentials (to a somewhat limited extent). If the target ν_* satisfies LSI, then the distribution π_t converges to the target ν_* in all three divergence measures defined in (1.4) exponentially fast, i.e.,

$$\text{LSI} \implies \begin{cases} \text{KL}(\pi_t|\nu_*) & \leq e^{-2t/\lambda} \text{KL}(\pi_0|\nu_*), \\ \chi^2(\pi_t|\nu_*) & \leq e^{-2t/\lambda} \chi^2(\pi_0|\nu_*), \\ R_\alpha(\pi_t|\nu_*) & \leq e^{-2t/\alpha\lambda} R_\alpha(\pi_0|\nu_*), \end{cases} \quad (1.5)$$

for all $t \geq 0$. Note that Chi-squared and KL *metrics* are closely related to the Rényi divergence (e.g. $\alpha = 2$ and $\alpha \rightarrow 1$ respectively); however, convergence in different metrics (e.g. for different values of α) may require different conditions on the potential (and on the target ν_*). In fact the exponential convergence in Chi-squared divergence as in (1.5) can be established under the Poincaré inequality [CLL19, CGL⁺20], which holds for a wider class of potentials [BBCG08] (see [CLL19, VW19] for the convergence of π_t in Rényi divergence under various conditions).

Under additional smoothness assumptions on the potential function f , the fast convergence of the Langevin diffusion (1.3) to equilibrium as in (1.5) can be translated to that of the LMC algorithm. In particular, implications of LSI on the convergence of LMC are relatively well-understood in KL divergence [Dal17b, DM17, VW19, EH20]. In addition to LSI, assuming further that the gradient of the potential is Lipschitz continuous, taking $\tilde{\mathcal{O}}(d/\epsilon)$ steps is sufficient to reach the ϵ -neighborhood of a d -dimensional target distribution ν_* in KL divergence [VW19]. However, the convergence properties in stronger notions of distance such as Chi-squared and Rényi divergence are not explored to the same degree. One exception is the recent work [GT20] where authors analyzed the convergence of LMC in α -Rényi divergence for strongly log-concave targets for $\alpha > 1$ and

obtained the rate estimate $\tilde{\mathcal{O}}(d/\epsilon^2)$, which implies the same rate of convergence in Chi-squared divergence by setting $\alpha = 2$. More specifically, their result implies a convergence estimate of $\tilde{\mathcal{O}}(d/\epsilon^2)$ in Chi-squared divergence for strongly convex potentials that have Lipschitz gradients.

Chi-squared divergence is particularly of interest because it conveniently upper bounds a variety of distance measures. For example, KL divergence (relative entropy), total variation (TV) distance and 2-Wasserstein (\mathcal{W}_2) metrics can be upper bounded as

$$\text{TV}(\rho, \nu_*) \leq \sqrt{\text{KL}(\rho|\nu_*)/2} \leq \sqrt{\chi^2(\rho|\nu_*)/2} \quad \text{and} \quad \mathcal{W}_2(\rho, \nu_*)^2/(2\lambda) \leq \chi^2(\rho|\nu_*). \quad (1.6)$$

For the former inequality above, see e.g. [Tsy08, Lemma 2.7] together with Csiszár-Kullback-Pinsker inequality [BV05], and the latter holds under LSI, see e.g. [Liu20, Theorem 1.1]. Therefore convergence in Chi-squared divergence implies convergence in these measures of distance as well. However, translating the rate estimate $\tilde{\mathcal{O}}(d/\epsilon^2)$ obtained in [GT20] using the above inequalities, one cannot recover the state-of-the-art convergence rates in these metrics. For example, the rate estimate $\tilde{\mathcal{O}}(d/\epsilon^2)$ in Chi-squared divergence implies the same rate in KL divergence, which is substantially slower than the well-known estimate $\tilde{\mathcal{O}}(d/\epsilon)$ under the same assumptions, i.e., strongly convex potentials with Lipschitz gradients (see e.g. [Dal17b, VW19, EH20]).

Our work bridges this gap in the convergence estimates, and further extends the analysis to potentials that exhibit strong dissipativity. Our contributions can be summarized as follows.

- For a first-order smooth potential f satisfying strong dissipativity in the following sense

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|, \quad \langle x - y, \nabla f(x) - \nabla f(y) \rangle \geq m \|x - y\|^2 - b,$$

where $L, m > 0$ and $b \geq 0$, we prove that taking $\tilde{\mathcal{O}}(\frac{\lambda^2 L^4}{m^2} \times \frac{b+d}{\epsilon})$ steps of LMC is sufficient to obtain an ϵ -accurate sample from a d -dimensional target in both Chi-squared and Rényi divergence, where λ is the LSI constant for $\nu_* = e^{-f}$. Interestingly, our results do not require warm-start to deal with exponential dimension dependency of the Chi-squared divergence at initialization, and can tolerate non-convexity as long as the tails of the potential is growing quadratically fast.

- When translated to KL divergence, TV and \mathcal{W}_2 metrics using the inequalities (1.6), we obtain the rate estimates $\tilde{\mathcal{O}}(\frac{\lambda^2 L^4}{m^2} \times \frac{b+d}{\epsilon})$, $\tilde{\mathcal{O}}(\frac{\lambda^2 L^4}{m^2} \times \frac{b+d}{\epsilon^2})$, and $\tilde{\mathcal{O}}(\frac{\lambda^3 L^4}{m^2} \times \frac{b+d}{\epsilon^2})$, respectively.
- For m -strongly convex potentials ($b = 0$), we have $\lambda^{-1} = m$; thus, our rate estimate established in Chi-squared divergence is able to recover the best-known rate estimates¹ for LMC in KL divergence, TV and \mathcal{W}_2 metrics, respectively given as $\tilde{\mathcal{O}}(d/\epsilon)$, $\tilde{\mathcal{O}}(d/\epsilon^2)$, and $\tilde{\mathcal{O}}(d/\epsilon^2)$.
- We further discuss sampling from non-convex potentials that are covered by our assumptions, namely smooth potentials exhibiting strong convexity outside of a compact region. By deriving bounds on their LSI constant λ , we establish rate estimates for LMC under various scenarios.

Our analysis builds on the prominent works by [VW19, GT20]. More specifically, we conduct a two-phase analysis: In the first phase, we extend the analysis provided in [GT20] to potentials that are strongly dissipative (e.g. strongly convex outside of a compact region) and control key quantities that impact the convergence of LMC for the interpolation process. In the second phase, we analyze a differential inequality for the Chi-squared (and Rényi) divergence, which resembles the (single-phase) analysis conducted by [VW19] for the KL divergence, to obtain our final rate estimate. Rest of the paper is organized as follows. We discuss related work and notation in the rest of this section. In Section 2, we motivate the two-phase analysis, and state two key lemmas

¹Additional second order smoothness is known to speed up the convergence in KL divergence and 2-Wasserstein distance [MFWB19, DK19]; however, our focus in this paper is first-order smoothness.

describing the characteristics of each phase. Section 3 contains the main results on the convergence of LMC. In Section 4, we provide examples and discuss the relative merits of certain non-convexity structures on our rate estimates. Finally in Section 5, we discuss future work. Majority of the proofs and the derivations are deferred to Appendix.

Related work. Started by the pioneering works [DM16, Dal17b, DM17], non-asymptotic analysis of LMC has drawn a lot of interest [Dal17a, CB18, CCAY⁺18, DM19, DMM19, VW19, DK19, BDMS19, LWME19, EH20]. It is known that $\tilde{\mathcal{O}}(d/\epsilon)$ steps of LMC yield an ϵ -accurate sample in KL divergence for strongly convex and first-order smooth potentials [CB18, DMM19]. This is still the best rate obtained in this setup, and recovers the fastest rates in total variation and 2-Wasserstein metrics [DM17, Dal17b, DM19]. For the same setting, [GT20] showed that $\tilde{\mathcal{O}}(d/\epsilon^2)$ steps are enough for ϵ -accurate solution in Rényi divergence. Recently, these global curvature assumptions are relaxed to growth conditions [CCAY⁺18, EMS18]. For example, [VW19] established convergence guarantees for LMC when sampling from targets distributions that satisfy a log-Sobolev inequality, and has a smooth potential. This corresponds to potentials with quadratic tails [BÉ85, BG99] up to finite perturbations [HS87]; thus, this result is able to deal with non-convex potentials while achieving the same rate of convergence $\tilde{\mathcal{O}}(d/\epsilon)$ in KL divergence. Finally, convergence of zigzag samplers is established in Chi-squared divergence under a warm-start condition, in order to deal with the ill behavior of this metric at initialization [LW20].

Notation. Throughout the paper, \log denotes the natural logarithm. For a real number $x \in \mathbb{R}$, we denote its absolute value with $|x|$. We denote the Euclidean norm of a vector $x \in \mathbb{R}^d$ with $\|x\|$. The gradient, divergence, and Laplacian of f are denoted by $\nabla f(x)$, $\nabla \cdot f(x)$ and $\Delta f(x)$, respectively.

We use $\mathbb{E}[x]$ to denote the expected value of a random variable or a vector x , where expectations are over all the randomness inside the brackets. For probability densities p, q on \mathbb{R}^d , we use $\text{KL}(p|q)$, $\chi^2(p|q)$, and $R_\alpha(p|q)$ to denote their KL (or relative entropy), Chi-squared, and Rényi divergence (for $\alpha > 1$), respectively, which are defined in (1.4). To ease the notation, we often use $\frac{p}{q}(x)$ instead of $p(x)/q(x)$. We denote the Borel σ -field of \mathbb{R}^d with $\mathcal{B}(\mathbb{R}^d)$. 2-Wasserstein metric and the total variation (TV) distance are defined respectively as

$$\mathcal{W}_2(p, q) = \inf_\nu \left(\int \|x - y\|^2 d\nu(p, q) \right)^{1/2}, \quad \text{and} \quad \text{TV}(p, q) = \sup_{A \in \mathcal{B}(\mathbb{R}^d)} \left| \int_A (p(x) - q(x)) dx \right|,$$

where in the first formula, infimum runs over the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ that has marginals with corresponding densities p and q . Multivariate Gaussian distribution with mean $\mu \in \mathbb{R}^d$ and covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$ is denoted with $\mathcal{N}(\mu, \Sigma)$.

2 Two-Phase Analysis: Motivation and Assumption

A representative analysis of LMC (see e.g. [VW19, EH20]) starts with the interpolation process,

$$d\tilde{x}_t = -\nabla f(x_{\lfloor t/\eta \rfloor}) dt + \sqrt{2} dB_t \quad \text{with} \quad \tilde{x}_0 = x_0. \quad (2.1)$$

Notice that the drift $\nabla f(x_{\lfloor t/\eta \rfloor})$ of the above process is evaluated at the LMC iterate $x_{\lfloor t/\eta \rfloor}$, and it is constant within each interval $t \in [k\eta, (k+1)\eta)$ for an integer k . Therefore, with the right coupling of the Brownian motion in (2.1) and the additive Gaussian in the LMC update (1.1), the solution of (2.1) produces the LMC iterates for $t = \eta k$ (i.e. $\tilde{x}_{k\eta} = x_k$) by simply interpolating the discrete algorithm to a continuous-time process. We denote by $\tilde{\rho}_t$ and ρ_k , the distributions of \tilde{x}_t and x_k , and we observe easily that $\tilde{\rho}_{k\eta} = \rho_k$. The advantage of analyzing the interpolation process is in its continuity in time, which allows one to work with the Fokker-Planck equation. Using this

property in Lemma 1, we show that the time derivative of $\chi^2(\tilde{\rho}_t|\nu_*)$ differs from the corresponding differential inequality for the continuous-time process by an additive error term.

In order to obtain a differential inequality in Chi-squared divergence, it is sufficient if the target satisfies a Poincaré inequality (PI), which is given as

$$\forall \rho, \quad \chi^2(\rho|\nu_*) \leq \lambda \int \left\| \nabla \frac{\rho(x)}{\nu_*(x)} \right\|^2 \nu_*(x) dx. \quad (\text{PI})$$

We note that the above condition holds under LSI with the same constant λ [Vil03], and emphasize that our final convergence results (in both Chi-squared and Rényi divergence) require the stronger condition LSI even though PI suffices for the following lemma.

