Dimensional Analysis

September-13-10
11:54 PM

Physics is the study of nature at it's most fundamental level.

Dimensional Analysis

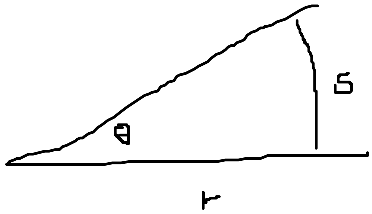
Physics concerns quantities with dimensions

Most fundamental dimensions:

Length:	L	Metre	Centimetre	Foot
Time	T	Second	Second	Second
Mass	M	Kilogram	Gram	Pound
		SI Units	*CGS Units*	*Imperial Units*

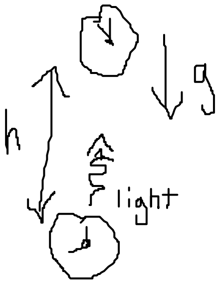
Square brackets [] around a variable indicate dimensionality:

[d] = L
[v] = L / T
[F] = [ma] = [m][a] = $\frac{ML}{T^2}$
[θ] = $\frac{L}{L} = 1$: Angle is a dimensionless quantity
Angle in radians is the ratio to arc length to radius.



$\theta = \frac{s}{r}$

How much does gravity slow down a clock?



$\frac{T}{t} = 1 + \varphi$
$|\varphi| = \frac{T}{T} = 1 \Rightarrow |h^x g^y c^z| = 1$
$L^x \left(\frac{L}{T^2}\right)^y \left(\frac{L}{T}\right)^z = 1$
$x + y + z = 0$
$2y + z = 0$
y = 1
z = -2
x = 1

$\frac{T}{t} = 1 + \varphi = 1 + (\frac{gh}{c^2})$

In any physics equation, all terms must have the same dimension.
Ex.
$x = vt + \frac{1}{2}at^2$
Quantities that are added/subtracted must have the same dimension
[x] = L
[v][t] = $\frac{L}{T}T = L$
$\left|\frac{1}{2}at^2\right| = \frac{L}{T^2}T = L$

Dimensional Analysis allows you to quickly check the validity of any equation.
$x \neq v^2 t + \frac{1}{2}at^2$

Can derive new equations:
Ex. What is the drag force on an object moving in a fluid?
Intuitively:
More area ⇒ more drag
More velocity ⇒ more drag
Denser fluid ⇒ more drag
$F \propto p^a v^b A^c$
$|F| = |p|^a |v|^b |A|^c$
$kg \times \frac{m}{s^2} = \left(\frac{kg}{m^2}\right)^a \times \left(\frac{m}{s}\right)^b \times m^c$
a = 1
b = 2
c = 1
$F \propto \rho v^2 A$
Quickly determined a general form of the equation. Proportionality constant requires more calculation.

Significant Figures

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Any measurement of a physical quantity is of limited accuracy.
How do we combine physics quantities to ensure we don't over/under estimate our answers.

Eg. Length of a pencil:
$L = 8.12 \pm 0.02$
8.1 is reliably known, 2 is estimated
Know the value is between 8.10 and 8.14

3 significant figures but only 2 reliably known digits.
Round off to 8.1 cm (2 significant figures). Reliable

Addition

Now a 2nd pencil of length 6.235cm is added
$L = 8.1 + 6.235 = 8.1 + 6.2 = 14.3\ cm$
Retain only the same number of digits to the right of the decimal point as the least accurate term.

Note: $8.1 \neq 8.100$

$3.249m - 3.241m = 0.008m = 8mm \neq 8.000mm$

Multiplication

$3 \times 4.1 \neq 12.3$
3 is really 3.0 ± 0.5
Any number that would round to 3

$2.5 \times 4.05 = 10.125$
$3.5 \times 4.15 = 14.525$
Average and round to the nearest one's place: 12.325
$3 \times 4.1 = 12$

Use the same number of significant figures as the least accurate quantity in the calculation.

<u>Error when Multiplying:</u>
To multiply x * y = z

$$\frac{\Delta z}{z} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2} \ or\ \Delta z = xy\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

<u>Alternatively</u>
$(3 \pm 0.5) \times (4.1 \pm 0.05)$
$= \left(3 \pm \frac{100}{6}\%\right) \times \left(4.1 \pm \frac{50}{41}\%\right) = \left(12.3 \pm \frac{2200}{123}\%\right)$
$= 12.3 \pm 2.2$

$(3 \pm 0.5) + (4.1 \pm 0.05) = 7.1 \pm 0.55$

Estimating

Process of guessing an answer to order-of-magnitude accuracy. How many cells are in a human brain?

How much does a human brain weigh?

The IDEAL Strategy

Interpret
- What is the question asking you?
- What are the applicable concepts?

Develop
- Draw a diagram
- Determine the relevant mathematical formulae

Evaluate
- Calculate the result using your math skills

Access
- Does the answer make sense?
- Significant figures? Dimensions? Overall magnitude?

Motion in 1 Dimension

September-20-10
9:35 AM

Kinematics

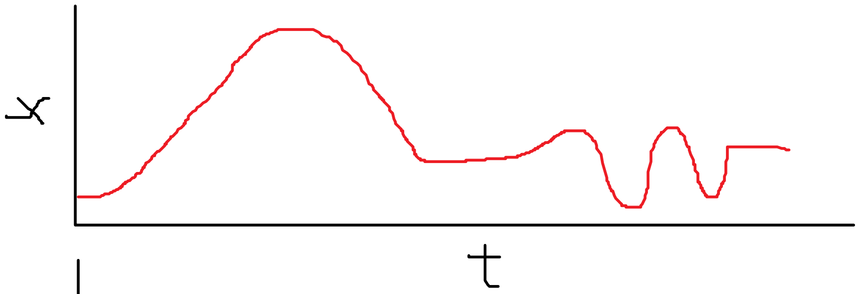
The study of motion without regard to its cause.

Dynamics

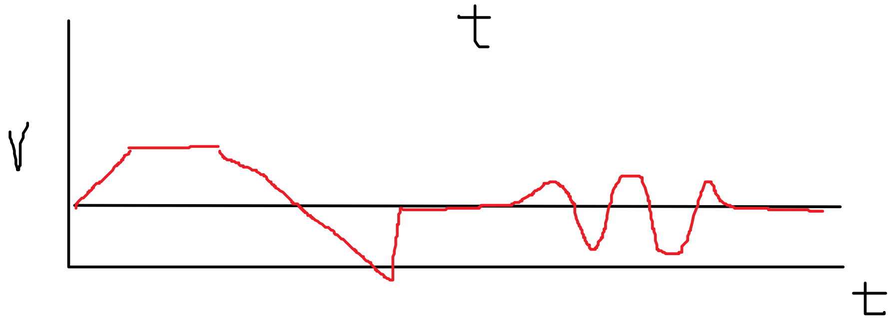
The study of the causes of motion.