Lemma 1. *If ν_* satisfies PI, then the following inequality governs the evolution of the Chi-squared divergence of the interpolated process (2.1) from the target*

$$\frac{d}{dt} \chi^2(\tilde{\rho}_t|\nu_*) \leq -\frac{3}{2\lambda} \chi^2(\tilde{\rho}_t|\nu_*) + 2\mathbb{E} \left[\frac{\tilde{\rho}_t}{\nu_*}(\tilde{x}_t)^2 \right]^{1/2} \mathbb{E} \left[\left\| \nabla f(\tilde{x}_t) - \nabla f(x_{\lfloor t/\eta \rfloor}) \right\|^4 \right]^{1/2}. \quad (2.2)$$

Proof. The proof follows from similar lines that lead to a differential inequality in KL divergence (see for example [VW19]). Let $\tilde{\rho}_{t|k}$ denote the distribution of \tilde{x}_t conditioned on x_k for $k = \lfloor t/\eta \rfloor$, which satisfies

$$\frac{\partial \tilde{\rho}_{t|k}(x)}{\partial t} = \nabla \cdot (\nabla f(x_k) \tilde{\rho}_{t|k}(x)) + \Delta \tilde{\rho}_{t|k}(x).$$

Taking expectation with respect to x_k we get

$$\frac{\partial \tilde{\rho}_t(x)}{\partial t} = \nabla \cdot \left(\tilde{\rho}_t(x) \left(\mathbb{E} [\nabla f(x_k) - \nabla f(x) | \tilde{x}_t = x] + \nabla \log \left(\frac{\tilde{\rho}_t(x)}{\nu_*(x)} \right) \right) \right). \quad (2.3)$$

Now we consider the time derivative of Chi-squared divergence of $\tilde{\rho}_t$ from the target ν_*

$$\begin{aligned} \frac{d}{dt} \chi^2(\tilde{\rho}_t|\nu_*) &= 2 \int \frac{\tilde{\rho}_t(x)}{\nu_*(x)} \times \nabla \cdot \left(\tilde{\rho}_t(x) \left(\mathbb{E} [\nabla f(x_k) - \nabla f(x) | \tilde{x}_t = x] + \nabla \log \left(\frac{\tilde{\rho}_t(x)}{\nu_*(x)} \right) \right) \right) dx \\ &\stackrel{1}{=} -2 \int \tilde{\rho}_t(x) \left\langle \nabla \frac{\tilde{\rho}_t(x)}{\nu_*(x)}, \mathbb{E} [\nabla f(x_k) - \nabla f(x) | \tilde{x}_t = x] + \nabla \log \left(\frac{\tilde{\rho}_t(x)}{\nu_*(x)} \right) \right\rangle dx \\ &\stackrel{2}{\leq} -\frac{3}{2} \int \left\| \nabla \frac{\tilde{\rho}_t(x)}{\nu_*(x)} \right\|^2 \nu_*(x) dx + 2 \int \mathbb{E} \left[\frac{\tilde{\rho}_t(x)}{\nu_*(x)} \left\| \nabla f(x) - \nabla f(x_k) \right\|^2 | \tilde{x}_t = x \right] \tilde{\rho}_t(x) dx \\ &\stackrel{3}{\leq} -\frac{3}{2\lambda} \chi^2(\tilde{\rho}_t|\nu_*) + 2\mathbb{E} \left[\frac{\tilde{\rho}_t(\tilde{x}_t)}{\nu_*(\tilde{x}_t)} \left\| \nabla f(\tilde{x}_t) - \nabla f(x_k) \right\|^2 \right], \end{aligned}$$

where step 1 follows from the divergence theorem, step 2 from $\langle a, b \rangle \leq \frac{1}{4} \|a\|^2 + \|b\|^2$ and in step 3, we used PI with ρ replaced by $\tilde{\rho}_t$. Finally, the result follows from the Cauchy-Schwartz inequality on the second term. \square

The above differential inequality will be used to establish a single step bound that can be iterated to yield the final convergence result. For this, one needs to control *i*) the additive error term in (2.2), namely $\mathbb{E}[\left\| \nabla f(\tilde{x}_t) - \nabla f(x_{\lfloor t/\eta \rfloor}) \right\|^4]$ under a smoothness condition on the potential function, and *ii*) the expected squared ratio of densities $\mathbb{E}[\frac{\tilde{\rho}_t}{\nu_*}(\tilde{x}_t)^2]$ which is harder to bound – indeed, it is exponential in dimension at initialization. Therefore, in order to avoid any warm-start assumption, we conduct our convergence analysis in two phases. In the first phase, we show that after taking N steps of LMC, the expected squared ratio (over the interpolation process) is bounded by an absolute

constant at time $N\eta$, and moreover it stays uniformly bounded for the time interval $[N\eta, 2N\eta]$. That is,

$$\mathbb{E} \left[\frac{\tilde{\rho}_T}{\nu_*} (\tilde{x}_T)^2 \right] \leq B \quad \text{for} \quad N\eta \leq T \leq 2N\eta, \quad (2.4)$$

where B is an absolute constant. The above opaque condition would ultimately imply that the LMC iterates also stay warm when the iteration counter belongs to the interval $[N, 2N]$.

Before describing the second phase of the analysis, we illustrate the above phenomenon on a simple Gaussian example where the expected squared ratio is exponential in dimension at initialization; yet, it stabilizes exponentially fast with the number of LMC iterations.

Motivating Example. In this toy example, the above uniform warmness condition (2.4) is verified for sampling from a Gaussian target e^{-f} using LMC with step size η . Assume for simplicity that $f(x) = \frac{1}{2}\|x\|^2 + C$, where C is the normalizing constant and $x_0 \sim \mathcal{N}(0, \sigma_0^2 \mathbf{I})$. For an integer $k \geq 0$, $t \in [0, \eta]$, and $T = k\eta + t$, it is easy to compute the distribution of the interpolation process

$$\tilde{\rho}_T = \mathcal{N}(0, \sigma_T^2 \mathbf{I}) \quad \text{where} \quad \sigma_T^2 := (1-t)^2 \sigma_{k\eta}^2 + 2t \quad \text{and} \quad \sigma_{k\eta}^2 := (1-\eta)^{2k} \sigma_0^2 + \frac{1-(1-\eta)^{2k}}{1-\eta/2}.$$

Moreover, elementary calculations yield that the expected squared ratio is given as

$$\mathbb{E} \left[\frac{\tilde{\rho}_T}{\nu_*} (\tilde{x}_T)^2 \right] = \frac{1}{(3/\sigma_T^2 - 2)^{d/2} \sigma_T^{3d}} \quad \text{whenever} \quad \sigma_T^2 < \frac{3}{2}.$$

For simplicity, let us initialize with $\sigma_0^2 = 0.5/(1-\eta/2)$. For a sufficiently small step size η , one can verify that $\sigma_{k\eta}^2$ is monotonically increasing and converges to $1/(1-\eta/2) < 1.5$. Moreover, $\sigma_0^2 > 1/2$ bounded away from 0. Therefore, at initialization (for $k = 0$) we have $\mathbb{E} \left[\frac{\tilde{\rho}_T}{\nu_*} (\tilde{x}_T)^2 \right] = e^{\mathcal{O}(d)}$. However, notice that if at any point along the iterations, the condition $1 - \frac{1}{d} \leq \sigma_{k\eta}^2 \leq 1 + \frac{1}{d}$ is satisfied, then we can show $\mathbb{E} \left[\frac{\tilde{\rho}_T}{\nu_*} (\tilde{x}_T)^2 \right] = \mathcal{O}(1)$ in the subsequent iterations, and accordingly B in (2.4) becomes $\mathcal{O}(1)$. Note that the denominator $\sigma \rightarrow (3/\sigma^2 - 2)^{d/2} \sigma^{3d}$ attains its minimum value in the interval $[1 - \frac{1}{d}, 1 + \frac{1}{d}]$ on its boundary (assuming $d > 3$). If this condition holds, the resulting upper bound on its inverse becomes $\mathcal{O}(1)$. On the other hand, reaching $1 - \frac{1}{d} \leq \sigma_T^2 \leq 1 + \frac{1}{d}$ in this setting is exponentially fast, which suggests conducting a two-phase analysis. First, we prove that the distribution $\tilde{\rho}_T$ gets close to target ν_* so that their expected squared ratio reduces to $\mathcal{O}(1)$, and stays warm in the subsequent iterates. Then in the next phase, we proceed the analysis with the differential inequality in Lemma 1 to obtain the final convergence estimate.

To formalize the above argument, we make the following assumptions on the potential function.

Assumption 1. *The potential function f is first-order smooth and strongly dissipative, i.e., $\forall x, y$*

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|, \quad \langle x - y, \nabla f(x) - \nabla f(y) \rangle \geq m \|x - y\|^2 - b,$$

for some constants $L, m > 0$ and $b \geq 0$, where we also define the condition number as $\kappa := L/m$.

The strong dissipativity condition is equivalent to [CCAY⁺18, Assumption 3], but it is presented in this form for convenience with later calculations. The target satisfies a LSI under strong dissipativity which can be easily deduced from [BÉ85, CGW10] (see Section C, cf. [RRT17, Prop 3.2]). For strongly convex potentials ($b = 0$), by the Bakry-Émery criterion we have $\lambda = 1/m$, i.e., the LSI constant is equal to the inverse of the strong convexity parameter of the potential. By plugging in $y = 0$, elementary algebra reveals that the strong dissipativity implies the standard 2-dissipativity condition $\langle x, \nabla f(x) \rangle \geq m' \|x\|^2 - b'$, which is commonly employed in recent analyses in sampling and non-convex optimization [RRT17, YBVE20, EH20]. While the stronger version does not cover all

the potentials covered by the standard dissipativity, it still allows for finite perturbations (similar to 2-dissipativity and LSI), which we discuss in detail in Section 4.

First phase. We establish the following bound on the expected squared ratio.

Lemma 2. *For $\alpha > 1$ and for a potential f satisfying Assumption 1, initialize the LMC algorithm with $x_0 = \mathcal{N}(0, \sigma^2 I)$ for $\sigma^2 < (1 + L)^{-1}$. If the step size satisfies $\eta \leq \frac{2}{\|\nabla f(0)\|^2} \wedge \frac{1 \wedge m}{4(1 \vee L^2)}$ and for some absolute constant c , the following conditions hold*

$$c\alpha^2 \kappa^2 L^2 N(b + d + \log N)\eta^2 \leq 1 \quad \text{and} \quad N\eta \geq 2\alpha\lambda \log(4\alpha d C_\sigma),$$

for the dimension free constant $C_\sigma = 1 + \frac{f(0) + \|\nabla f(0)\|^2}{d} - \log(\sigma^2[(1 + L) \wedge 2\pi])$, then we have

$$\mathbb{E} \left[\frac{\tilde{\rho}_T}{\nu_*} (\tilde{x}_T)^{2\alpha-2} \right] \leq 14\alpha^{\frac{1}{4}}, \quad \forall T \in [N\eta, 2N\eta].$$

The above result extends the results of [GT20] to the interpolation process, and it is key to our analysis. Its proof is deferred to Section A. We use the above bound for $\alpha = 2$ for the Chi-squared divergence, but the case $\alpha > 2$ will be useful when we extend the results to the Rényi divergence. After taking N iterations of LMC, the expected density ratio $\mathbb{E} \left[\frac{\tilde{\rho}_T}{\nu_*} (\tilde{x}_T)^{2\alpha-2} \right]$ is bounded and it stays bounded in the subsequent N iterations, as in (2.4).

Second phase. In the next stage of the analysis, we adapt the strategy used in [VW19] to the differential inequality in Chi-squared divergence only for the iterations ranging from step N to $2N$. Since for these iterations, we have a uniform bound on the expected squared ratio over the interpolation process (by Lemma 2), we can simply integrate the differential inequality in Lemma 1 to obtain the following single step bound.

Lemma 3. *Instantiate the assumptions of Lemma 2 for $\alpha = 2$. If further $\eta \leq \frac{\lambda}{2}$, then for any iteration number k such that $2N \geq k \geq N$, the following inequality controls the evolution of LMC in Chi-squared divergence*

$$\chi^2(\rho_{k+1} | \nu_*) \leq \left(1 - \frac{3\eta}{4\lambda}\right) \chi^2(\rho_k | \nu_*) + c\beta L^2(b + d)\eta^2, \quad (2.5)$$

where c is an absolute constant and β is a dimension free parameter defined as

$$\beta^2 := 1 + \frac{\sigma^4(1 + 2/d) + 6\sigma^2 \|x^*\|^2/d + \|x^*\|^4/d^2}{(1 + b/d)^2} + \frac{\sigma^2 + \|x^*\|^2/d}{1 + b/d}.$$

The proof follows by integrating the differential inequality derived in Lemma 1 for $t \in [0, \eta]$ after using the uniform bound on the expected squared ratio given by Lemma 2, which we defer to Section A. The key innovation of the above result is that the additive error term in (2.5), the second term on the right hand side, has $\mathcal{O}(d\eta^2)$ dependence for the LMC iterations ranging from N to $2N$. This is because the constant β is uniformly bounded in dimension.

3 Main Results

In this section, we first provide results on the convergence of LMC (1.1) in Chi-squared divergence, by simply iterating the single step bound in Lemma 3 from iteration N to $2N$. Then, using a different differential inequality but similar arguments, we extend the convergence to the Rényi

divergence. Before we present the main technical results of this paper, we note that the following (unnormalized) potential

$$f(x) = \frac{1}{2}\|x\|^2 + \frac{5}{4}\cos(\|x\|)$$

serves as a canonical example for our framework. This non-convex potential satisfies strong dissipativity and it has a Lipschitz gradient; thus, Assumption 1 is satisfied. The resulting target also satisfies LSI with constant e^5 . We will discuss several non-trivial examples in Section 4.

Theorem 4. *Let the potential f satisfy Assumption 1 and suppose we run $2N$ iterations of LMC (1.1) with step size η to sample from $\nu_* = e^{-f}$. If we initialize x_0 with $\mathcal{N}(0, \sigma^2 \mathbf{I})$ for some $\sigma^2 < (1 + L)^{-1}$, in order to get $\chi^2(\rho_{2N}|\nu_*) \leq \epsilon$, it is sufficient if the following inequalities hold*

$$\begin{aligned} \eta &\leq \frac{2}{\|\nabla f(0)\|^2} \wedge \frac{1 \wedge m}{4(1 \vee L^2)} \wedge \frac{\lambda}{2} \wedge \frac{c_3}{\beta \lambda L^2} \times \frac{\epsilon}{b + d} \\ N\eta &\geq c_2 \lambda \log(144dC_\sigma/\epsilon) \\ c_1 &\geq \kappa^2 L^2 (b + d + \log N) N \eta^2, \end{aligned} \tag{3.1}$$

where λ is the LSI constant of the target, c_1, c_2, c_3 are absolute constants, and C_σ and β are dimension free constants defined respectively in Lemmas 2 and 3.

Consequently, if we choose

$$\begin{aligned} \eta &= \frac{c_3}{\beta} \times \frac{m^2}{\lambda L^4 \log(\frac{L^4 \lambda^2}{m^2})} \times \frac{\epsilon}{b + d} \times \frac{1}{\log(144dC_\sigma/\epsilon)^2} \\ N &= \frac{c_2 \beta}{c_3} \times \frac{\lambda^2 L^4 \log(\frac{\lambda^2 L^4}{m^2})}{m^2} \times \frac{b + d}{\epsilon} \times \log(144dC_\sigma/\epsilon)^3, \end{aligned} \tag{3.2}$$

then $2N$ iterations of LMC yield $\chi^2(\rho_{2N}|\nu_*) \leq \epsilon$ for $N \geq 2$ and for $\epsilon > 0$ sufficiently small.

The above theorem implies that $\tilde{\mathcal{O}}(\lambda^2 L^4 / m^2 \times d / \epsilon)$ steps of LMC algorithm is sufficient to obtain an ϵ -accurate sample in Chi-squared divergence. The rate estimate given in (3.2) can be slightly improved to $\mathcal{O}(d/\epsilon \log(d/\epsilon) \log(d)^2)$ in terms of ϵ dependency at the expense of introducing more complicated expressions; yet, we use (3.2) for simplified exposition.

The theorem states that if the step size and the number of iterations satisfy the three conditions given in (3.1), the LMC algorithm is guaranteed to produce an ϵ -accurate sample in exactly $2N$ iterations. We emphasize that this result may not hold for the subsequent iterates of LMC because of the last condition (3.1). This is due to the delicate bound we construct using Lemma 2, which will be violated as $N \rightarrow \infty$ for any fixed step size η . Rate estimates with this restriction are frequent in the literature [SL19, GT20, EH20]. This typically occurs when there is a diverging bound on a quantity that appears in the convergence analysis; in [EH20] this is due to the diverging moment estimates, in [GT20] it is due to the Rényi divergence between the clipped LMC and the clipped Langevin diffusion, and in [SL19] it is due to the sum of the expected squared difference between 1-step exact Langevin diffusion and the corresponding LMC iterates (ergodicity of the algorithm is established in another work [HBE20]). In our case, the source of this is the expected squared ratio bounded in Lemma 2. We note that LMC is ergodic [MSH02] and its subsequent iterates after iteration $2N$ remain ϵ -accurate in KL divergence and 2-Wasserstein distance from the target under LSI [VW19]. Therefore it is likely that the case $N \rightarrow \infty$ can be covered with a different proof technique; but the analysis used in the current paper introduces this artifact.

We can translate our rate estimate in Chi-squared divergence using (1.6), to obtain guarantees in KL divergence, TV, and \mathcal{W}_2 metrics.

Corollary 5. *Instantiate the assumptions and the notation in Theorem 4. Table 1 summarizes the convergence rate estimates in various measures of distance.*

DISTANCE	ϵ_{χ^2}	N	η
χ^2	ϵ	$\tilde{\mathcal{O}}\left(\frac{\lambda^2 L^4}{m^2} \times \frac{b+d}{\epsilon}\right)$	$\tilde{\mathcal{O}}\left(\frac{m^2}{\lambda L^4} \times \frac{\epsilon}{b+d}\right)$
KL	ϵ	$\tilde{\mathcal{O}}\left(\frac{\lambda^2 L^4}{m^2} \times \frac{b+d}{\epsilon}\right)$	$\tilde{\mathcal{O}}\left(\frac{m^2}{\lambda L^4} \times \frac{\epsilon}{b+d}\right)$
TV	$2\epsilon^2$	$\tilde{\mathcal{O}}\left(\frac{\lambda^2 L^4}{m^2} \times \frac{b+d}{\epsilon^2}\right)$	$\tilde{\mathcal{O}}\left(\frac{m^2}{\lambda L^4} \times \frac{\epsilon^2}{b+d}\right)$
\mathcal{W}_2	$\epsilon^2/2\lambda$	$\tilde{\mathcal{O}}\left(\frac{\lambda^3 L^4}{m^2} \times \frac{b+d}{\epsilon^2}\right)$	$\tilde{\mathcal{O}}\left(\frac{m^2}{\lambda^2 L^4} \times \frac{\epsilon^2}{b+d}\right)$

Table 1: Translation of the rate estimate in Chi-squared divergence to various measures of distance by choosing an appropriate accuracy level ϵ_{χ^2} in Theorem 4.

For strongly convex potentials (i.e. $\lambda = 1/m$ and $b = 0$), the above rate estimates recover the state-of-the-art estimates in all of the above measures of distance in both accuracy ϵ and dimension d . In other words, there is no loss in converting the rates using the inequalities (1.6). However, we note that the known condition dependency is $\tilde{\mathcal{O}}(\kappa^2)$, in for example KL divergence [VW19], and our estimate produces $\tilde{\mathcal{O}}(\kappa^4)$; thus, there is room for improvement in this dependency.

3.1 Extending convergence to the Rényi Divergence for $\alpha > 1$

The results presented in the previous section for the Chi-squared divergence can be extended to the Rényi divergence with minimal effort. The key is to establish a differential inequality in this measure of distance as given below (cf. Lemma 1), which will be solved and iterated to yield a convergence rate in the Rényi divergence. Contrary to Lemma 1 which was established under PI, the following result is established under LSI.

Lemma 6. *If ν_* satisfies LSI and $\alpha > 1$, then the following inequality controls the evolution of the α -Rényi divergence of the interpolated process from the target*

$$\frac{d}{dt} R_\alpha(\tilde{\rho}_t | \nu_*) \leq -\frac{3}{2\alpha\lambda} R_\alpha(\tilde{\rho}_t | \nu_*) + \alpha \mathbb{E} \left[\frac{\tilde{\rho}_t}{\nu_*}(\tilde{x}_t)^{2\alpha-2} \right]^{\frac{1}{2}} \mathbb{E} \left[\|\nabla f(\tilde{x}_t) - \nabla f(x_{[t/\eta]})\|^4 \right]^{\frac{1}{2}}.$$

The proof of the above statement is similar to that of Lemma 1, and deferred to Section A.1. To iterate the bound obtained using Lemma 6, we again conduct a two-phase analysis. In the first phase, we use Lemma 2; N steps of LMC implies that $\mathbb{E} \left[\frac{\tilde{\rho}_t}{\nu_*}(\tilde{x}_t)^{2\alpha-2} \right]$ is bounded by $\mathcal{O}(\alpha^{0.25})$, and stays bounded in the subsequent N iterations. In the second phase, we use the following generalization of the single-step bound in Lemma 3 for the Rényi divergence.

Lemma 7. *Under the assumptions of Lemma 2, and if we additionally have $\eta \leq \frac{2\alpha\lambda}{3}$, then for any iteration k such that $k \in [N, 2N]$, LMC satisfies the following bound*

$$R_\alpha(\rho_{k+1} | \nu_*) \leq \left(1 - \frac{3\eta}{4\alpha\lambda}\right) R_\alpha(\rho_k | \nu_*) + c\beta L^2(b+d)\alpha^{9/8}\eta^2, \quad (3.3)$$

where c is an absolute constant and β is defined in Lemma 3.

The immediate consequence of this lemma is a bound on the Rényi divergence, which is stated in the following theorem.

Theorem 8. *For $\alpha > 1$ and for a potential f satisfying Assumption 1, suppose we run $2N$ iterations of LMC (1.1) with step size η to sample from $\nu_* = e^{-f}$. If we initialize x_0 with $\mathcal{N}(0, \sigma^2 \mathbf{I})$ for some $\sigma^2 < (1 + L)^{-1}$, in order to get $R_\alpha(\rho_{2N} | \nu_*) \leq \epsilon$, it is sufficient if the following inequalities hold*

$$\begin{aligned} \eta &\leq \frac{2}{\|\nabla f(0)\|^2} \wedge \frac{1 \wedge m}{4(1 \vee L^2)} \wedge \frac{2\alpha\lambda}{3} \wedge \frac{c_3}{\beta\lambda L^2 \alpha^{17/8}} \times \frac{\epsilon}{(b+d)} \\ N\eta &\geq c_2\alpha\lambda \log\left(\frac{6\alpha^2 d C_\sigma}{(\alpha-1)\epsilon}\right) \\ c_1 &\geq \alpha^2 \kappa^2 L^2 (b+d + \log N) N\eta^2, \end{aligned}$$

where λ is the LSI constant of the target, c_1, c_2, c_3 are absolute constants, and C_σ and β are dimension free constants defined respectively in Lemmas 2 and 3.

Consequently, if we choose

$$\begin{aligned} \eta &= \frac{c_3}{\beta} \times \frac{m^2}{L^4 \lambda \log\left(\frac{L^4 \lambda^2}{m^2}\right)} \times \frac{1}{\alpha^3} \times \frac{\epsilon}{b+d} \times \frac{1}{\log\left(\frac{6\alpha^2 d C_\sigma}{(\alpha-1)\epsilon}\right)^2} \\ N &= \frac{c_2 \beta}{c_3} \times \frac{L^4 \lambda^2 \log\left(\frac{L^4 \lambda^2}{m^2}\right)}{m^2} \times \alpha^4 \times \frac{b+d}{\epsilon} \times \log\left(\frac{6\alpha^2 d C_\sigma}{(\alpha-1)\epsilon}\right)^3, \end{aligned}$$

then, $2N$ steps of LMC yield $R_\alpha(\rho_{2N} | \nu_*) \leq \epsilon$ for $N \geq 2$ and for a sufficiently small $\epsilon > 0$.

The above theorem is similar to Theorem 4; therefore the same remarks also apply to this result. One important difference is the α dependency of the rate $\mathcal{O}(\alpha^4 \log(\frac{\alpha^2}{\alpha-1})^3)$, which diverges as $\alpha \rightarrow 1$ and $\alpha \rightarrow \infty$. If one is interested in $\alpha \approx 1$, then using the monotonicity of Rényi divergence, one can obtain better rate estimates, for example by bounding R_α by R_2 .

The above result also implies the same rate estimate for the Chi-squared divergence; however, several key steps for the latter require milder conditions (cf. Lemmas 1 and 6). Therefore relaxing the conditions of Lemma 2 would directly improve the set of feasible potentials in the Chi-squared divergence, which is not true for the Rényi divergence due to the conditions of Lemma 6. Also see Section 5 for a detailed discussion on this direction.

4 Examples

Assumption 1 implies that the target satisfies a log-Sobolev inequality. This can be deduced from the results of [BÉ85, CGW10], and a derivation is provided in Section C (cf. [RRT17, Prop 3.2]). However, under more specific curvature conditions on the potential, one can obtain better estimates (in terms of dimension) on the LSI constant. We discuss a few interesting cases below.

4.1 Strongly convex and first-order smooth potentials

Potentials that are in this category satisfy $LI \succeq \nabla^2 f(x) \succeq m\mathbf{I}$. It is easy to see that Assumption 1 holds for the same parameters L, m , and $b = 0$. Moreover, due to the Bakry-Émery criterion, the target $\nu_* = e^{-f}$ satisfies LSI with constant $\lambda = 1/m$.

While strongly convex potentials have been most frequently studied in prior work, the known rate in Chi-squared and Rényi divergence is $\mathcal{O}(d/\epsilon^2)$ [GT20]. Our analysis instead obtains $\mathcal{O}(d/\epsilon)$; this represents a significant improvement to the known convergence rate in these metrics. When

translated to KL divergence (using (1.4)), our rate recovers the state-of-the-art rates for the same class of potentials [Dal17b, DM17].

Despite their apparent simplicity, this function class contains numerous practical applications.

Ex 1: Gaussian mixtures. In this case, we consider sampling from potentials of the form

$$\nu_*(x) \propto \exp\left(-\frac{1}{2}\|x-a\|^2\right) + \exp\left(-\frac{1}{2}\|x+a\|^2\right)$$

with $a \in \mathbb{R}^d$ a parameter controlling the modal separation. These potentials have strongly convex densities if $\|a\|_2 < 1$, with first, second and third order derivatives all being Lipschitz [Dal17b].

Ex 2: Bayesian logistic regression. Consider data samples $V = \{v_i\}_{i=1}^n \in \mathbb{R}^{n \times d}$, $y = \{y_i\}_{i=1}^n \in \mathbb{R}^n$, and a Bernoulli distribution $\mathbb{P}(y = 1|v) = 1/(1 + \exp(-\langle x, v \rangle))$ with parameter $x \in \mathbb{R}^d$. If we use the prior $x \sim \mathcal{N}(0, \alpha \Sigma_V^{-1})$ where $\Sigma_V = V^\top V/n$, then the resulting posterior is

$$\nu_*(x) \propto \exp\left(y^\top Wx - \sum_{i=1}^n \log(1 + \exp(\langle x, v_i \rangle)) - \frac{\alpha}{2} \left\| \Sigma_V^{1/2} x \right\|^2\right).$$

Again, the potential is strongly convex, with Lipschitz derivatives up to the third order [Dal17b].

Ex 3: Bounded perturbations. We also admit potentials of the form $f = f_{\text{sc}} + f_{\text{p}}$ such that f_{sc} is m_0 -strongly convex with L_0 -Lipschitz gradient, and f_{p} satisfies $|f_{\text{p}}| \vee \|\nabla f_{\text{p}}\| \vee \|\nabla^2 f_{\text{p}}\| \leq B$. Then

$$\begin{aligned} \langle \nabla f(x) - \nabla f(y), x - y \rangle &= \langle \nabla f_{\text{sc}}(x) - \nabla f_{\text{sc}}(y), x - y \rangle + \langle \nabla f_{\text{p}}(x) - \nabla f_{\text{p}}(y), x - y \rangle \\ &\geq m_0 \|x - y\|^2 - 2B \|x - y\| \geq m \|x - y\|^2 - b, \end{aligned}$$

where the last inequality holds for $m = m_0/2$ and $b = 2B^2/m$. We also have $\|\nabla^2 f\| \leq \|\nabla^2 f_{\text{sc}}\| + \|\nabla^2 f_{\text{p}}\| \leq L_0 + B := L$. Thus, Assumption 1 holds for finite perturbations of strongly convex and smooth potentials. Further, by the Holley-Stroock lemma [HS87], LSI is satisfied for $\lambda = m_0^{-1} e^{2B}$.

4.2 Strong convexity outside a ball

For a second class of examples, consider potentials that are strongly convex outside a ball. If we assume that f has continuous and upper bounded Hessian $\nabla^2 f(x) \preceq M_0 \mathbf{I}$, this assumption means

$$\inf_{\|x\| \geq r} \nabla^2 f(x) \succeq m_0 \mathbf{I}, \quad \inf_{\|x\| < r} \nabla^2 f(x) \succeq -k \mathbf{I}, \quad (4.1)$$

where $m_0 > 0$ is the convexity parameter, and $r \geq 0, k > 0$. This condition implies f is first-order smooth with $L := M_0 \vee k$. The first condition in (4.1) is enough to verify strong dissipativity; we write $y = x + s_3 z$ for $s_3 = \|x - y\|, \|z\| = 1$, and let $s_1, s_2 \in [0, s_3]$ such that $\|x + sz\| \leq r \iff s \in [s_1, s_2]$. Then, strong dissipativity follows from

$$\begin{aligned} \langle \nabla f(x) - \nabla f(y), x - y \rangle &= \int_{s \in [0, s_1] \cup [s_2, s_3]} \langle \nabla^2 f(x + sz) z, s_3 z \rangle ds + \langle \nabla f(x + s_2 z) - \nabla f(x + s_1 z), s_3 z \rangle \\ &\geq m_0 (s_3 - 2r) s_3 - 2L r s_3 \geq \frac{m_0}{2} \|x - y\|^2 - \frac{2}{m_0} (m_0 r + L r)^2, \end{aligned}$$

where outside of the ball we use the strong convexity, and inside we use smoothness.