Displacement: Net change in position (vector)
Velocity: Time rate of change in position (vector)
Distance: Total path length travelled (scalar)
Speed: Magnitude of velocity (scalar)

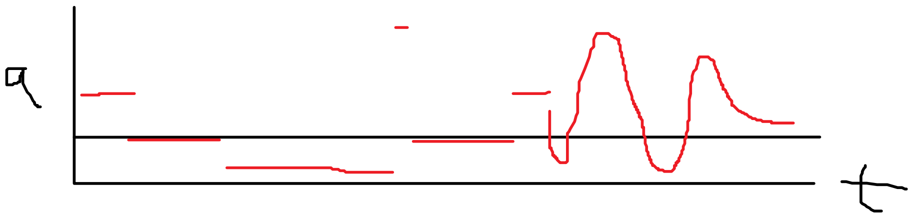
Average velocity $v_{av} = \frac{\Delta x}{\Delta t}$
Instantaneous velocity is the derivative of position with respect to time
$v = \frac{dx}{dt}$



$v = \frac{\mathrm{d}x}{\mathrm{d}t}$



$a = \frac{\mathrm{d}v}{\mathrm{d}t}$



$$v(t) = \frac{dx}{dt} \Rightarrow dx = v(t)dt$$

dx = change in displacement from time t to time t + dt
= velocity at time t multiplied by interval dt
= v(t)dt

$$\Delta x = x_{final} - x_{inital} = \int_{x_{initial}}^{x_{final}} dx = \int_{x_{initial}}^{x_{final}} v(t)dt$$

= accumulated area under v(t) vs. t graph between initial and final times.

Kinematics Equations

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For Constant Acceleration

$$\Delta x = v_i t + \frac{1}{2} a t^2$$

$$x = x_i + v_i t + \frac{1}{2} a t^2$$

This is a special case when $a = \frac{dv}{dt}$ is constant.
In this case v(t) is a straight line

$$v = v_i + a\Delta t$$

$$x_f = x_i + \frac{1}{2}(v_i + v_f)t$$

$$\Delta x = v_{av} t$$

$$x_f = x_i + \frac{v_f^2 - v_i^2}{2a}$$

$$v = v_0 + a\Delta t$$

$$v_{av} = \frac{1}{2}(v + v_0)$$

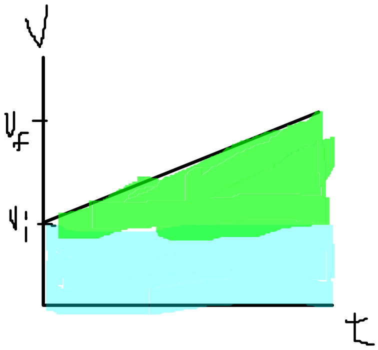
Syntax

Often the initial position/velocity x_i/v_i is written as x_0/v_0.

What is an integral? The infinite summation of infinitely small measures.

Position/Velocity

$$x = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n$$
$$= \frac{\Delta x_1}{\Delta t_1}\Delta t_1 + \frac{\Delta x_2}{\Delta t_2}\Delta t_2 + \cdots \frac{\Delta x_n}{\Delta t_n}\Delta t_n$$
$$= \sum_J v_{av} \times \Delta t_J$$
$$= \int v dt$$



Acceleration/Velocity

$$\Delta v = \int_{t_{initial}}^{t_{final}} a(t) dt$$

For constant a(t)
$$v = v_{initial} + a_{av}\Delta t$$

$$\Delta x = accumulated\ area$$
$$= v_i t + \frac{1}{2}(v_f - v_i)$$
$$= v_i t + \frac{1}{2}\left(\frac{v_f - v_i}{t}\right) t^2$$
$$= v_i t + \frac{1}{2} a_{av} t^2$$

Kinematics Equation Const. Accel

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Equation	Contains	#
$v = v_0 + at$	v, a, t; no x	2.7
$x = x_0 + \frac{1}{2}(v_0 + v)t$	x, v, t; no a	2.9
$x = x_0 + v_0 t + \frac{1}{2}at^2$	x, a, t; no v	2.10
$v^2 = v_0^2 + 2a(x - x_0)$	x, v, a; no t	2.11

All of these follow from the definitions:

$$a = \frac{dv}{dt} \leftrightarrow \Delta \mathrm{v} = \int_{t_{initial}}^{t_{final}} a(t)dt$$

$$v = \frac{dx}{dt} \leftrightarrow \Delta x = \int_{t_{initial}}^{t_{final}} a(t)dt$$

Vectors Operations

September-24-10
10:01 AM

Scalar

A quantity having only magnitude
e.g. Temperature, M ass

Vector

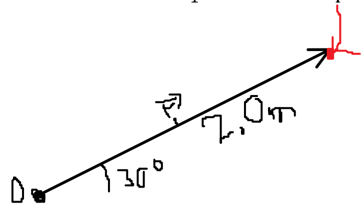
A quantity with both magnitude and direction
e.g. Position, Velocity

Unit Vector

Any dimensionless vector of length 1, denote with a ^

For a general vector use an arrow:

The vector $\vec{r_1}$ describes the position of this point.



O is the arbitrary origin

(x, y, z) = Cartesian coordinate system, origin "O"
$\hat{\imath}, \hat{\jmath}, \hat{k}$= 3 unit vectors based at O and pointing in the positive (x, y, z) directions respectively ⇒
These are called basis vectors.

Any vector $\vec{A}$ can be written as $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$
A_x, A_y, A_z are components. Components have units - the basis vectors do now.
The components each have the same dimensions as the vector.

Adding Vectors

$\vec{r_1} + \Delta\vec{r} = \vec{r_2} \Rightarrow \Delta\vec{r} = \vec{r_2} - \vec{r_1}$
Add tip to tail

Vector addition is commutative
$\vec{A} + \vec{B} = \vec{B} + \vec{A}$

Vector addition is also associative:
$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

Dot Product (inner product)

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$
$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$
$\Rightarrow \vec{A} \cdot \vec{B} \equiv A_xB_x + A_yB_y + A_zB_z = \vec{B} \cdot \vec{A}$
The dot product of two vectors is a scalar

$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$
$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$

Useful rule:
$\vec{A} \cdot \vec{B} = 0 \Leftrightarrow \vec{A} \perp \vec{B}$

Vector Magnitudes

$|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2} = A \geq 0$

Cross Product (outer product)

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$
$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$
$= (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$
The cross product of two vectors is a (pseudo) vector.

Basis vector cross products
$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$
$\hat{\imath} \times \hat{\jmath} = \hat{k}$
$\hat{\jmath} \times \hat{k} = \hat{\imath}$
$\hat{k} \times \hat{\imath} = \hat{\jmath}$

Useful Rule:
$\vec{A} \times \vec{B} \Leftrightarrow \vec{A} \parallel \vec{B}$

Motion and Vectors

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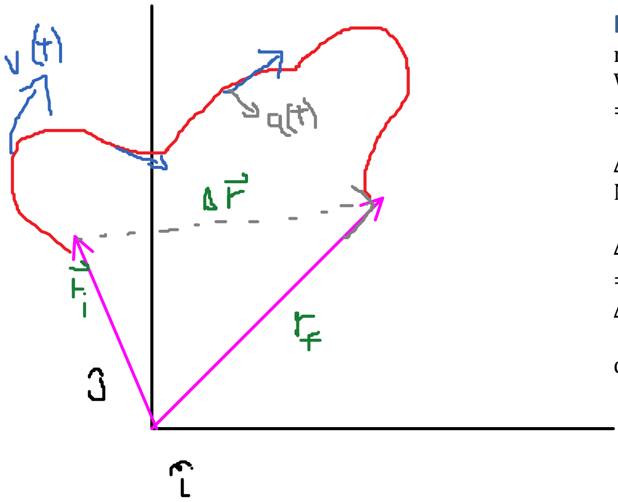

Position

r(t) = position vector
Vector from origin to position of particle at time t
=x(t)i+y(t)j

Δr = displacement vector
Net change in position from initial to final time

$\Delta r = r_f - r_i$
$= (x_f - x_i)i + (y_f - y_r)j$
$\Delta xi + \Delta yi$

d = distance traveled ≠ Δr

Note:

Vector is not the same as magnitude which is not the same as average.