For LSI, we replace f with a strongly convex function $\tilde{f}(x) = f(x) + 0.5(k + m_0) \|x\|^2 \mathbf{1}_{\{\|x\| \leq r\}}$, so that $\nabla^2 \tilde{f} \succeq m_0 \mathbf{I}$ everywhere. Then $\|f - \tilde{f}\|_\infty \leq 0.5(k + m_0) r^2$, and by the Holley-Stroock perturbation lemma [HS87], $\lambda \leq \lambda_{\tilde{f}} \cdot \exp((k + m_0) r^2) = m_0^{-1} \exp((k + m_0) r^2)$.

Ex: Student's t-Regression with Gaussian prior. Consider the function with $\alpha > 0$

$$f(x) = \frac{1}{2} \log(1 + \|x\|^2) + \frac{\alpha}{2} \|x\|^2 + \text{constant}. \quad (4.2)$$

The gradient is $\frac{x}{1+\|x\|^2} + \alpha x$, which is $(\alpha + 1)$ -Lipschitz, and Hessian is $\alpha I + \frac{(1+\|x\|^2)I - 2xx^\top}{(1+\|x\|^2)^2}$. When $\alpha < 1/8$ is sufficiently small, this function is non-convex. However, if we take the radius to be $\|x\| \geq 1/\sqrt{\alpha}$; we find strong convexity outside the ball with $m_0 = \alpha^2(\alpha + 3)/(\alpha + 1)^2$, and $k = -1/8$. Strong dissipativity is satisfied, and LSI holds with constant $\frac{(\alpha+1)^2}{\alpha^2(\alpha+3)} \exp\left(\frac{\alpha^2(\alpha+3)}{\alpha(\alpha+1)^2} + \frac{1}{8\alpha}\right)$.

For instance, when data is heavy tailed, Student's t-distribution is used to model the errors. Under a Gaussian prior on the coefficients $x \sim \mathcal{N}(0, \alpha I)$, the posterior distribution has the potential

$$f(x) = \sum_{i=1}^n \log(1 + (y_i - \langle v_i, x \rangle)^2) + \frac{\alpha}{2} \|x\|^2 + \text{constant},$$

where $\{v_i\}_{i=1}^n, \{y_i\}_{i=1}^n$ are data samples as before. Notice that the potential has the same form as (4.2). Under suitable assumption on data, one can use the same steps above to verify our conditions.

4.3 Non-uniform strong convexity

Finally, we consider functions which are similar to the previous section, but with variable convexity

$$\inf_{\|x\| \geq r} \nabla^2 f(x) \succeq m_0(r) I.$$

Then if $\sup_{r \geq 0} m_0(r) > 0$, strong dissipativity holds for the same reason as in the prior subsection, since we need only fix some $r > 0$, where $m_0(r) > 0$ to recover the first inequality in (4.1). LSI is satisfied as well [CW97] with the constant λ bounded by

$$\lambda \leq \frac{a_0^2}{2} \exp\left(\int_0^{a_0} r m_0(r) dr - 1\right) \text{ where } a_0 \text{ uniquely solves } \int_0^{a_0} m_0(r) dr = 2/a_0.$$

Ex: Heavy-tailed regression with corrupted noise. Consider the following function

$$f(x) = -\frac{1}{2} \log\left(\beta + \exp(-\|x\|^2)\right) + \frac{\alpha}{2} \|x\|^2 + \text{constant}, \quad (4.3)$$

where $\beta > 0, \alpha > 0$. The gradient is $\frac{x}{\beta \exp(\|x\|^2) + 1} + \alpha x$, which is $\alpha + \frac{1}{1+\beta}$ -Lipschitz, and the Hessian is $\frac{\beta \exp(\|x\|^2)(I - 2xx^\top) + I}{(\beta \exp(\|x\|^2) + 1)^2} + \alpha I$. In this case, $m_0(r) = \alpha - \frac{\beta \exp(r^2)(2r^2 - 1) - 1}{(\beta \exp(r^2) + 1)^2} \geq \alpha - \frac{r^2}{\beta \exp(r^2)}$. Since $\frac{r^2}{\beta \exp(r^2)} \rightarrow 0$, this quantity eventually becomes positive. By our argumentation, the function is strongly dissipative and we can solve for the LSI constant through numerical integration.

For an instance of this, consider linear regression on data $\{v_i\}_{i=1}^n, \{y_i\}_{i=1}^n$, with corrupted Gaussian noise such that with small probability the noise is instead sampled from some uniform distribution (this arises in visual reconstruction problems [BZ87]). Assuming a prior $x \sim \mathcal{N}(0, \alpha I)$, we obtain the following potential for the posterior distribution

$$f(x) = -\frac{1}{2} \sum_{i=1}^n \log\left(\beta + \exp(-(y_i - \langle v_i, x \rangle)^2)\right) + \frac{\alpha}{2} \|x\|^2 + \text{constant}.$$

This potential has the same structure as (4.3) and our assumptions can be verified using the same steps, under suitable conditions on the data.

5 Conclusion

In this paper, we analyzed the convergence of unadjusted LMC algorithm for a class of potentials that are first-order smooth and satisfy strong dissipativity. We used the Fokker-Planck equation of the interpolated process alongside the smoothness assumptions to obtain a differential inequality which in turn yielded a single step bound to be iterated to obtain our main convergence results. In the case of strongly convex potentials, the obtained rates improve upon the existing rates in Chi-squared and Rényi divergence, and recover the state-of-the-art rates in KL, TV, as well as \mathcal{W}_2 .

There are several important future directions to consider, among which we highlight a few here.

- Although we assumed LSI throughout the paper, Poincaré inequality is sufficient to establish the differential inequality in Lemma 1. The restriction is due to the strong dissipativity assumption, which enforces a quadratic growth on the potential function; thus, LSI must hold. This also enforces at least linear growth on the gradient, which prevents us from considering weakly smooth potentials that satisfy Hölder continuity. Relaxing this assumption to weak dissipativity may allow one to rely on Poincaré inequality, which permits weakly smooth potentials that have at least linear growth. A promising set of conditions in this context is, for some $\theta < 1$ and $\gamma \in [1, 2)$

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|^\theta, \quad \langle x, \nabla f(x) \rangle \geq m \|x\|^\gamma - b.$$

We note that this setting is already considered in [EH20] under the KL divergence.

- We mentioned that the rate estimates presented in this paper do not hold for the subsequent iterates, similar to [SL19, GT20, EH20]. This is an artifact of the proof technique we rely on, and hopefully can be remedied in the future work.
- Working out a simple Gaussian toy example, one can verify that the LMC algorithm converges to the target in $\tilde{\mathcal{O}}(\sqrt{d/\epsilon})$ iterations in Chi-squared divergence. This is $\tilde{\mathcal{O}}(\sqrt{d/\epsilon})$ better than the rate estimate we obtained in the current paper, and is due to the additional second-order smoothness of the Gaussian potential. Therefore, achieving this rate under second-order smoothness is an interesting direction left for future work.
- Similar techniques can be utilized to establish convergence rates for other numerical schemes; the ones that satisfy certain optimality criteria are of particular interest [SL19, CLW20].

Finally, we note that many of the bounds in the paper can be improved, at the expense of introducing some additional complexity into the results.

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A Proofs of the Main Results

Proof of Lemma 2. We use Cauchy-Schwarz inequality to get

$$\mathbb{E}_{\tilde{\rho}_T} \left[\frac{\tilde{\rho}_T}{\nu_*}(x)^{2\alpha-2} \right] \leq \mathbb{E}_{\nu_*} \left[\frac{\pi_T}{\nu_*}(x)^{4\alpha-3} \right]^{\frac{1}{2}} \mathbb{E}_{\pi_T} \left[\frac{\tilde{\rho}_T}{\pi_T}(x)^{4\alpha-2} \right]^{\frac{1}{2}}.$$

For the first term, we then apply Lemmas 25 and 26, and obtain that when

$$N\eta \geq \frac{(4\alpha - 3)\lambda}{2} \log \log \mathbb{E}_{\nu_*} \left[\frac{\rho_0}{\nu_*}(x)^{4\alpha-3} \right],$$

we have $\mathbb{E}_{\nu_*} \left[\frac{\pi_T}{\nu_*}(x)^{4\alpha-3} \right] \leq e$ for $T \geq N\eta$. We can apply a crude bound of $\mathbb{E}_{\nu_*} \left[\frac{\rho_0}{\nu_*}(x)^{4\alpha-3} \right] \leq \exp(4\alpha d C_\sigma)$, where C_σ is as in the remark after Lemma 26.

For the second term, we apply Lemma 14 with $16\alpha - 10$ and number of iterations $2N$, under the condition that

$$c\alpha^2 \kappa^2 L^2 (b + d + \log 2N) N\eta^2 \leq 1.$$

Then we get $\mathbb{E}_{\pi_T} \left[\frac{\tilde{\rho}_T}{\pi_T}(x)^{4\alpha-2} \right] \leq 20e\sqrt{\alpha}$; combining these two terms yields the final bound. \square

Proof of Lemma 3. Suppose x^* is the global minimizer of f (therefore $\nabla f(x^*) = 0$). From Lemma 10 and $\eta \leq m/L^2$ we have

$$\|x - \eta \nabla f(x) - x^*\|^2 \leq (1 - m\eta) \|x - x^*\|^2 + 2b\eta.$$

Using the previous inequality with the fact that Gaussian has zero odd moments and Lemma 22 we get

$$\mathbb{E} \left[\|x_k - x^*\|^2 \right] \leq \mathbb{E} \left[\|x_0 - x^*\|^2 \right] + \frac{2(b+d)}{m}.$$

Doing the same for power 4 we get the following

$$\mathbb{E} \left[\|x_{k+1} - x^*\|^4 \right] \leq (1 - m\eta) \mathbb{E} \left[\|x_k - x^*\|^4 \right] + 12\eta(b+d) \mathbb{E} \left[\|x_k - x^*\|^2 \right] + 12\eta^2(b+d)^2.$$

Plugging the bound on $\mathbb{E} [\|x_k - x^*\|^2]$ back in the previous inequality and using Lemma 22, we get the following

$$\mathbb{E} [\|x_k - x^*\|^4] \leq \left[\frac{\mathbb{E} [\|x_0 - x^*\|^4]}{(b+d)^2} + \frac{12}{m} \left(\frac{\mathbb{E} [\|x_0 - x^*\|^2]}{b+d} + \frac{2}{m} + \eta \right) \right] (b+d)^2.$$

For $t \leq \eta$, we write

$$\begin{aligned} \mathbb{E} \left[\left\| t \nabla f(x_k) + \sqrt{2} B_t \right\|^4 \right] &\leq 8\eta^4 \mathbb{E} [\|\nabla f(x_k)\|^4] + 32\mathbb{E} [\|B_t\|^4] \\ &\leq 8\eta^4 L^4 \mathbb{E} [\|x_k - x^*\|^4] + 96\eta^2 d^2. \end{aligned}$$

By combining the last two inequalities we get the following.

$$\begin{aligned} &\mathbb{E} \left[\left\| t \nabla f(x_k) + \sqrt{2} B_t \right\|^4 \right] \\ &\leq \left(96 + 8\eta^2 L^4 \left[\frac{\mathbb{E} [\|x_0 - x^*\|^4]}{(b+d)^2} + \frac{12\mathbb{E} [\|x_0 - x^*\|^2]}{m(b+d)} + \frac{24}{m^2} + \frac{12\eta}{m} \right] \right) (b+d)^2 \eta^2, \end{aligned}$$

using the definition of β and moments of Gaussian and the bound on η we get

$$\mathbb{E} \left[\left\| t \nabla f(x_k) + \sqrt{2} B_t \right\|^4 \right]^{1/2} \leq c\beta(b+d)\eta, \quad (\text{A.1})$$

for some universal constant c . Now if the conditions of Lemma 2 hold, then we have the expected ratio of the densities bounded by an absolute constant for $N \leq k\eta + t \leq 2N$. Combining this with the previous bound, we obtain

$$\mathbb{E} \left[\frac{\tilde{\rho}_{k\eta+t}}{\nu_*} (\tilde{x}_{k\eta+t})^2 \right]^{1/2} \mathbb{E} [\|\nabla f(\tilde{x}_{k\eta+t}) - \nabla f(x_k)\|^4]^{1/2} \leq c\beta L^2(b+d)\eta$$

where c is an absolute constant.

We plug in the derived upper bounds back in (2.2) to get

$$\frac{d}{dt} \chi^2(\tilde{\rho}_{k\eta+t} | \nu_*) \leq -\frac{3}{2\lambda} \chi^2(\tilde{\rho}_{k\eta+t} | \nu_*) + c\beta L^2(b+d)\eta.$$

Integrating this differential inequality and using $t \leq \eta$ results in the following single step bound

$$\chi^2(\rho_{k+1} | \nu_*) \leq \left(1 - \frac{3\eta}{4\lambda} \right) \chi^2(\rho_k | \nu_*) + c\beta L^2(b+d)\eta^2,$$

where we used $\eta \leq \frac{\lambda}{2}$ and that $e^{-x} \leq 1 - x/2$ for $x \in [0, 1]$, and as before we absorbed universal constants into c . \square

Proof of Theorem 4. In the first phase, by Lemma 2, we used the first N steps of LMC to show that the expected squared ratio of densities evaluated at the interpolation process is bounded by an absolute constant and it stays bounded for the subsequent N iterations. This allows us to iterate the single-step bound provided by Lemma 3, for which we need to bound its starting point, the LMC iteration N . We write

$$\chi^2(\rho_N | \nu_*) = \mathbb{E}_{\nu_*} \left[\frac{\rho_N}{\nu_*} (x)^2 \right] - 1 \leq \mathbb{E}_{\pi_T} \left[\frac{\rho_N}{\pi_{N\eta}} (x)^4 \right]^{1/2} \mathbb{E}_{\nu_*} \left[\frac{\pi_{N\eta}}{\nu_*} (x)^3 \right]^{1/2} - 1,$$

by Cauchy-Schwartz inequality. We bound the right hand side term by term with an absolute constant. For the first term we use Lemma 14 for $\alpha = 14$. In order for it to be bounded by $22e$, it is sufficient if the following holds that $c_1 \kappa^2 L^2 N(b + d + \log N) \eta^2 \leq 1$, where c_1 is some universal constant. The second term can be written as

$$\mathbb{E}_{\nu_*} \left[\frac{\pi N \eta}{\nu_*} (x)^3 \right] = \mathbb{E}_{\nu_*} \left[\frac{\pi N \eta}{\nu_*} (x)^{4\alpha-3} \right] \text{ for } \alpha = 3/2,$$

for which we already obtained an upper bound in the proof of Lemma 2. That is, whenever $N\eta \geq \frac{3\lambda}{2} \log(6dC_\sigma)$, the right hand side is bounded by e . Thus, we have $\chi^2(\rho_N|\nu_*) \leq 12$.

In the second phase of the analysis, we use the differential inequality. Iterating the single step bound in Lemma 3 with the help of Lemma 22, together with the upper bound on the initialization of the second phase, we get

$$\chi^2(\rho_{2N}|\nu_*) \leq \exp\left(-\frac{3\eta N}{4\lambda}\right) 12 + c\lambda L^2 \beta(b+d)\eta,$$

where we used the bound on Chi-square divergence in step N and absorbed absolute constants into c . To make this less than ϵ , it suffices if the following additional inequalities hold

$$N\eta \geq \frac{4}{3}\lambda \log \frac{24}{\epsilon}, \quad \eta \leq \frac{1}{2c\beta\lambda L^2} \times \frac{\epsilon}{b+d}$$

for some absolute constant c . Combining the above conditions with the conditions of used Lemmas, and simplifying the statements, we obtain the inequalities in (3.1).

For the last statement, it suffices to check the inequalities in (3.1) for the given choice of step size and the number of iterations. \square

A.1 Proofs for the Rényi Divergence

Proof of Lemma 6. Define the following quantities for $\alpha \geq 1$ between two densities p, q

$$F_\alpha(p|q) = \mathbb{E}_q \left[\frac{p}{q}(x)^\alpha \right], \quad G_\alpha(p|q) = \mathbb{E}_q \left[\frac{p}{q}(x)^{\alpha-2} \left\| \nabla \frac{p}{q}(x) \right\|^2 \right].$$

Then we have $(\alpha - 1)R_\alpha(p|q) = \log F_\alpha(p|q)$. Note that $F_\alpha(p|q) \geq 1$ for any p, q . We have the following result for the Langevin diffusion.

Lemma 9. (Adapted from [VW19, Lemma 5]) *If q satisfies LSI with constant λ , then*

$$\frac{G_\alpha(p|q)}{F_\alpha(p|q)} \geq \frac{2}{\alpha^2 \lambda} R_\alpha(p|q).$$

The above lemma was used in [VW19] to prove the exponential convergence of the Langevin diffusion. We use it in a similar way, but for the interpolation process.

Consider the dynamics in (2.3) and write to express the time-derivative of F_α as

$$\begin{aligned} \frac{d}{dt} F_\alpha(\tilde{\rho}_t|\nu_*) &= \alpha \int \frac{\tilde{\rho}_t}{\nu_*}(x)^{\alpha-1} \frac{\partial \tilde{\rho}_t(x)}{\partial t} dx \\ &= \alpha \int \frac{\tilde{\rho}_t}{\nu_*}(x)^{\alpha-1} \nabla \cdot \left(\tilde{\rho}_t(x) \left(\mathbb{E}[\nabla f(x_k) - \nabla f(x)|\tilde{x}_t = x] + \nabla \log \frac{\tilde{\rho}_t}{\nu_*}(x) \right) \right) dx \\ &\stackrel{1}{=} \alpha(\alpha - 1) \int \frac{\tilde{\rho}_t}{\nu_*}(x)^{\alpha-2} \left\langle \nabla \frac{\tilde{\rho}_t}{\nu_*}(x), \mathbb{E}[\nabla f(x) - \nabla f(x_k)|\tilde{x}_t = x] - \nabla \log \frac{\tilde{\rho}_t}{\nu_*}(x) \right\rangle \tilde{\rho}_t(x) dx, \end{aligned}$$

where in 1 we use the divergence theorem. For the first term, we write

$$\begin{aligned}
& \int \frac{\tilde{\rho}_t}{\nu_*}(x)^{\alpha-2} \left\langle \nabla \frac{\tilde{\rho}_t}{\nu_*}(x) \sqrt{\frac{\nu_*(x)}{\tilde{\rho}_t(x)}}, \sqrt{\frac{\tilde{\rho}_t(x)}{\nu_*(x)}} \mathbb{E}[\nabla f(x) - \nabla f(x_k) | \tilde{x}_t = x] \right\rangle \tilde{\rho}_t(x) dx \\
& \stackrel{1}{\leq} \frac{1}{4} \int \frac{\tilde{\rho}_t}{\nu_*}(x)^{\alpha-2} \left\| \nabla \frac{\tilde{\rho}_t}{\nu_*}(x) \right\|^2 \nu_*(x) dx \\
& \quad + \int \mathbb{E} \left[\frac{\tilde{\rho}_t}{\nu_*}(x)^{\alpha-1} \|\nabla f(x) - \nabla f(x_k)\|^2 | \tilde{x}_t = x \right] \tilde{\rho}_t(x) dx \\
& = \frac{1}{4} G_\alpha(\tilde{\rho}_t | \nu_*) + \mathbb{E} \left[\frac{\tilde{\rho}_t}{\nu_*}(\tilde{x}_t)^{\alpha-1} \|\nabla f(\tilde{x}_t) - \nabla f(x_k)\|^2 \right],
\end{aligned}$$

where in 1 we used that $\langle a, b \rangle \leq \frac{1}{4} \|a\|^2 + \|b\|^2$. For the second term, we find

$$\int \frac{\tilde{\rho}_t}{\nu_*}(x)^{\alpha-2} \left\langle \nabla \frac{\tilde{\rho}_t}{\nu_*}(x), -\nabla \log \frac{\tilde{\rho}_t}{\nu_*}(x) \right\rangle \tilde{\rho}_t(x) dx = -G_\alpha(\tilde{\rho}_t | \nu_*)$$

Combining terms, we get

$$\begin{aligned}
\frac{d}{dt} R_\alpha(\tilde{\rho}_t | \nu_*) &= \frac{1}{(\alpha-1)F_\alpha(\tilde{\rho}_t | \nu_*)} \frac{dF_\alpha(\tilde{\rho}_t | \nu_*)}{dt} \\
&\leq -\frac{3\alpha}{4} \frac{G_\alpha(\tilde{\rho}_t | \nu_*)}{F_\alpha(\tilde{\rho}_t | \nu_*)} + \alpha \frac{\mathbb{E} \left[\frac{\tilde{\rho}_t}{\nu_*}(\tilde{x}_t)^{\alpha-1} \|\nabla f(\tilde{x}_t) - \nabla f(x_k)\|^2 \right]}{F_\alpha(\tilde{\rho}_t | \nu_*)} \\
&\stackrel{1}{\leq} -\frac{3\alpha}{4} \frac{G_\alpha(\tilde{\rho}_t | \nu_*)}{F_\alpha(\tilde{\rho}_t | \nu_*)} + \alpha \mathbb{E} \left[\frac{\tilde{\rho}_t}{\nu_*}(\tilde{x}_t)^{2\alpha-2} \right]^{\frac{1}{2}} \mathbb{E} \left[\|\nabla f(\tilde{x}_t) - \nabla f(x_k)\|^4 \right]^{\frac{1}{2}},
\end{aligned}$$

where in 1 we use a Cauchy-Schwarz inequality and that $F_\alpha(\tilde{\rho}_t | \nu_*) \geq 1$. It remains to apply Lemma 9 on the first term. \square

Proof of Lemma 7. The bound on the second term in (3.3) is obtained directly from (A.1), as

$$\mathbb{E} \left[\|\nabla f(\tilde{x}_{k\eta+t}) - \nabla f(x_k)\|^4 \right]^{\frac{1}{2}} \leq c\beta L^2(b+d)\eta.$$

So combining this with Lemma 2, for any $k \in [N, 2N]$, we get

$$\mathbb{E} \left[\frac{\tilde{\rho}_{k\eta+t}}{\nu_*}(\tilde{x}_{k\eta+t})^{2\alpha-2} \right]^{\frac{1}{2}} \mathbb{E} \left[\|\nabla f(\tilde{x}_{k\eta+t}) - \nabla f(x_k)\|^4 \right]^{\frac{1}{2}} \leq c\beta L^2(b+d)\alpha^{1/8}\eta.$$

Substitution into Lemma 6 yields

$$\frac{d}{dt} R_\alpha(\tilde{\rho}_t | \nu_*) \leq -\frac{3}{2\alpha\lambda} R_\alpha(\tilde{\rho}_t | \nu_*) + c\beta L^2(b+d)\alpha^{9/8}\eta.$$

It remains to integrate this for $t \leq \eta$, and apply $e^{-x} \leq (1 - \frac{x}{2})$ for $x \in [0, 1]$, and $\eta \leq \frac{2\alpha\lambda}{3}$. \square

Proof of Theorem 8. In the first phase, in accordance with the proof of Theorem 4, we will need to ensure that the density ratio at step N remains bounded. So we again write

$$\mathbb{E}_{\nu_*} \left[\frac{\tilde{\rho}_{N\eta}}{\nu_*}(x)^\alpha \right] \leq \mathbb{E}_{\pi_{N\eta}} \left[\frac{\tilde{\rho}_{N\eta}}{\pi_{N\eta}}(x)^{2\alpha} \right]^{\frac{1}{2}} \mathbb{E}_{\nu_*} \left[\frac{\pi_{N\eta}}{\nu_*}(x)^{2\alpha-1} \right]^{\frac{1}{2}}.$$

For the first term, we can use Lemma 14 with $8\alpha - 2$, so that again we have the condition

$$c\alpha^2\kappa^2L^2(b+d+\log N)N\eta^2 \leq 1,$$

to guarantee that the first term is bounded by $7\alpha^{0.25}$. The second term is bounded via a Hölder inequality

$$\mathbb{E}_{\nu_*} \left[\frac{\pi N \eta}{\nu_*}(x)^{2\alpha-1} \right] \leq \mathbb{E}_{\nu_*} \left[\frac{\pi N \eta}{\nu_*}(x)^{4\alpha-3} \right]^{\frac{2\alpha-1}{4\alpha-3}},$$

which is simply bounded by $\exp(\frac{2\alpha-1}{4\alpha-3})$ under the conditions of Lemma 2. Consequently the Rényi divergence at $k = N$ is bounded by

$$R_\alpha(\rho_N|\nu_*) \leq \frac{\log(12\alpha)}{\alpha-1}.$$

In the second phase, we simply iterate the differential inequality in Lemma 7 and apply Lemma 21 to get

$$R_\alpha(\rho_{2N}|\nu_*) \leq \exp\left(-\frac{3N\eta}{4\alpha\lambda}\right) \frac{\log(12\alpha)}{\alpha-1} + c\beta\lambda L^2(b+d)\alpha^{17/8}\eta.$$

Thus we obtain ϵ accuracy if the following inequalities hold

$$N\eta \geq \frac{4}{3}\alpha\lambda \log \frac{2\log(12\alpha)}{(\alpha-1)\epsilon}, \quad \eta \leq \frac{1}{2c\beta\lambda L^2\alpha^{17/8}} \times \frac{\epsilon}{b+d}$$

for some absolute constant c . Combining these conditions with the conditions of other lemmas used above and simplifying the expressions using $\alpha, C_\sigma > 1 \geq \epsilon$, we conclude the proof of the first part for the given choice of η and N , and choosing a sufficiently small ϵ .

The second part of the theorem follows from verifying the conditions in the first part, and choosing a suitably small accuracy ϵ . □

B Main Technical Lemmas

Let X_t show the clipped interpolation process, with step size η , which is defined similar to interpolation process with the following exception: if for any $s \leq t$ we have $\|X_s - X_{\lfloor s/\eta \rfloor \eta}\| > r$ we change X_t to \perp . We define X_t^j similarly, with step size η/j for $j \in \mathbb{N}$ and define X_t' similarly for the continuous time process. We use the same r and η for all processes, in other words we change X_t^j and X_t' to \perp when for some $s \leq t$ we have $\|X_s^j - X_{\lfloor s/\eta \rfloor \eta}^j\| > r$ and $\|X_s' - X_{\lfloor s/\eta \rfloor \eta}'\| > r$, respectively. We will refer to these processes as clipped processes that are started from the same distribution. Let $P_t(x), P_t^j(x)$ and $Q_t(x)$ show the density of X_t, X_t^j and X_t' at x .

We prove a bound on the probability of jump on both continuous time and discrete time process that will be used to remove the bounded movement assumption. This part is an extension of Lemma 13 in [GT20]. First we prove both continuous-time and discrete-time processes satisfy a semi-contraction inequality.