Velocity

$\vec{v}(t) = velocity\ vector$
$= \frac{dr(t)}{dt} = \frac{d}{dt}|x(t)i + y(t)j|$
$v_x(t)i + v_y(t)j$

This is true because i and j are constant.
$v(t) = \frac{dr(t)}{dt} = \lim_{\Delta x \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t}$

$v(t) = speed\ scalar = magnitude\ of\ velocity\ vector$
$= |\vec{v}(t)| = \sqrt{\vec{v}(t) \cdot \vec{v}(t)} = \sqrt{v_x^2 + v_y^2} \geq 0$
$v(t) = average\ speed\ vector = \frac{d}{\Delta t}$

We often drop the t when calculating vectors and motion
$r = xi + yj + zk$
$v = v_x i + v_y j + v_z k$
$a = a_x i + a_y j + a_z k$
i, j, k no not have units!

1D: Acceleration induces changes in speed
2D and 3D: Acceleration induces changes the velocity (and usually the speed)
Circular motion necessitates acceleration.

Acceleration perpendicular to the motion does not change the speed

Motion in 3D is just a superposition of the three independent 1D motions

Example

A windsurfer is sailing at 7.2 m/s when a gust of wind lasting 5.5 s blasts her at a 60° angle to the direction of motion with an acceleration of $1.0\frac{m}{s^2}$
What is her displacement during this time?

$$a = 1.0\frac{m}{s}\begin{cases} a_x = 1.0 \times \cos 60 = 0.50\frac{m}{s^2} \\ a_y = 1.0 \times \sin 60 = 0.87\frac{m}{s^2} \end{cases}$$

$$r_f = r_i + v_i t + \frac{1}{2}at^2$$

$$x_f = 0 + 7.2t + \frac{1}{2}(0.50)t^2 = 47.2m$$
$$y_f = 0 + 0t + \frac{1}{2}(0.87)t^2 = 13.2\ m$$
$$r = 47.2i + 13.2j\ m$$

Vector Kinematics

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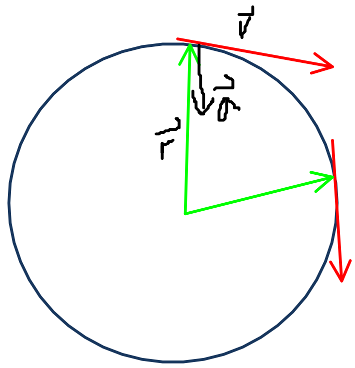
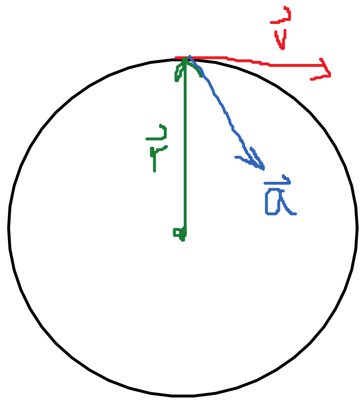
Vector Equations

$r_f = r_i + v_i t + \frac{1}{2}at^2$
$v_f = v_i + at$

Scalar Equation

$a \cdot \Delta r = \frac{|v_f|^2 - |v_i|^2}{2}$





Projectile Motion

Constant Gravitational Field: So ignore air resistance, and height small compared to Earth's radius

Split into components, x velocity is constant, y velocity is changing

To find range, solve for t first when $r_y(t) = 0$ then use this t to calculate $r_x(t)$ which will be the range.

For a given start angle and returning to the starting height, the range can be described by:

$x = \frac{v_0^2}{g}\sin 2\theta$

Uniform Circular Motion

An object moving with a constant speed $|v|$ on a circular path is uniform circular motion.

As r rotates through an angle θ so des v, so both are similar isosceles triangles

$\frac{|\Delta v|}{v} = \frac{|\Delta r|}{r} \Rightarrow \frac{v\Delta t}{r} = \frac{|\Delta v|}{v} \Rightarrow \frac{|\Delta v|}{\Delta t} = \frac{v^2}{r} \Rightarrow \frac{dv}{dt} = \frac{v^2}{r}$
$a = \frac{v^2}{r}$
Acceleration under uniform circular motion

Acceleration is always towards to centre of the circle
$a = \frac{v^2}{r}\dot{r}$
Where $\dot{r}$ is a unit vector of direction r

T = period of UCP
= distance/speed
$T = \frac{2\pi r}{v}$

f = frequency of UCM
$f = \frac{1}{T} = \frac{v}{2\pi r}$

Non-Uniform Circular Motion

What if the speed around the circle is not constant?

There is still acceleration perpendicular to the velocity bending the path into a circle:
$a_\perp = -\frac{v^2}{r}\dot{r}$
There is also a tangential acceleration acting in the direction of the motion to speed up the particle.
E.g. For car going down a curved hill, break acceleration due to gravity into radial acceleration based on the radius of curvature at that point, and tangential velocity increasing the speed of the car.

Newton's Laws

October-04-10
9:39 AM

Reference Frame (RF)

Space [rulers for coordinate system, eg (x, y, z) and time [clock, t]

$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Newton's Three Laws

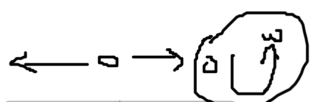
1. **Inertia**
   A body in uniform motion remains in uniform motion, and a body at rest remains at rest unless acted upon by a nonzero net force.

2. **Change**
   The rate at which a body's momentum changes is equal to the net force acting on a body.
   $\vec{F}_{net} = \frac{d\vec{p}}{dt}$

3. **Reaction**
   If a body A exerts a force on a body B, then B exerts an oppositely directed force of equal magnitude on A.

Conservation of Momentum

In an interaction between two bodies
$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$



Centrifugal	Centripetal
Fictitious	Real

Types of Reference Frames

Inertial (IRF)

Newton's Laws hold
- Non-rotating
- Non-accelerating
- Example: RF attached to
  - Isolated body - theoretical idealization
  - Non-rotating relative to distant stars (Mach's Principle)

  Imagine an object very far from any strong gravitational field and not rotating relative to a distant star.