Lemma 10. *If f satisfies Assumption 1 and z_t and z'_t are two instances of Langevin diffusion with synchronously coupled Brownian motion, then we have the following*

$$\|z_t - z'_t\| \leq e^{-mt} \|z_0 - z'_0\| + \sqrt{\frac{b}{m} (1 - e^{-2mt})}.$$

Furthermore, if $\eta \leq m/L^2$, then gradient descent satisfies the following

$$\|(x - \eta \nabla f(x)) - (y - \eta \nabla f(y))\|^2 \leq (1 - m\eta) \|x - y\|^2 + 2\eta b.$$

Proof. For discrete-time, the result follows from elementary calculations. For continuous-time, by coupling the Brownian motions and subtracting we get the following,

$$\frac{d}{dt}(z_t - z'_t) = -(\nabla f(z_t) - \nabla f(z'_t)),$$

then we differentiate $\|z_t - z'_t\|^2$ with respect to time to get

$$\frac{d}{dt} \|z_t - z'_t\|^2 = -2\langle z_t - z'_t, \nabla f(z_t) - \nabla f(z'_t) \rangle \leq -2m \|z_t - z'_t\|^2 + 2b,$$

where the last step follows from strong dissipativity. Solving this differential inequality and using $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, concludes the proof. \square

Lemma 11. *Suppose the potential satisfies Assumption 1 and $x_0 = z_0 \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$, for $\sigma^2 < (L+1)^{-1}$. If the step size is small enough, $\eta \leq \frac{1 \wedge m}{4(1 \vee L^2)} \wedge \frac{2}{\|\nabla f(0)\|^2}$, then each of the following jump conditions, denoted with \mathcal{E}_δ^1 and \mathcal{E}_δ^2 , happens with probability at least $1 - \delta$.*

$$\begin{aligned} \forall t \leq N\eta : \|\tilde{x}_t - \tilde{x}_{\lfloor t/\eta \rfloor \eta}\| &\leq (2\kappa + 1) \left(1 + \sqrt{b} + \sqrt{d} + 2\sqrt{\log(2(N+1)/\delta)}\right) \sqrt{2\eta}, \\ \forall t \leq N\eta : \|z_t - z_{\lfloor t/\eta \rfloor \eta}\| &\leq (5\kappa + 1) \left(1 + \sqrt{b} + \sqrt{d} + 2\sqrt{\log(2(N+1)/\delta)}\right) \sqrt{2\eta}. \end{aligned}$$

Remark. The RHS of both of the bounds can be written as $c\kappa \left(\sqrt{b} + \sqrt{d} + \sqrt{\log N/\delta}\right) \sqrt{\eta}$, for a universal constant c (note that $\kappa > 1$).

Proof. In order to ease the notation we will use B_u^t to denote $B_{t+u} - B_t$. From Lemma 20 we know that

$$\sqrt{2} \sup_{s \leq \eta} \|B_s^{k\eta}\| \leq \sqrt{2\eta} \left(\sqrt{d} + 2\sqrt{\log \frac{2(N+1)}{\delta}} \right),$$

with probability at least $1 - \frac{\delta}{N+1}$, for all $k \leq N$. We also note that the initial distribution $x_0 \sim \rho_0$ satisfies the following (see Lemma 19)

$$\mathbb{P} \left[\|x_0\| \leq \frac{2\sqrt{2}}{m\sqrt{\eta}} \left(1 + \sqrt{b} + \sqrt{d} + 2\sqrt{\log \frac{2(N+1)}{\delta}} \right) \right] \geq 1 - \frac{\delta}{N+1}.$$

First, we prove the discrete-time case. By plugging $y = 0$ into Lemma 10 and taking the square root, we get the following

$$\|x - \eta \nabla f(x)\| \leq \left(1 - \frac{m\eta}{2}\right) \|x\| + \sqrt{2b\eta} + \eta \|\nabla f(0)\|,$$

thus, we can write

$$\begin{aligned}\|x_{k+1}\| &\leq \|x_k - \eta \nabla f(x_k)\| + \sqrt{2} \left\| B_\eta^{k\eta} \right\| \\ &\leq \left(1 - \frac{m\eta}{2}\right) \|x_k\| + \sqrt{2\eta}(1 + \sqrt{b}) + \sqrt{2} \sup_{s \leq \eta} \left\| B_s^{k\eta} \right\|,\end{aligned}$$

where we used $\eta^2 \|\nabla f(0)\|^2 \leq 2\eta$. This, combined with the high probability bound on x_0 and supremum of Brownian motion with the aid of union bound implies the following happens for $k \leq N$, with probability at least $1 - \delta$.

$$\|x_k\| \leq \frac{2\sqrt{2}}{m\sqrt{\eta}} \left(1 + \sqrt{b} + \sqrt{d} + 2\sqrt{\log \frac{2(N+1)}{\delta}}\right).$$

Let $k = \lfloor t/\eta \rfloor$, we write

$$\begin{aligned}\|\tilde{x}_t - \tilde{x}_{\eta \lfloor t/\eta \rfloor}\| &\leq \eta \|\nabla f(x_k)\| + \sqrt{2} \sup_{t \leq \eta} \left\| B_t^{k\eta} \right\| \\ &\leq \eta L \|x_k\| + \eta \|\nabla f(0)\| + \sqrt{2} \sup_{t \leq \eta} \left\| B_t^{k\eta} \right\| \\ &\leq \eta L \|x_k\| + \sqrt{2\eta} + \sqrt{2} \sup_{t \leq \eta} \left\| B_t^{k\eta} \right\|,\end{aligned}$$

combining this with the high probability bound on the Brownian motion and $\|x_k\|$ (note that there is no need for union bound as this event is subset of the high probability event on the norm of the Brownian motion) concludes the proof for the discrete case.

We use a similar structure for proving the continuous-time case. Let z'_u denote the continuous time Langevin dynamics started at $z'_0 = 0$. We write

$$\|z'_u\| \leq \int_0^u \|\nabla f(z'_s)\| ds + \sqrt{2} \|B_u\| \leq L \int_0^u \|z'_s\| ds + \eta \|\nabla f(0)\| + \sup_{s \leq \eta} \sqrt{2} \|B_s\|,$$

for $u \leq \eta$. Using Grönwall inequality (Lemma 21), we get the following (for $u \leq \eta$)

$$\|z'_u\| \leq e^{Lu} \left(\eta \|\nabla f(0)\| + \sup_{s \leq \eta} \sqrt{2} \|B_s\| \right) \leq 2 \left(\eta \|\nabla f(0)\| + \sup_{s \leq \eta} \sqrt{2} \|B_s\| \right).$$

where the last inequality is because $u \leq \eta \leq \log 2/L$. Plug $z'_0 = 0$ in Lemma 10 and shift z_0 to z_t , using the semi-group property, to get the following

$$\begin{aligned}\|z_{t+u}\| &\leq e^{-mu} \|z_t\| + \|z'_u\| + \sqrt{\frac{b}{m}} (1 - e^{-2mu}) \\ &\leq (1 - mu/2) \|z_t\| + 2 \left(\sqrt{2\eta} + \sup_{s \leq \eta} \sqrt{2} \|B_s^t\| \right) + \sqrt{2b\eta},\end{aligned}\tag{B.1}$$

where the last inequality follows from $4mu \leq 4m\eta \leq 1$, and $\eta^2 \|\nabla f(0)\|^2 \leq 2\eta$. We plug $u = \eta$ in the previous inequality and use the high probability bound on the Brownian motion and $z_0 = x_0$ with union bound to get $\|z_{k\eta}\| \leq \frac{4\sqrt{2}}{m\sqrt{\eta}} (1 + \sqrt{b} + \sqrt{d} + 2\sqrt{\log \frac{2(N+1)}{\delta}})$ with probability at least $1 - \delta$ for all $k \leq N$.

Next, we modify (B.1) as follows

$$\|z_{t+u}\| \leq \|z_t\| + 2 \left(\sqrt{2\eta} + \sup_{s \leq \eta} \sqrt{2} \|B_s\| \right) + \sqrt{2b\eta}.$$

Using this inequality with previous high probability bound on $\|z_{k\eta}\|$ shows that the following happens with probability at least $1 - \delta$ for all $t \leq N\eta$.

$$\|z_t\| \leq \frac{5\sqrt{2}}{m\sqrt{\eta}} \left(1 + \sqrt{b} + \sqrt{d} + 2\sqrt{\log \frac{2(N+1)}{\delta}} \right),$$

where we used $\eta \leq 1/4m$. Finally we use the following inequality to connect this tail bound to probability of jump

$$\begin{aligned} \|z_t - z_{\lfloor t/\eta \rfloor \eta}\| &= \left\| \int_{\lfloor t/\eta \rfloor \eta}^t -\nabla f(z_s) ds + \sqrt{2} dB_s \right\| \\ &\leq \eta L \sup_{s \leq N\eta} \|z_s\| + \eta \|\nabla f(0)\| + \sqrt{2} \sup_{s \leq \eta} \|B_s^{\lfloor t/\eta \rfloor \eta}\|. \end{aligned}$$

Combining previous inequality with high probability bounds on $\|z_t\|$ and the Brownian motion and using $\eta^2 \|\nabla f(0)\|^2 \leq 2\eta$ concludes the proof for the continuous-time case (where again, there is no need for union bound as this event is subset of the high probability event on the norm of the Brownian motion).

Finally, we collect all the upper bounds on η in a more compact form

$$\eta \leq \frac{1 \wedge m}{4(1 \vee L^2)} \wedge \frac{2}{\|\nabla f(0)\|^2} \leq \frac{2}{\|\nabla f(0)\|^2} \wedge \frac{4}{m\sigma^2} \wedge \frac{m}{L^2} \wedge \frac{1}{4m} \wedge \frac{\log 2}{L}.$$

□

Now we prove the continuous-time and discrete-time clipped processes stay sufficiently close to each other. In order to prove that we first state a lemma that describes the behavior of discrete chain as step size approaches zero. We will show that the Rényi divergence of the discrete-time process and the continuous-time process converges to zero as the step size approaches zero. The proof relies on the Girsanov theorem and Lemma 11.

Lemma 12. *Let Γ_T^η, Π_T be the distribution of the paths on $[0, T]$ of the interpolated time process with step-size η , and continuous time process respectively. Starting from $x_0 = z_0 = \mathcal{N}(0, \sigma^2 \mathbf{I})$ for $\sigma^2 \leq (L+1)^{-1}$, and under Assumption 1, for any $T > 0$, $\alpha \geq 1$*

$$\lim_{\eta \rightarrow 0} R_\alpha(\Gamma_T^\eta | \Pi_T) = 0.$$

Proof. Denote our underlying probability space by (Ω, \mathcal{F}, P) ; let $\mathcal{Z} = \{f | f : [0, T] \rightarrow \mathbb{R}^d\}$ be the space of possible paths. We use the notations $\tilde{x}(\omega)$ and $z(\omega)$ to denote one realization of discrete and continuous time process. Using Girsanov's Theorem (see Lemma 24), we define the measure Q

$$\begin{aligned} \frac{dQ}{dP}(\omega) &= N(\tilde{x}(\cdot)) \\ &\triangleq \exp \left(-\frac{1}{\sqrt{2}} \int_0^T (\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta}))^\top dB_s - \frac{1}{4} \int_0^T \|\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta})\|^2 ds \right), \end{aligned}$$

such that the P -law of z equals the Q -law of \tilde{x} . Note that Novikov condition can be checked by using Lemmas 11 and 18 and smoothness of potential. Thus, we can write the following

$$\begin{aligned}\Pi_T(A) &= P(\{\omega|z(\omega) \in A\}) \\ &= Q(\{\omega|\tilde{x}(\omega) \in A\}) \\ &= \int \mathbb{1}_{\{\tilde{x}(\omega) \in A\}} N(\tilde{x}(\omega)) dP(\omega) \\ &= \int_{\tilde{x} \in A} N(\tilde{x}) d\Gamma_T^\eta(\tilde{x}),\end{aligned}$$

for any measurable $A \in \mathcal{Z}$. This implies the following about the Radon–Nikodym derivative

$$\frac{d\Pi_T}{d\Gamma_T^\eta}(\tilde{x}) = N(\tilde{x}).$$

Raising to power $-(\alpha - 1)$ and using the definition of N and taking expectation we get

$$\begin{aligned}\mathbb{E}_{\Pi_T} \left[\left(\frac{d\Gamma_T^\eta}{d\Pi_T} \right)^\alpha \right] &= \mathbb{E}_{\Gamma_T^\eta} \left[\left(\frac{d\Gamma_T^\eta}{d\Pi_T} \right)^{\alpha-1} \right] \\ &= \mathbb{E} \left[\exp \left(\frac{\alpha-1}{\sqrt{2}} \int_0^T (\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta}))^\top dB_s + \frac{\alpha-1}{4} \int_0^T \|\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta})\|^2 ds \right) \right].\end{aligned}$$

If we define $M_s = \sqrt{2}(\alpha - 1) (\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta}))$, this expectation is equal to the following

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T M_s^\top dB_s - \frac{1}{4} \int_0^T \|M_s\|^2 ds + \int_0^T \left(\frac{1}{4} + \frac{1}{8(\alpha-1)} \right) \|M_s\|^2 ds \right) \right].$$

Subsequently, the following holds by Cauchy-Schwartz

$$\begin{aligned}\mathbb{E}_{\Pi_T} \left[\left(\frac{\Gamma_T^\eta}{\Pi_T} \right)^\alpha \right] &\leq \mathbb{E} \left[\exp \left(\int_0^T M_s^\top dB_s - \frac{1}{2} \int_0^T \|M_s\|^2 ds \right) \right]^{1/2} \\ &\quad \times \mathbb{E} \left[\exp \left\{ \left((\alpha-1)^2 + \frac{\alpha-1}{2} \right) \int_0^T \|\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta})\|^2 ds \right\} \right]^{1/2} \\ &\leq \mathbb{E} \left[\exp \left(\alpha^2 \int_0^T \|\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta})\|^2 ds \right) \right]^{1/2},\end{aligned}$$

where the last inequality step follows from Lemma 23 and again Novikov's condition holds as argued before. We use Assumption 1, and the first event in Lemma 11 to get

$$\begin{aligned}\int_0^T \|\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta})\|^2 ds &\leq L^2 \int_0^T \|\tilde{x}_s - \tilde{x}_{\lfloor s/\eta \rfloor \eta}\|^2 ds \\ &\leq KTL^2 \kappa^2 \left(b + d + \log \frac{T}{\eta \delta} \right) \eta,\end{aligned}$$

with probability at least $1 - \delta$, if η , is sufficiently small for some universal constant K . Letting the event in which this bound holds be called \mathcal{E}_δ , by calculating the conditional expectation we get

$$\begin{aligned} & \mathbb{E} \left[\exp \left(c\alpha^2 \int_0^T \|\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta})\|^2 ds \right) | \mathcal{E}_\delta \right] \\ & \leq \exp \left(Kc\alpha^2 T L^2 \kappa^2 \left(b + d + \log \frac{T}{\eta\delta} \right) \eta \right) \\ & \leq \delta^{-\gamma_c(\eta)} \exp \left(Kc\alpha^2 T L^2 \kappa^2 (b + d + \log T - \log \eta) \eta \right), \end{aligned}$$

where $\gamma_c(\eta) = Kc\alpha^2 T L^2 \kappa^2 \eta$, and we absorbed constants into K . For any fixed c , we know $\lim_{\eta \rightarrow 0} \gamma_c(\eta) = 0$, therefore for small enough η , we have $\gamma_c(\eta) < 1$, thus we can apply Lemma 18 with $\theta = c$ and $\gamma = \gamma_c(\eta)$ to get

$$\mathbb{E} \left[\exp \left(\alpha^2 \int_0^T \|\nabla f(\tilde{x}_s) - \nabla f(\tilde{x}_{\lfloor s/\eta \rfloor \eta})\|^2 ds \right) \right] \leq \frac{2^{\frac{2}{c}} c}{c-1} \exp \left(K\alpha^2 T L^2 \kappa^2 (b + d + \log T - \log \eta) \eta \right).$$