Non-Inertial

Newton's Laws do not hold
- Rotating and/or accelerating with respect to an IRF
- Yields fictitious forces
- Earth is a non-inertial reference frame. Why?
  - Interacts with moon, sun
  - Rotates on its axis

  But, Earth is nearly an IRF because fictitious forces are so small relative to g
- Some example non-inertial frames
  - Elevator accelerating upwards
  - Car turning a corner
  - Rotating merry-go-round

Newton's Three Laws

First Law

- The natural (force-free) state of motion of any body is that of zero acceleration or constant velocity. Not zero velocity
- We do not need to explain motion.
- What needs explanation is changes in motion - where the 2nd law is relevant.
- Inertia: The tendency of a body to resist changes in motion.
- Aside: In general relativity the effect of gravity is not a force but instead a generalization of the first law: the natural state of any body in the presence of gravity is zero acceleration, which is the "shortest path" in a "curved spacetime."

Second Law

- Momentum: Mass × Velocity
- Mass: A measure of resistance a body exhibits to changes in its velocity (inertial mass)
  - Intrinsic property of the body - same everywhere.
  - Typically constant - but not always (e.g. Rocket ejecting fuel; bomb exploding)
- Mass ≠ Weight
  - Mass = resistance to acceleration (regardless of whether gravity is present)
  - Weight = gravitational force on the body by another body
  - $\frac{m_1}{m_2} \equiv \frac{|\vec{a}_2|}{|\vec{a}_1|}$
- Force: That which changes the velocity of a body
  - Force changes motion; it does not cause it
  - 2 Everyday kinds: Field force and Contact force
  - Force is a vector quantity with magnitude and direction

    $\vec{F}_{net} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$
    (provided the body's mass stays constant)
- All masses can be determined given a known mass and a constant force.
  - This force is the Earth's gravitational field
  - There is a 1kg standard- all other masses are defined relative to it.
- 1 Newton: The quantity of net force required to accelerate 1 kg by 1 $\frac{m}{s^2}$
  - $N = \frac{kgm}{s^2}$
- 2nd law is valid only in IRF's
- In non-IRFs, must add in "fictitious forces: to get the 2nd law to be valid.
  - Observer on rotating disk sees no acceleration, Observer on ground sees UCM
- Acceleration depends on the net force only
- 2nd law is valid for all bodies in all states in motion, provided forces are correctly identified
- If all forces sum to zero, then a = 0
  - Body is in *equilibrium*: behaves as through there were no forces at all.

Third Law

In an interaction:
$\vec{F}_{AonB} = -\vec{F}_{BonA}$
$m_1\vec{a}_1 + m_2\vec{a}_2 = 0$
$\frac{m_1 d\vec{v}_1}{dt} + \frac{m_2 d\vec{v}_2}{dt} = 0$

$$\frac{d}{dt}(m_1\dot{v}_1 + m_2\dot{v}_2) = 0$$
$$\frac{d}{dt}(\dot{p}_1 + \dot{p}_2) = 0$$

Conservation of momentum.

- Action/Reaction pairs are
  - Always present, in any interaction (contact and field)
  - Act on different objects (never on the same object)

Free-Body Diagrams

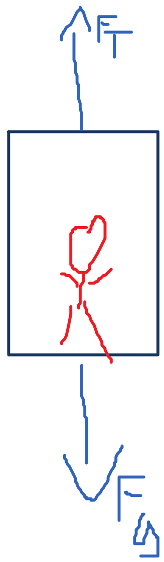
October-08-10
9:31 AM

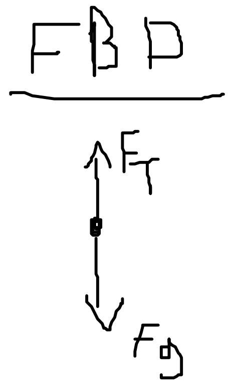
Normal Force

Force of contact of a surface - perpendicular to the surface.

The normal force must always be cancelled by some other force in the direction perpendicular to the surface.

1. Identify the object of interest and all of the forces acting on it.
2. Represent the object as a dot
3. Draw the vectors *only for those forces that act on the object* with all tails starting on the dot.





Suppose an elevator of mass 740kg is accelerating upwards at $1.1m/s^2$ What is the tension in the cable?

$F_{net}\hat{j} = ma\hat{j}$
$\vec{F_t}\hat{j} + F_g = ma\hat{j}$
$\vec{F_t}\hat{j} = m(a+g)\hat{j}$
$\vec{F_t}\hat{j} = 740kg(1.1\,m\backslash s^2 + 9.8m\backslash s^2)\hat{j}$
$\vec{F_t} = 8.1kN$

Forces
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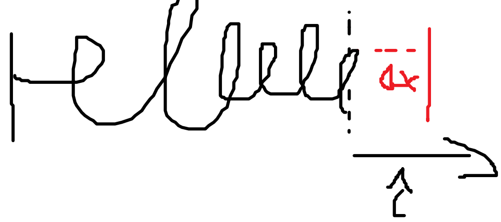
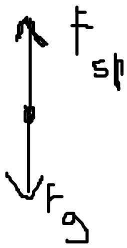
Hooke's Law
For an ideal spring
$\vec{F}_s = -kx\hat{\imath}$

Rest position




A helicopter lifts a 35kg load of concrete on a spring scale whose constant $k = 3.4\frac{kN}{m}$. How much does the spring compress
a) When it is at rest?
b) When it accelerates upward at $1.9\ \frac{m}{s^2}$

$F_{sp} + F_g = m\vec{a}$
a) $F_{sp} = -F_g$
$F_g = kx\hat{\imath}$

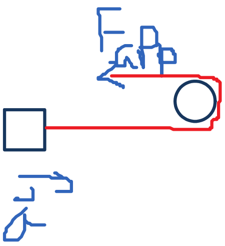
$$x = \frac{mg}{k} = 35kg \times \frac{9.8\frac{m}{s^2}}{3400\frac{N}{m}} = 10\ cm$$

b) $a = 1.9 \Rightarrow x = 35kg \times \frac{1.9\frac{m}{s} + 9.8\frac{m}{s}}{3400\frac{N}{m}} = 12\ cm$

Gravity as a Force
$\vec{F}_g = G \times \frac{M_\oplus m}{r^2}\hat{r}$
$|\vec{F}_g| = \frac{GM_\oplus m}{(R_\oplus + h)^2} \approx \frac{GM_\oplus m}{R_\oplus^2}$
$= mg\ for\ h \ll R_\oplus$

$$a = \frac{|\vec{F}_g|}{m} = \frac{mg}{m} = g$$

Gravitational mass = Inertial mass
Q: Why this cancellation?
A: The Equivalence Principle
Einstein: Gravity locally can't be distinguished from acceleration.



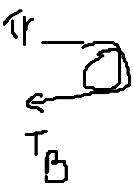
Friction
When an object is at rest, static friction opposes the applied force

Kinetic friction opposes the relative motion

Pulleys
Frictionless surface, massless rope, no slipping

$\vec{F}_{net} = m\vec{a} \Rightarrow F_n = Mg \Rightarrow T_B = Ma$

As M accelerates, pulley turns faster
⇒ This requires a net torque on the pulley



Ropes apply opposite torques on the pulley.
Need $T_r > T_b$ to apply a torque to accelerate the pulley, and therefore accelerate the rope system
If $T_r = T_B$ then no net torque and no acceleration
Deal at first with massless pulleys.

Circular Motion

October-18-10
10:14 AM

Uniform Circular Motion

A body moving through space on a circular path of radius r with constant speed v experience a centripetal acceleration only.
2nd law then implies that all forces acting on the body - whatever they are- must vectorially add up to

$m\vec{a}_c$: $\sum \vec{F} = m\vec{a}_c = -\frac{mv^2}{r}\hat{r}$

Banked Curve

What frictional force is needed to keep the car on the track?
$\hat{k} - upwards, \hat{r}\ - outward\ from\ circle\ of\ curve$
$\vec{n} = n\cos(\theta)\,\hat{k} - n\sin(\theta)\,\hat{r}$
$\vec{f} = -f\sin(\theta)\,\hat{k} - f\cos(\theta)\,\hat{r}$
$\vec{F_g} = m\vec{g} = -mg\hat{k}$

$$\vec{F}_{net} = m\vec{a}_c \Rightarrow \vec{n} + \vec{f} + \vec{F_g} = -\frac{mv^2}{r}\hat{r} = \begin{cases} n\cos(\theta) - f\sin(\theta) - mg = 0 \\ -n\sin(\theta) - f\cos(\theta) = -\frac{mv^2}{r} \end{cases}$$

Rewrite in Matrix Form
$MV = V'$
$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} n \\ f \end{pmatrix} = \begin{pmatrix} mg \\ \frac{mv^2}{r} \end{pmatrix}$$
$V = V'M^{-1}$

This is a special kind of matrix called a rotation matrix
The inverse of a rotation matrix, is the negative angle of the same matrix
$$M^{-1} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$V = \begin{pmatrix} n \\ f \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} mg \\ \frac{mv^2}{r} \end{pmatrix}$$
$$f = -mg\sin(\theta) + \frac{mv^2}{r}\cos(\theta)$$

How fast can it go without sliding?

Not Banked

$$f \le \mu_s n \Rightarrow \frac{mv^2}{r} \le \mu_s mg \Rightarrow v \le \sqrt{\mu_s rg} = v_{max}$$

Banked

$$f \le \mu_s n \Rightarrow -mg\sin(\theta) + \frac{mv^2}{r}\cos(\theta) \le \mu_s mg \Rightarrow v \le \sqrt{\frac{g(\mu_s + \sin\theta)}{r\cos(\theta)}}$$

Non-Uniform Circular Motion

$$\vec{a} = \vec{a}_c + \vec{a}_t = -\frac{v^2}{r}\hat{r} + \frac{dv}{dt}\hat{v}$$

Drag Forces

Something falling through fluid:
Buoyant Force: proportional to the weight of fluid displaced
$\Rightarrow \vec{B} = -\rho Vg\hat{j}$
$\vec{F_g} + \vec{B} = (mg\ - \rho Vg)\hat{j} = (m - \rho V)g\hat{j}$

Resistive Force: $\vec{R} = -b\vec{v}$
$\vec{F_g} + \vec{B} + \vec{R} = m\vec{a}$, neglect buoyancy
$mg - bv = m\frac{dv}{dt}$, as v increases, acceleration a decreases due to increasing restive force
Solve: Let $V = v - \frac{mg}{b} \Rightarrow -bV = m\frac{dV}{dt}$, since $\frac{dV}{dt} = \frac{dv}{dt}$

$$\frac{dV}{dt} = -\frac{b}{m}V$$
Guess:
$$V = V_0 \times e^{kt}v \Rightarrow \frac{dV}{dt} = V_0ke^{kt} = -\frac{b}{m}V_0e^{kt} \Rightarrow k = -\frac{b}{m}$$
$$V = V_0 \times e^{-\frac{b}{m}t}$$

Work and Energy

October-20-10
10:00 AM

Energy and Energy Transfer

Energy is neither created no destroyed: it is only transferred from one place to another or from one form to another.

Work

Work is a way of transferring energy from one system to another

$W = F\Delta x$

$$= (ma)\left(\frac{v_f^2 - v_i^2}{2a}\right) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

For an object moving in one dimension, the work W done on the object by constant applied force $\vec{F}$ is

$W = F_x \Delta x$

Where F_x is the component of the force acting in the direction of the displacement

Suppose system A is a point particle of mass m movin through space under the action of several applied forces

System B is the rest of the universe

$$W = \int_i^f \vec{F}_a \cdot d\vec{r}$$

Work = Force × Distance

Work is measured in Newton-Meters or Joules

Work is the area under a force-vs.-position curve

Pulling a Spring

$$W = \int_i^f \vec{F} . d\vec{r}$$

$$W = \int_0^x kx\, dx$$

$$W = \frac{1}{2}kx^2$$

Work and Gravity

$$W = \int_i^f \vec{F} . d\vec{r} = F\Delta y = mg\Delta y$$

E_A goes down by $\Delta E = W$ (chemical energy of person decreases if lifting object)

E_B goes up by $\Delta E = W$ (potential energy in gravitational field increases - when lifting object in Earth's gravity)

Kinetic Energy

Work is additive

$$W = \int_i^f (\vec{F}_a + \vec{F}_b) \cdot d\vec{r} = \int_i^f \vec{F}_a \cdot d\vec{r} + \int_i^f \vec{F}_b \cdot d\vec{r} = W_a + W_b$$

$$\sum_i W = \Delta K$$

$$K = \frac{1}{2}mv^2$$

Power

$$P = \frac{\Delta W}{\Delta t}$$

$$P = \lim_{\Delta t \to 0} \frac{\Delta \mathrm{W}}{\Delta \mathrm{t}} = \frac{dW}{dt} \Rightarrow W = \int_i^f \vec{F} \cdot d\vec{r} \Rightarrow dW = \vec{F} \cdot d\vec{r}$$

$P = \vec{F} \cdot \vec{v}$

Joules per second = Watt

Conservative Force

October-25-10
10:16 AM

The total work done by a conservative force active over any closed path is zero.

$$\oint \vec{F} \cdot d\vec{r} = 0$$

The change ΔU_{AB} in potential energy associated with a conservative force is the negative of the work done by that force as it acts over any path from point A to point B

Spring (Simple Harmonic Oscillator)

$$\Delta U = -\int_{x_1}^{x_2} \vec{F} \cdot d\vec{r} = -\int_{x_1}^{x_2} (-kx)dx = \frac{1}{2}k(x_2^2 - x_1^2)$$

Energy stored in a spring
$U_{spring} = \frac{1}{2}kx^2$
(from equilibrium)

Gradients

For any conservative force, $\vec{F}(x, y, z)$ we can find a potential $U(x, y, z)$ such that $\vec{F} = -\vec{\nabla}U$
Every force can be written as the derivative of a potential, where the potential is a gradient

$$\vec{\nabla}U = \frac{dU}{di}\hat{\imath} + \frac{dU}{dj}\hat{\jmath} + \frac{dU}{dk}\hat{k}$$

$$F_x = -\frac{dU}{dx} \quad F_y = -\frac{dU}{dy} \quad F_z = -\frac{dU}{dz}$$

$$W = \int_i^f \vec{F}(r)\, dr$$

Gravity

$$U = mgy + c$$

$$F_y = -\frac{dU}{dy} = -mg$$

The constant is arbitrary because

a) The derivative of a constant is zero
b) Work depends only on potential difference.