We substitute this into our earlier bound,

$$\mathbb{E}_{\Pi_T} \left[\left(\frac{\Gamma_T^\eta}{\Pi_T} \right)^\alpha \right] \leq \frac{2^{\frac{2}{c}} c}{c-1} \exp \left(K\alpha^2 T L^2 \kappa^2 (b + d + \log T - \log \eta) \eta \right).$$

We take the limit as $\eta \rightarrow 0$ to get

$$\lim_{\eta \rightarrow 0} \mathbb{E}_{\Pi_T} \left[\left(\frac{\Gamma_T^\eta}{\Pi_T} \right)^\alpha \right] \leq \frac{2^{\frac{2}{c}} c}{c-1}.$$

Finally, we take another limit as $c \rightarrow \infty$ to get

$$\lim_{\eta \rightarrow 0} \mathbb{E}_{\Pi_T} \left[\left(\frac{\Gamma_T^\eta}{\Pi_T} \right)^\alpha \right] \leq 1.$$

Substituting this into the definition of the Rényi divergence and using continuity of log at 1 concludes the proof. \square

Now using the previous lemma, we extend [GT20, Corollary 11] to the interpolation process under strong dissipativity.

Lemma 13. *If $P_0 = Q_0 = \mathcal{N}(0, \sigma^2 \mathbf{I})$ for $\sigma^2 < (L+1)^{-1}$, then the following holds for $\alpha \geq 2$, when Assumption 1 is satisfied.*

$$\mathbb{E}_{Q_T} \left[\frac{P_T}{Q_T}(x)^\alpha \right] \leq \exp \left(T\alpha(\alpha-1)L^2 r^2 \right) \tag{B.2}$$

Proof. We use the same argument as in [GT20, Lemma 10]. By sampling both X_t and X'_t at multiples of η/j and at the final moment (T), we get the following tuples.

$$X_{0-T} = \{X_{i\eta/j}\}_{0 \leq i < jT/\eta} + \{X_T\}, \quad X_{0-T}^j = \{X_{i\eta/j}^j\}_{0 \leq i < jT/\eta} + \{X_T^j\},$$

where by $+$ we mean appending the element to the end of the tuple. In order to use Lemma 17, we consider functions ϕ_1 and ϕ_2 that append one new sample to the tuples of sampled clipped processes. For example ϕ_1 gets $\{X_{i\eta/j}\}_{0 \leq i < k}$ and applies Langevin update rule along with the clipping criteria

for step size η/j using the gradient at the last multiple of η to produce $\{X_{i\eta/j}\}_{0 \leq i \leq k}$. ϕ_2 is defined similarly but uses gradient at last multiple of η/j . Note that we get X_{0-T} and X_{0-T}^j by multiple applications of ϕ_1 and ϕ_2 , except for the final iterate. Assume \tilde{X} is a deterministic tuple (i.e. point mass) we bound $R_\alpha(\phi_1(\tilde{X})|\phi_2(\tilde{X}))$. If \tilde{X} contains \perp then this is zero, therefore we assume \tilde{X} does not contain jumps larger than r and by data processing inequality (Lemma 16) we can ignore clipping done by ϕ_1 and ϕ_2 . Since \tilde{X} was a point mass, both $\phi_1(\tilde{X})$ and $\phi_2(\tilde{X})$ are Gaussians with possibly different means, which cannot differ more than $Lr\eta/j$, because of smoothness of potential and assumption that jumps are smaller than r . Thus, Lemma 15 implies $R_\alpha(\phi_1(\tilde{X})|\phi_2(\tilde{X})) \leq \alpha L^2 r^2 \eta / 4j$. Let $T = k\eta/j + t'$ such that $t' < \eta/j$. We can apply Lemma 17 for k times to get that the Rényi divergence between $\{X_{i\eta/j}\}_{0 \leq i < jT/\eta}$ and $\{X_{i\eta/j}^j\}_{0 \leq i < jT/\eta}$ is bounded by $\frac{\alpha L^2 r^2 (T-t')}{4}$. Now modifying ϕ_1 and ϕ_2 to use time t' instead of η/j , by the same argument as before and using Lemma 17 once more, we can conclude that

$$R_\alpha(X_{0-T}|X_{0-T}^j) \leq \frac{\alpha L^2 r^2 T}{4}.$$

To go back to P_T and Q_T we write

$$R_\alpha(P_T|Q_T) \leq \frac{\alpha - 0.5}{\alpha - 1} R_{2\alpha}(P_T|P_T^j) + R_{2\alpha-1}(P_T^j|Q_T),$$

where we used Cauchy-Schwartz inequality. Taking the limit as $j \rightarrow \infty$ and using $\alpha \geq 2$, we get

$$R_\alpha(P_T|Q_T) \leq 2 \lim_{j \rightarrow \infty} R_{2\alpha}(P_T|P_T^j) + \lim_{j \rightarrow \infty} R_{2\alpha-1}(P_T^j|Q_T).$$

The second term in RHS converges to 0, since by data processing inequality (see Lemma 16) we have

$$\lim_{j \rightarrow \infty} R_{2\alpha-1}(P_T^j|Q_T) \leq \lim_{j \rightarrow \infty} R_{2\alpha-1}(\Gamma_T^{\eta/j}|\Pi_T) = 0,$$

where the last step is due to Lemma 12. Therefore, we get the following

$$R_\alpha(P_T|Q_T) \leq 2 \lim_{j \rightarrow \infty} R_{2\alpha}(P_T|P_T^j) \leq 2 \lim_{j \rightarrow \infty} R_{2\alpha}(X_{0-T}|X_{0-T}^j) \leq T\alpha L^2 r^2,$$

where the second inequality follows from data processing inequality (Lemma 16). This in turn implies (B.2). \square

Finally, we combine the previous results to go back to the unclipped process.

Lemma 14. *Suppose Assumption 1 holds and $x_0 = z_0 = \mathcal{N}(0, \sigma^2 \mathbf{I})$ for $\sigma^2 < (L+1)^{-1}$. If $\eta \leq \frac{1 \wedge m}{4(1 \vee L^2)} \wedge \frac{2}{\|\nabla f(0)\|^2}$, and for some universal constant c we have $N\eta^2 \leq \frac{1}{c\kappa^2 L^2 \alpha^2}$, then for any $T \leq N\eta$ and $\alpha \geq 2$, we have the following*

$$\mathbb{E}_{\pi_T} \left[\frac{\tilde{\rho}_T}{\pi_T} (x)^{\frac{\alpha}{4} + \frac{1}{2}} \right] \leq \frac{5\alpha + 10}{\sqrt{\alpha}} \times \exp(cT\kappa^2 L^2 \alpha^2 (b + d + \log(N))\eta).$$

Proof. Considering the events $\mathcal{E}_{\delta_1}^1, \mathcal{E}_{\delta_2}^2$, we plug the following value in (B.2)

$$r = c\kappa(\sqrt{b} + \sqrt{d} + \sqrt{\log(N/\delta_1)} + \sqrt{\log(N/\delta_2)})\sqrt{\eta},$$

where c is a universal constant such that the remark after Lemma 11 holds. This implies $r^2 \leq c\kappa^2 (b + d + \log(N/\delta_1) + \log(N/\delta_2))\eta$, where we absorbed universal constants into c . We write

$$\begin{aligned}\mathbb{E}_{\pi_T} \left[\frac{P_T}{\pi_T}(x)^\alpha \middle| \mathcal{E}_{\delta_2}^2 \right] &\leq \frac{1}{1 - \delta_2} \mathbb{E}_{Q_T} \left[\frac{P_T}{Q_T}(x)^\alpha \right] \\ &\leq 2 \exp(T\alpha(\alpha - 1)L^2r^2) \\ &\leq \frac{2 \exp(cT\kappa^2L^2\alpha^2(b + d + \log(N))\eta)}{(\delta_1\delta_2)^{cT\kappa^2L^2\alpha^2\eta}},\end{aligned}$$

where the first step follows from $\pi_T(x) \geq Q_T(x)$ (for $x \in \mathbb{R}^d$), and the second from Lemma 13. In order to utilize Lemma 18 we set $\gamma = cT\kappa^2L^2\alpha^2\eta$ and we need $\gamma < 1$ which combined with $T \leq N\eta$ shows that it is sufficient if we have

$$N\eta^2 \leq \frac{1}{c\kappa^2L^2\alpha^2}.$$

Lemma 18 implies

$$\mathbb{E}_{\pi_T} \left[\frac{P_T}{\pi_T}(x)^{\frac{\alpha}{2}} \right] \leq 4\sqrt{2} \frac{\exp(cT\kappa^2L^2\alpha^2(b + d + \log(N))\eta)}{\delta_1^{cT\kappa^2L^2\alpha^2\eta}},$$

where universal constants are again absorbed into c . For replacing P_T with $\tilde{\rho}_T$ we write

$$\begin{aligned}\mathbb{E}_{\pi_T} \left[\frac{P_T}{\pi_T}(x)^{\frac{\alpha}{2}} \right] &= \int_{\mathbb{R}^d} \frac{P_T(x)^{\alpha/2}}{\pi_T(x)^{\alpha/2-1}} dx \\ &= \frac{\alpha}{2} \int_{\mathbb{R}^d} \int_0^{P_T(x)} \frac{y^{\alpha/2-1}}{\pi_T(x)^{\alpha/2-1}} dy dx \\ &= \frac{\alpha}{2} \int_{\mathbb{R}^d} \left(\int_0^{\tilde{\rho}_T(x)} \frac{y^{\alpha/2-1} \mathbb{1}_{\{y \leq P_T(x)\}}}{\pi_T(x)^{\alpha/2-1}} \times \frac{1}{\tilde{\rho}_T(x)} dy \right) \tilde{\rho}_T(x) dx \\ &= \frac{\alpha}{2} \mathbb{E}_{x \sim \tilde{\rho}_T, y \sim U(0, \tilde{\rho}_T(x))} \left[\frac{y^{\alpha/2-1}}{\pi_T(x)^{\alpha/2-1}} \middle| y \leq P_T(x) \right] \\ &\quad \times \mathbb{P}_{x \sim \tilde{\rho}_T, y \sim U(0, \tilde{\rho}_T(x))} [y \leq P_T(x)].\end{aligned}$$

We consider RHS term by term. For the first term we write

$$\mathbb{E}_{x \sim \tilde{\rho}_T, y \sim U(0, \tilde{\rho}_T(x))} \left[\frac{y^{\alpha/2-1}}{\pi_T(x)^{\alpha/2-1}} \middle| y \leq P_T(x) \right] = \mathbb{E}_{x \sim \tilde{\rho}_T, y \sim U(0, \tilde{\rho}_T(x))} \left[\frac{y^{\alpha/2-1}}{\pi_T(x)^{\alpha/2-1}} \middle| \mathcal{E}_{\delta_1} \right]$$

with the right coupling between y and the path x . For the second term we have

$$\mathbb{P}_{x \sim \tilde{\rho}_T, y \sim U(0, \tilde{\rho}_T(x))} [y \leq P_T(x)] = \int_{\mathbb{R}^d} \int_0^{P_T(x)} \frac{dy}{\tilde{\rho}_T(x)} \tilde{\rho}_T(x) dx = \int_{\mathbb{R}^d} P_T(x) dx \geq 1 - \delta_1 \geq \frac{1}{2}.$$

Putting these together we get

$$\begin{aligned}\mathbb{E}_{x \sim \tilde{\rho}_T, y \sim U(0, \tilde{\rho}_T(x))} \left[\frac{y^{\alpha/2-1}}{\pi_T(x)^{\alpha/2-1}} \middle| \mathcal{E}_{\delta_1} \right] &\leq \frac{4}{\alpha} \mathbb{E}_{\pi_T} \left[\frac{P_T}{\pi_T}(x)^{\alpha/2} \right] \\ &\leq \frac{16\sqrt{2}}{\alpha} \frac{\exp(cT\kappa^2L^2\alpha^2(b + d + \log(N))\eta)}{\delta_1^{cT\kappa^2L^2\alpha^2\eta}}.\end{aligned}$$

Using Lemma 18 another time, we need to set $\gamma = cT\kappa^2L^2\alpha^2\eta$. The condition $\gamma < 1$ is already satisfied as we used this lemma before (with a different c). Therefore, we get the following (universal constants are absorbed into c again.)

$$\mathbb{E}_{x \sim \tilde{\rho}_T, y \sim U(0, \tilde{\rho}_T(x))} \left[\frac{y^{\frac{\alpha}{4} - \frac{1}{2}}}{\pi_T(x)^{\frac{\alpha}{4} - \frac{1}{2}}} \right] \leq \frac{2^{17/4}}{\sqrt{\alpha}} \exp(cT\kappa^2L^2\alpha^2(b + d + \log(N))\eta).$$