Spring

$$U = \frac{1}{2}kx^2 + c$$

$$F_x = \frac{dU}{dx} = -kx$$

Gravity

October-29-10
10:00 AM

History of Gravitation

1543 - Nicolaus Copernicus proposes that planets orbit the Sun, no the Earth
1593 - Tycho Brache accurately maps out the motions of the planets
1601 - Johannes Kepler analyzes these observations and proposes planets move on ellipses, not circles.
1610 - Galileo Galilei discovers moons orbiting Jupiter, the phases of Venus, sunspots, and mountains on the moon
1664 - Isaac Newton proposes a universal law of gravitation.

Kepler's Laws - Know for Final

First Law: The orbit of planets is elliptical, with the Sun at one focus.
Second Law: The area swept out between the planet and the sun in equal times are equal.
Third: The square of the orbital period is proportional to the cube of the semimajor axis.

Universal Gravitation

$$F = \frac{Gm_1m_2}{r^2}$$

$$\vec{F}_{12} = -\frac{Gm_1m_2}{r^2}\hat{r}, \qquad \vec{F}_{21} = \frac{Gm_1m_2}{r^2}\hat{r}$$

Henry Cavendish

Set out to measure the density of the earth.

Did so by measuring the strength of gravity. Set up a torsion balance with heavy lead balls to measure the gravitational attraction between the balls.

Orbit

$$\vec{F}_g = -\frac{GMm}{r^2}\hat{r} = -\frac{mv^2}{r}\hat{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

Orbital Period

$$v = \frac{2\pi r}{T} \Rightarrow T^2 = \frac{4\pi^2r^3}{GM}$$

Which matches with Kepler's Law

Geosynchronous Orbits

Orbits for which the period is 24 hours

$$r = \sqrt[3]{\frac{GM_ET^2}{4\pi^2}} = 44200\ km\ approximately$$

Parabolic Motion Under Gravity

$$\vec{r}_f = \vec{r}_i + \vec{v}_i t - \frac{1}{2}g\hat{j}t^2$$

Orbits due to an Inverse Square law

Closed (Circle)
Closed (ellipse)
Borderline (parabola)
Open (hyperbola)

These are all results of the general motion equation:

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{GMm}{r^2}\hat{r}$$

Gravitational Potential

November-01-10
10:00 AM

$U_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} \frac{GMm}{r^2} dr = -\frac{GMm}{r_2} + \frac{GMm}{r_1}$

$U_{12} = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

The zero of the potential

$U_{12} = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

Take $r_1 \to \infty, r_2 = r$

$U = -\frac{GMm}{r}$

The difference between an open orbit and a closed orbit is whether or not the Gravitational Potential Energy is greater or less than zero.

Escape Speed

11.2 km/s

For a body of mass m and radius r
Escape velocity is

$v_{esc} = \sqrt{\frac{2GM}{R}}$

What if the escape velocity is greater than the speed of light?

$c = \sqrt{\frac{2GM}{R}} \Rightarrow R_{BH} = \frac{2GM_{BH}}{c^2}$

Black Hole Surface Gravity

$g_{BH} = \frac{GM_{BH}}{R_{BH}^2} = \frac{c^4}{4GM_{BH}}$

So the larger the black hole, the smaller the black hole

Energy in Circular Orbits

$K = \frac{1}{2}mv^2, U = -\frac{GMm}{R}$

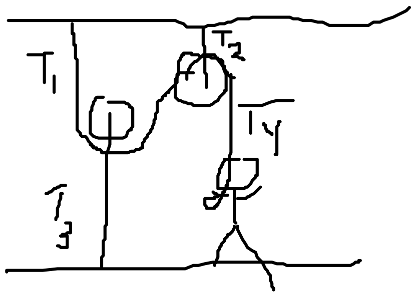
$\frac{mv^2}{R} = \frac{GMm}{R^2} \Rightarrow v^2 = \frac{GM}{R}$

$E = K + U = \frac{GMm}{2R} - \frac{GMm}{R} = -\frac{GMm}{R}$

So high kinetic energy (and therefore higher speed) corresponds to a lower total energy.

Pulleys

November-03-10
9:43 AM



$T_1 = T_4$
$T_3 = 2T_1$
$T_2 = 2T_4$
$T_3 + T_4 = mg \Rightarrow 3T_4 = mg$
$T_1 = T_4 = \frac{1}{3}mg$
$T_3 = T_2 = \frac{2}{3}mg$

When working with pulleys, consider each individual pulley as a FBT. If in equilibrium, will have two forces in one direction balanced by one in the other.
The tensions on rope passing over the pulley will be the same on each side.

Gravity as a Field Theory

November-05-10
9:49 AM

$$U = -\frac{GM_E m}{|\vec{r}|} \Rightarrow \vec{F} = -\nabla U = -\frac{dU}{dr}\hat{r} = -\frac{GMm}{r^2}\hat{r}$$

Gravitational field:

$$\vec{g} = -\frac{GM}{r^2}\hat{r}$$

Way say that the gravitational field of Earth (or any massive body) pervades all of space.
Any other massive body responds to the gravitational field in its local vicinity.

Gravitational Potential:

$$\varphi = \frac{U}{m} = -\frac{GM}{|\vec{r}|} \Rightarrow \vec{g} = -\nabla\varphi = -\frac{GM}{r^2}r$$

Force on a body of mass m:

$$\vec{F} = mg = -\frac{GMm}{r^2}r$$

A field theory answers two questions:

1. How do bodies generate force?
2. How do bodies respond to force?

The unifying link is the potential field. For a body of mass m its gravitational potential is $\varphi = -\frac{Gm}{|\vec{r}|}$

Systems of Particles

November-08-10
9:31 AM

Isolated System

No external forces

Conservation of Linear Momentum

When the net external force on a system is zero the total momentum of the system - which is the vector sum of the individual momenta of the constituent particles - remains constant.

Consider a system of two particles
The internal forces will cancel
The change of momentum of the system is due only to the external forces.

$$\vec{F}^{ext} = M\frac{d^2\vec{r}_{CM}}{dt^2}$$

Total Mass:

$$M = \sum_i m_i$$

Center of Mass:

$$\vec{r}_{CM} = \frac{1}{M}\sum_i m_i\vec{r}_i$$

The point at $\vec{r}_{CM}$ is called the Centre of Mass of the system. It behaves just as though it were a a particle of mass M obeying Newton's 2nd law with external forces $\vec{F}^{ext}$ acting on it.

Finding the Centre of Mass

For two bodies

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{2} m_i\vec{r}_i = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M}\sum_i m_i\vec{v}_i$$

System Momentum:

$$\vec{P} = M\vec{v}_{CM}$$

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M}\sum_i m_i\vec{a}_i$$

Continuous Distribution of Mass

$$M = \lim_{N\to\infty}\sum_{i=1}^{N} \square$$

$$Uniform\ Distribution \Rightarrow \frac{mass\ of\ strip}{total\ mass} = \frac{total\ area\ of\ strip}{total\ area}$$

Energy

$$K_{total} = K_{CM} + K_{internal}$$

$$K_{CM} = \frac{1}{2}Mv_{CM}^2$$

$$K_{internal} = \frac{1}{2}\sum_I m_I(v_I^{rel})^2$$

v_I^{rel} is the velocity of each mass relative to the center of mass

Impulse

November-10-10
10:06 AM

Consider a single body moving on a trajectory

$$\frac{d\vec{p}}{dt} = \sum F^{\rightarrow}$$

$$d\vec{p} = \left(\sum \vec{F}\right) dt$$

$$\int_i^f d\vec{p} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} \left(\sum \vec{F}\right) dt = \vec{J}$$

Collision

A brief, intense interaction between bodies.
Momentum is conserved during collision.