Finally, we write

$$\begin{aligned} \mathbb{E}_{\pi_T} \left[\frac{\tilde{\rho}_T}{\pi_T}(x)^{\frac{\alpha}{4} + \frac{1}{2}} \right] &= \left(\frac{\alpha}{4} + \frac{1}{2} \right) \mathbb{E}_{x \sim \tilde{\rho}_T, y \sim U(0, \tilde{\rho}_T(x))} \left[\frac{y^{\frac{\alpha}{4} - \frac{1}{2}}}{\pi_T(x)^{\frac{\alpha}{4} - \frac{1}{2}}} \right] \\ &\leq \left(\frac{\alpha}{4} + \frac{1}{2} \right) \frac{2^{17/4}}{\sqrt{\alpha}} \exp(cT\kappa^2L^2\alpha^2(b + d + \log(N))\eta). \end{aligned}$$

□

C Logarithmic Sobolev Inequality under Assumption 1

For some $m, b > 0$, assume that the following holds

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq m\|x - y\|^2 - b \quad \text{for all } x, y \in \mathbb{R}^d.$$

Define the Lyapunov function $W(x) = \exp\{\frac{\gamma}{2}\|x - x_*\|^2\}$ for some γ and a critical point x_* of f . We have

$$\nabla W(x) = \gamma(x - x_*)W(x) \quad \text{and} \quad \Delta W(x) = \gamma(d + \gamma\|x - x_*\|^2)W(x)$$

and consequently

$$\begin{aligned} \frac{LW(x)}{W(x)} &= \gamma(d + \gamma\|x - x_*\|^2 - \langle x - x_*, f(x) \rangle) \\ &\leq \gamma(d + b + (\gamma - m)\|x - x_*\|^2). \end{aligned}$$

Next, choosing $\gamma = m/2$ and defining $R^2 = \frac{2}{m}(d + b + 1)$, one can show that the right hand side above is upper bounded by

$$-\frac{m}{2} + \frac{m}{2}(d + b)\mathbb{1}_{\{\|x - x_*\| \leq R\}}.$$

Thus, the target $\nu_* = e^{-f}$ satisfies the Lyapunov condition given in [BBCG08], and consequently satisfies a Poincaré inequality with a constant upper bounded with

$$\lambda \leq \frac{2}{m} \left(1 + c \frac{m}{2} (d + b) e^{\text{Osc}_R(f)} \right)$$

where $\text{Osc}_R(f) = \sup_{\|x - x_*\| \leq R} f(x) - \inf_{\|x - x_*\| \leq R} f(x)$ and c is an absolute constant. This bound is of order $\mathcal{O}(de^{\text{Osc}_R(f)})$. Further using the results of [CGW10], one can show that LSI holds for this class of potentials with a constant bounded by $\mathcal{O}(d^2e^{\text{Osc}_R(f)})$.

D Useful Lemmas

Lemma 15. (From [VEH14]) The Rényi divergence of two Gaussians can be calculated as follows

$$R_\alpha(\mathcal{N}(0, \sigma^2 \mathbf{I}) | \mathcal{N}(x, \sigma^2 \mathbf{I})) = \frac{\alpha \|x\|^2}{2\sigma^2}.$$

Lemma 16. (Data processing inequality, From [VEH14]) Suppose $x_1 \sim \rho_1$ and $x_2 \sim \rho_2$. For any function f , let $f(x_1) \sim \pi_1$ and $f(x_2) \sim \pi_2$, then $R_\alpha(\pi_1 | \pi_2) \leq R_\alpha(\rho_2 | \rho_2)$.

Lemma 17. (From [Mir17]) Let $\Delta(S_1), \Delta(S_2)$ be the space of probability measures on S_1, S_2 let $\phi_1, \phi_1' : \Delta(S_1) \rightarrow \Delta(S_2)$ and $\phi_2, \phi_2' : \Delta(S_2) \rightarrow \mathfrak{P}$ be maps such that for any distributions δ that is a point mass (on either $\Delta(S_1)$ or $\Delta(S_2)$) we have $R_\alpha(\phi_i(\delta) | \phi_i'(\delta)) \leq \epsilon_i$. Then, for any probability measure $\rho \in \Delta(S_1)$ we have $R_\alpha(\phi_2(\phi_1(\rho)) | \phi_2'(\phi_1'(\rho))) \leq \epsilon_1 + \epsilon_2$.

Lemma 18. (Adapted from [GT20, Lemma 14]) Let $Y > 0$ (a.s.), $\gamma < 1$ and $\theta > 1 + \gamma$. If for all $0 < \delta < 1/2$ an event \mathcal{E}_δ has probability at least $1 - \delta$, and $\mathbb{E}[Y^\theta | \mathcal{E}_\delta] \leq \frac{\beta}{\delta^\gamma}$, then $\mathbb{E}[Y] \leq 2^{2/\theta} \beta^{1/\theta} \frac{\theta}{\theta-1}$. In particular, if $\theta = 2$, we get: $\mathbb{E}[Y] \leq 4\sqrt{\beta}$.

Lemma 19. For $W \sim \mathcal{N}(0, \mathbf{I})$ we have the following tail bound for $x \geq 0$

$$\mathbb{P}\left[\|W\| \geq \sqrt{d} + x\right] \leq \exp(-x^2/2),$$

Proof. Suppose $W = (w_1, \dots, w_d)$ and denote $\sqrt{d} + x$ with a . We write

$$\mathbb{P}[\|W\| \geq a] = \mathbb{P}\left[\exp(t\|W\|^2) \geq \exp(ta^2)\right] \leq \frac{\mathbb{E}[\exp(tw_1^2)]^d}{\exp(ta^2)} = \exp\left(-ta^2 - \frac{d}{2} \ln(1 - 2t)\right),$$

for all $t < 1/2$, therefore we can plug $t = \frac{1}{2} \left(1 - \frac{d}{a^2}\right)$, and put $a = \sqrt{d} + x$ back to get

$$\mathbb{P}\left[\|W\| \geq \sqrt{d} + x\right] \leq \exp(-x^2/2) \times \exp\left(-d \left(\frac{x}{\sqrt{d}} - \ln\left(1 + \frac{x}{\sqrt{d}}\right)\right)\right) \leq \exp(-x^2/2),$$

where the last inequality holds since $\frac{x}{\sqrt{d}} - \ln\left(1 + \frac{x}{\sqrt{d}}\right) \geq 0$. \square

Lemma 20. For d -dimensional Brownian motion B_t we have ($x \geq 0$)

$$\mathbb{P}\left[\sup_{s \leq t} \|B_s\| \geq \sqrt{t}(\sqrt{d} + x)\right] \leq 2 \exp(-x^2/4).$$

Proof. Let r denote $\sqrt{t}(\sqrt{d} + x)$ and τ denote the first exit time of B_t out of the ball of radius r around origin. Note that $\tau < t$ coincides with $\sup_{s \leq t} \|B_s\| > r$, furthermore $\|B_\tau\| = r$. We write

$$\begin{aligned} \mathbb{P}\left(\sup_{s \leq t} \|B_s\| \geq r\right) &\leq \mathbb{P}(\|B_t\| \geq r) + \mathbb{P}(\tau < t, \|B_t\| < r) \\ &= \mathbb{P}(\|B_t\| \geq r) + \mathbb{E}\left[\mathbb{1}_{\{\tau < t\}} \mathbb{P}(\|B_t - B_\tau + B_\tau\| < r | \tau, B_\tau)\right] \\ &\leq \mathbb{P}(\|B_t\| \geq r) + \mathbb{E}\left[\mathbb{1}_{\{\tau < t\}} \mathbb{P}(\langle B_t - B_\tau, B_\tau \rangle < 0 | \tau, B_\tau)\right] \\ &= \mathbb{P}(\|B_t\| \geq r) + \mathbb{P}\left(\sup_{s \leq t} \|B_s\| \geq r\right) / 2, \end{aligned}$$

where the last step follows from independence of updates and normality. Rearranging and using Lemma 19 concludes the proof. \square

Lemma 21. (Grönwall inequality [Bel43]) For a function v satisfying $v(t) \leq C + A \int_0^t v(s) ds$, for $0 \leq t \leq T$ with $A > 0$. The following holds: $v(t) \leq Ce^{At}$.

Lemma 22. For a real sequence $\{\theta_k\}_{k \geq 0}$, if we have $\theta_k \leq (1 - a)\theta_{k-1} + h$ for some $a \in (0, 1)$, and $h \geq 0$, then $\theta_k \leq e^{-ak}\theta_0 + h/a$.

Proof. Recursion on $\theta_k \leq (1 - a)\theta_{k-1} + h$ yields

$$\theta_k \leq (1 - a)^k \theta_0 + h(1 + (1 - a) + (1 - a)^2 + \dots + (1 - a)^{k-1}) \leq (1 - a)^k \theta_0 + \frac{h}{a}.$$

Using the fact that $1 - a \leq e^{-a}$ completes the proof. \square

Lemma 23. (Exponential Martingale Theorem [IW14, Chapter III, Theorem 5.3]) Let B_t be a Brownian motion and \mathcal{F}_t its associated filtration. If for an \mathcal{F}_t -adapted stochastic process M_t and some $T \geq 0$, the following (Novikov's) condition holds

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T \|M_s\|^2 ds \right) \right] < \infty,$$

then $\exp \left(\int_0^t M_s^\top dB_s - \frac{1}{2} \int_0^t \|M_s\|^2 ds \right)$ is an exponential Martingale and in particular its expectation is equal to 1 for all $t \leq T$.

Lemma 24. (Girsanov Theorem, Adapted from [Oks13, Theorem 8.6.8]) Let $x_t, y_t \in \mathbb{R}^d$ be defined as follows

$$\begin{aligned} dx_t(\omega) &= b(x_t(\omega))dt + \sqrt{2}dB_t(\omega), \\ dy_t(\omega) &= \gamma(\omega, t)dt + \sqrt{2}dB_t(\omega), \end{aligned}$$

such that $y_0 = x_0$ and ω is an element of underlying probability space Ω . Let $\{\mathcal{F}_t\}$ be the natural filtration for B_t and P be the measure such that B_t is Brownian with respect to P and let

$$M_t(\omega) \triangleq \exp \left(-\frac{1}{\sqrt{2}} \int_0^t (\gamma(\omega, s) - b(y_s(\omega)))^\top dB_s(\omega) - \frac{1}{4} \int_0^t \|\gamma(\omega, s) - b(y_s(\omega))\|^2 ds \right).$$

If M_t is a martingale with respect to \mathcal{F}_t , in particular if $\gamma(\omega, s) - b(y_s(\omega))$, satisfies Novikov's condition, then on \mathcal{F}_T we have a unique measure Q such that

$$\frac{dQ}{dP}(\omega) = M_T(\omega),$$

with the property that the Q -law of y is equal to the P -law of x , where $x(\omega)$ and $y(\omega)$ are one realization of x_t and y_t on $[0, T]$.

Finally, we state two helper lemmas. The first lemma shows the convergence of continuous time process when the target satisfies LSI.

Lemma 25 (Adapted from Theorem 3 in [VW19]). If $f = -\log \nu_*$ satisfies (LSI), then the following holds

$$\log \mathbb{E}_{\nu_*} \left[\frac{\pi_T}{\nu_*} (x)^\alpha \right] \leq e^{-\frac{2T}{\alpha\lambda}} \log \mathbb{E}_{\nu_*} \left[\frac{\pi_0}{\nu_*} (x)^\alpha \right].$$

We briefly remark that the analog of the above lemma in Chi-squared divergence requires only the (PI).

In the second helper lemma, we prove that initializing with a normal distribution with a sufficiently small variance will cause $\mathbb{E}_{\nu_*} \left[\frac{\rho_0}{\nu_*}(x)^\alpha \right]$ to be of order $\tilde{O}(c^{\alpha d})$ for some constant c .

Lemma 26. *Suppose f is L -smooth and $\alpha \geq 2$, then the following holds for $\nu_* = e^{-f}$ and $\rho_0 = \mathcal{N}(0, \sigma^2 \mathbf{I})$ when $\sigma^2 < (L+1)^{-1}$.*

$$\mathbb{E}_{\nu_*} \left[\frac{\rho_0}{\nu_*}(x)^\alpha \right] \leq \frac{\exp\left((\alpha-1)\left(f(0) + \frac{\|\nabla f(0)\|^2}{2}\right)\right)}{(2\pi\sigma^2)^{\frac{\alpha d}{2}}} \left(\frac{2\pi}{\frac{\alpha}{\sigma^2} - (\alpha-1)(L+1)}\right)^{\frac{d}{2}}.$$

Remark. For the sake of simplicity, we use the following crude bound

$$\mathbb{E}_{\nu_*} \left[\frac{\rho_0}{\nu_*}(x)^\alpha \right] \leq e^{\alpha d C_\sigma} \quad \text{with} \quad C_\sigma = 1 + \frac{f(0) + \|\nabla f(0)\|^2}{d} - \log(\sigma^2[(1+L) \wedge 2\pi])$$

where C_σ is a dimension free constant that does not depend on α .

Proof. For any x we have

$$f(x) \leq f(0) + \frac{\|\nabla f(0)\|^2}{2} + \left(\frac{L+1}{2}\right) \|x\|^2.$$

Thus, we can write

$$\begin{aligned} \mathbb{E}_{\nu_*} \left[\frac{\rho_0}{\nu_*}(x)^\alpha \right] &\leq \frac{1}{(2\pi\sigma^2)^{\frac{\alpha d}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{\alpha \|x\|^2}{2\sigma^2} + (\alpha-1)f(x)\right) dx \\ &\leq \frac{\exp\left((\alpha-1)\left(f(0) + \frac{\|\nabla f(0)\|^2}{2}\right)\right)}{(2\pi\sigma^2)^{\frac{\alpha d}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2} \left(\frac{\alpha}{\sigma^2} - (\alpha-1)(L+1)\right) \|x\|^2\right) dx \\ &\leq \frac{\exp\left((\alpha-1)\left(f(0) + \frac{\|\nabla f(0)\|^2}{2}\right)\right)}{(2\pi\sigma^2)^{\frac{\alpha d}{2}}} \left(\frac{2\pi}{\frac{\alpha}{\sigma^2} - (\alpha-1)(L+1)}\right)^{\frac{d}{2}}. \end{aligned}$$

□