Brief

Interaction time is short compared to all other time scales.

Intense

Forces during collision are far larger than any external forces on the system (thus can neglect external forces during collision)

Kinds of Collisions

Elastic

Kinetic Energy is Conserved
Bounce off each other perfectly, no energy loss.

Inelastic (Sticky)

Kinetic Energy is Not Conserved - Kinetic Energy is reduced
Stick to each other to some degree, energy loss in collision to the environment.

Explosive

Kinetic Energy is Not Conserved - Kinetic Energy is increased
Potential energy is released during the collision, transformed to kinetic energy.

Collisions

November-12-10
9:44 AM

Momentum

$M(V - V') = m(v' - \mathrm{v})$

Kinetic Energy

$\frac{1}{2}MV^2 + \frac{1}{2}mv^2 = \frac{1}{2}MV' + \frac{1}{2}mv'$
$V + V' = v' + v$

Coefficient of Restitution

$e = -\frac{V' - v'}{V - v}$
e < 1 inelastic
e = 1 elastic
e > 1 explosive

$v' = \frac{M(1+e)}{m+M}V + \frac{m - eM}{m+M}v$
$V' = \frac{m(1+e)}{m+M}v + \frac{M - em}{m+M}V$

Notice if e = 0 then v' = V'

Head on Collision

$\vec{P}_{initial} = MV\hat{\imath}$
$\vec{p}_{initial} = mv\hat{\imath}$

$\vec{P}_{final} = MV'\hat{\imath}$
$\vec{p}_{final} = mv'\hat{\imath}$

$\vec{P}_{initial} + \vec{p}_{initial} = \vec{P}_{final} + \vec{p}_{final}$
$MV + mv = MV' + mv'$
$M(V - V') = m(v' - v)$
One equation but two unknowns.

Elastic

If a perfectly elastic collision we can assume energy conservation
$K_{initial} = K_{final}$
$\frac{1}{2}MV^2 + \frac{1}{2}mv^2 = \frac{1}{2}MV' + \frac{1}{2}mv'$
$M(V^2 - V'^2) = m(v'^2 - v^2)$
$M(V - V')(V + V') = m(v' - v')(v' + v)$
But $M(V - V') = m(v' - v)$
so
$V + V' = v' + v$ or
$V' - v' = -(V - v)$

Inelastic

$\frac{1}{2}mv^2 + \frac{1}{2}MV^2 > \frac{1}{2}mv'^2 + \frac{1}{2}MV'^2$
$\Rightarrow V' - v > -(V - v)$
Define
$e = -\frac{V' - v'}{V - v}$

Rotational Dynamics

November-15-10
9:38 AM

Average angular velocity

$\omega = \frac{\Delta\theta}{\Delta t}$

Instantaneous angular velocity:

$\omega = \frac{d\theta}{dt}$

Average angular acceleration

$\alpha = \frac{\Delta\omega}{\Delta t}$

Instantaneous angular acceleration

$\alpha = \frac{d\omega}{dt}$

Angle in Radians

$\theta = \frac{s}{r}$

Tangential displacement

$s = r\Delta\theta$

Average tangential velocity

$v_t = \frac{\Delta s}{\Delta t} = r\omega$

Instantaneous tangential velocity

$v_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$

Average tangential velocity

$a_t = \frac{\Delta v_t}{\Delta t} = r\alpha$

Instantaneous tangential velocity

$a_t = \frac{dv_t}{dt} = r\alpha$

Angular Velocity and Acceleration

Convention: Clockwise is positive, widdershins is negative.
Axis or rotation is perpendicular to rotation

The average angular velocity is the same for all points on the rotating object, the average tangential velocity is different depending on how far the point is from the pivot.

Circular motion, acceleration inwards

$a_r = \frac{v_t^2}{r} = r\omega^2$

Constant Angular Acceleration

For constant angular acceleration α, the same kinds of equations hold as for constant linear acceleration $\acute{a}$

$x = x_i + v_i t + \frac{1}{2}at^2$

$s = s_i + v_{t_i}t + \frac{1}{2}a_t t^2$

$r\theta = r\theta_i = r\omega_i t + \frac{1}{2}r\alpha t^2 \Rightarrow \theta = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$

$v = v_i + at$

$v_t = v_{t_i} + a_t t$

$r\omega = r\omega_i + r\alpha t \Rightarrow \omega = \omega_i + \alpha t$

$v_f^2 - v_i^2 = 2a(x_f - x_i)$

$\left(v_{t_f}^2 - v_{t_i}^2\right) = 2a_t(s_f - s_i)$

$(r\omega_f)^2 - (r\omega_i)^2 = 2r\alpha(r\theta_f - r\theta_i) \Rightarrow \omega_f^2 - \omega_i^2 = 2\alpha(\theta_f - \theta_i)$

Torque

When applying a force to a point distance r from the pivot point

$F_{||} = F\cos\phi$, this parallel component does not apply a torque

$F_\perp = F\sin\phi$

$\tau = F_\perp r = rF\sin\phi$

τ is called the torque, it causes angular acceleration

$\acute{\tau} = \acute{r} \times \acute{F} = rF\sin\phi\,\acute{n}$

Moment of Inertia

November-17-10
11:35 AM

Newton's Law Analogue for Rotation

$\tau = I\alpha$
Where $I = mr^2$ for one mass

Moment of Inertia

$$I = \sum_I m_I r_I^2$$

$F_\perp = ma_t = mr\alpha$
$\tau = rF_\perp$
$\Rightarrow \tau = mr^2\alpha$

$\tau = I\alpha$

Continuous Mass Distribution

$$I = \int r^2\, dm$$

Example, moment of inertia of meter stick

Continuous so:

$$\frac{dm}{M} = \frac{dl}{L}$$

$$I = \int r^2\, dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \left(\frac{M}{L} dx\right) = \frac{M}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx = \frac{M}{L}\left(\frac{1}{3}x^3\right)\Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{ML^3}{24L} - \frac{M(-L)^3}{24L} = \frac{ML^2}{12}$$

If axis is at one end:

$$I = \int_0^L x^2 \frac{M}{L} dx = \frac{ML^2}{3}$$

Moment of inertia of Ring

$$\frac{dm}{M} = \frac{d\theta}{2\pi}$$

$$I = \int_0^{2\pi} r^2 \left(\frac{M}{2\pi}\, d\theta\right) = \frac{M}{2\pi}\int_0^{2\pi} r^2\, d\theta = \frac{M}{2\pi}(2\pi R^2) = MR^2$$

Moment of Inertia of Disk

$$\frac{dm}{M} = \frac{A_{ring}}{A_{total}} = \frac{2\pi r\, dr}{\pi R^2}$$

$$dm = \frac{2Mr\, dr}{R^2}$$

$$I = \int_0^R r^2\, dm = \int_0^R \frac{r^2 2Mr}{R^2} dr = \frac{2M}{R^2}\int_0^R r^3\, dr = \frac{2M}{R^2}\left(\frac{1}{4}R^4\right) = \frac{1}{2}MR^2$$

Rotational Kinetic Energy

Each particle in the body is in circular motion

$$K = \sum_I K_I = \sum_I \frac{1}{2} m_I v_{t_I}^2 = \sum_I \frac{1}{2} m_I r_I^2 \omega^2 = \frac{1}{2} I\omega^2$$

Compare

$$K^{trans} = \frac{1}{2}Mv_{CM}^2, K^{rot} = \frac{1}{2}I\omega^2$$

Work-Energy Theorem

$$dK = \frac{1}{2}(dm)(\omega r)^2$$

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta = \Delta K^{rot} = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

Parallel Axis Theorem

November-19-10
9:44 AM

For a body rotating about some axis

$I = Md^2 + I_{CM}$
M - Total mass
d - distance form axis or rot. to CM
I_{CM} rotational inertial relative to CM

Each particle in the body is in circular motion about some instantaneous axis of rotation.

$$I = \sum_I m_I r_I^2 = \sum_I m_I (d + r_I^{\sim})^2 = \sum_I m_I \left(d^2 + 2dr_I^{\sim} + r_I^{\sim 2}\right)$$
$$= d^2 \sum_I m_I + \sum_I m_I r_I^{\sim 2} + 2d \sum_I m_I r_I^{\sim}$$
$$2d \sum_I m_I r_I^{\sim} = 0 \; since \; origin \; is \; at \; CM$$
$$I = Md^2 + I_{CM}$$

d is the distance from the axis of rotation to the CM
$r^{\sim}$ is the distance from the point to the centre of mass

Ex

Rotating a Sphere from its edge
d = R
$$I = \frac{2}{5}MR^2 + Md^2 = \frac{7}{5}MR^2$$

Wheels

Wheels combine both rotational and translational motion
Without slipping, a wheel will move θR after a rotation of θ

$$K^{trans} = \frac{1}{2}Mv^2$$
$$K^{rot} = \frac{1}{2}I\omega^2$$
$$v_{CM} = R\omega$$
$$K^{tot} = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I\omega^2 = \frac{3}{2}I\omega^2$$

Angular Motion Vectors

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9:35 AM

Matrices and Determinants

$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
$\det A \coloneqq ad - bc$

$V = \begin{vmatrix} V_{11} & V_{12} & V_{12} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix}$

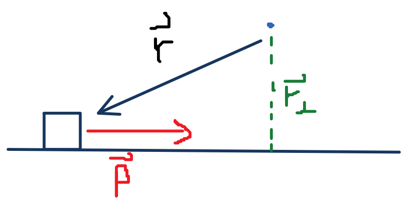
$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

Pseudo Vector

Pseudo vectors behave oppositely to vectors under reflection.

Angular Momentum

$L = \vec{r} \times \vec{p} = I\vec{\omega}$
$\frac{d\vec{L}}{dt} = \sum_I \vec{\tau}_I$
Rotational version of newton's second law.



$L = pr_\perp$
$p \text{ and } r_\perp$ are always constant so L is conserved

Right hand rule for rotation vectors

$\vec{\omega} = \frac{d\vec{\theta}}{dt}$
$\vec{\theta} = \theta\vec{n}$
$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

Angular acceleration is a measure of both the change in direction and change in magnitude of the angular velocity.

Cross-Product and the Right-Hand Rule

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$
$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
$= (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$
The cross product is a pseudo vector

Angular Momentum

$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}$

$\frac{d\vec{L}}{dt} = \frac{dr}{dr} \times \vec{p} + \vec{r} \times \frac{dp}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \sum_I \vec{F}_I = \sum_I \vec{\tau}_I$
$\vec{v}$ and $\vec{p}$ are parallel so $\vec{v} \times \vec{p} = 0$

For constant L, decreasing I increases ω.

Angular Momentum of a System

$\vec{L} = \sum \vec{\tau}^{ext}$

provided the internal forces all act along the axis connecting the two particles.

Conservation of Angular Momentum

When the next external torque on a system is zero the total angular momentum of the system - which is the vector sum of the individual angular momenta of the constituent particles - remain constant.

Hoop Example

A hoop of radius R and mass M is attached to an axle at the bottom. I tis given a tiny push. What is
a) it's angular velocity at its lowest point?
b) the spend of the lowest point?

Solve using energy conservation
$U_{initial} = MgR$
$K_{initial} = 0$
$U_{final} = -MgR$
$K_{final} = \frac{1}{2}I\omega^2 = \frac{1}{2}(2MR^2)\omega^2 = MR^2\omega^2$
$K_{final} + U_{final} = K_{initial} + U_{initial}$
$MR^2\omega^2 = 0 + MgR + MgR = 2MgR$

$\omega = \sqrt{\frac{2g}{\mathrm{R}}}$

$|\vec{v}| = 2R\omega = \sqrt{8gR}$

Angular Momentum in Non-circular motion

Angular momentum is $r \times p$, so something moving in a straight line will have angular momentum relative to some point that is not in its path of motion.

Precession

$\frac{d\vec{L}}{dt} = \sum \vec{\tau}^{ext}$

External torque not parallel to angular momentum

The axis of rotation traces out a circle. This is called **precession**.

$\frac{d\vec{L}}{dt} = \sum \vec{\tau}^{ext} = \vec{r} \times \vec{F}_g = Mgrn$

$d\vec{L} = |\vec{L}|d\theta n$

$$\frac{d\vec{L}}{dt} = |\vec{L}|\frac{d\theta}{dt}n = Mgrn$$
$$\Omega = \frac{d\theta}{dt} = \frac{Mgr}{|\vec{L}|} = \frac{Mgr}{I_s\omega_s}$$
$$\Omega = \frac{Mgr}{I_s\omega_s}$$
Good if $\Omega \ll \omega_s$

Static Equilibrium

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11:33 AM

Conditions

$$\sum \vec{\tau}^{ext} = 0$$

$$\sum \vec{F}^{ext} = 0$$

When looking at forces: Gravity acts downward from the centre of mass

Centre of Gravity

$$\vec{\tau} = \sum \vec{r} \times \vec{F_g} = \sum \vec{r} \times m\vec{g} = \sum (m\vec{r}) \times \vec{g} = \sum \frac{m\vec{r}}{M} \times \vec{M}g$$

$$\vec{\tau}_g = x_{cm} \times M\vec{g}$$

Equilibrium:

$$\frac{dU}{dx} = 0$$

Potential energy is not changing with change in position. A maximum or minimum in potential energy.

Stable Equilibrium:

$$\frac{d^2U}{dx^2} > 0$$

potential energy is at a local minimum

Unstable Equilibrium:

$$\frac{d^2U}{dx^2} < 0$$

potential energy is at a local maximum

Neutral Equilibrium:

$$\frac{d^2U}{dx^2} = 0$$

Metastable Equilibrium:

Potential energy is a local minimum, but there are lower energies nearby.

The Cosmological Constant Problem

The expansion rate of the universe relative to the cosmological constant is at an unstable equilibrium between contracting and rapid expansion.

Metastable