
Graph Cuts is a Max-Product Algorithm

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Abstract

The maximum a posteriori (MAP) configuration of binary variable models with submodular graph-structured energy functions can be found efficiently and exactly by graph cuts. Max-product belief propagation (MP) has been shown to be suboptimal on this class of energy functions by a canonical counterexample where MP converges to a suboptimal fixed point (Kulesza & Pereira, 2008).

In this work, we show that under a particular scheduling and damping scheme, MP is equivalent to graph cuts, and thus optimal. We explain the apparent contradiction by showing that with proper scheduling and damping, MP always converges to an optimal fixed point. Thus, the canonical counterexample only shows the suboptimality of MP with a particular suboptimal choice of schedule and damping. With proper choices, MP is optimal.

1 Introduction

Maximum a posteriori (MAP) inference in probabilistic graphical models is a fundamental machine learning task with applications to fields such as computer vision and computational biology. There are various algorithms designed to solve MAP problems, each providing different problem-dependent theoretical guarantees and empirical performance. It is often difficult to choose which algorithm to use in a particular application. In some cases, however, there is a “gold-standard” algorithm that clearly outperforms competing algorithms, such as the case of graph cuts for binary submodular problems.¹ A popular and more gen-

¹From here on, we drop “graph-structured” and refer to the energy functions just as binary submodular. Unless explicitly specified otherwise, we always assume that energies are defined on a simple graph.

eral, but also occasionally erratic, algorithm is max-product belief propagation (MP).

Our aim in this work is to establish the precise relationship between MP and graph cuts, namely that graph cuts is a special case of MP. To do so, we map analogous aspects of the algorithms to each other: message scheduling in MP to selecting augmenting paths in graph cuts; passing messages on a chain to pushing flow through an augmenting path; message damping to limiting flow to be the bottleneck capacity of an augmenting path; and letting messages reinforce themselves on a loopy graph to connected components decoding scheme of graph cuts.

This equivalence implies strong statements regarding the optimality of MP on binary submodular energies defined on graphs with arbitrary topology, which may appear to contradict much of what is known about MP—all empirical results showing MP to be suboptimal on binary submodular problems, and the theoretical results of Kulesza and Pereira (2008); Wainwright and Jordan (2008) which show analytically that MP converges to the wrong solution. We analyze this issue in depth and show there is no contradiction, but implicit in the previous analysis and experiments is a suboptimal choice of scheduling and damping, leading the algorithms to converge to bad fixed points. Our results give a more complete characterization of these issues, showing (a) there is always an optimal fixed point for binary submodular energy functions (existence), and (b) with proper scheduling and damping MP can always be made to converge to an optimal fixed point (construction).

The *existence* of the optimal MP fixed point can alternatively be derived as a consequence of the analysis of the zero temperature limit of convexified sum-product in Weiss et al. (2007) along with the well-known fact that the standard linear program relaxation is tight for binary submodular energies. Our proof of the existence of the fixed point, then, is an alternative, more direct proof. However, we believe our *construction* of

the fixed point to be novel and significant, particularly due to the fact that the construction comes from simply running max-product within the standard algorithmic degrees of freedom of damping and scheduling.

Our analysis is significant for many reasons. Two of the most important are as follows. First, it shows that previous constructions of MP fixed points for binary submodular energy functions critically depend on the particular schedule, damping, and initialization. Though there exist suboptimal fixed points, there also always exist optimal fixed points, and with proper care, the bad fixed points can always be avoided. Second, it simplifies the space of MAP inference algorithms, making explicit the connection between two popular and seemingly distinct algorithms. The mapping improves our understanding of message scheduling and gives insight into how graph cut-like algorithms might be developed for more general settings.

2 Background and Notation

We are interested in finding maximizing assignments of distributions $P(\mathbf{x}) \propto e^{-E(\mathbf{x})}$ where $\mathbf{x} = \{x_1, \dots, x_M\} \in \{0, 1\}^M$. We can equivalently seek to minimize the energy E , and for the sake of exposition we choose to present the analysis in terms of energies².

We restrict our attention to submodular energy functions over binary variables. *Graph-structured* energy functions are defined on a simple graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each node is associated with a variable x . Potential functions Θ_i and Θ_{ij} map configurations of individual variables and pairs of variables whose corresponding nodes share an edge, respectively, to real values. We write this energy function as

$$E(\mathbf{x}; \Theta) = \sum_{i \in \mathcal{V}} \Theta_i(x_i) + \sum_{ij \in \mathcal{E}} \Theta_{ij}(x_i, x_j). \quad (1)$$

E is said to be *submodular* if and only if for all $ij \in \mathcal{E}$, $\Theta_{ij}(0, 0) + \Theta_{ij}(1, 1) \leq \Theta_{ij}(0, 1) + \Theta_{ij}(1, 0)$. We use the shorthand notation $[\theta_i^0, \theta_i^1] = [\Theta_i(0), \Theta_i(1)]$.

When E is submodular, it is always possible to represent all pairwise potentials in the canonical form

$$\begin{bmatrix} \Theta_{ij}(0, 0) & \Theta_{ij}(0, 1) \\ \Theta_{ij}(1, 0) & \Theta_{ij}(1, 1) \end{bmatrix} = \begin{bmatrix} 0 & \theta_{ij}^{01} \\ \theta_{ij}^{10} & 0 \end{bmatrix}$$

with $\theta_{ij}^{01}, \theta_{ij}^{10} \geq 0$ without changing the energy of any assignment. Further, by adding a constant κ_i to θ_i^0 and θ_i^1 , we can ensure that one entry of each unary

²This makes “max-product” a bit of a misnomer, since in reality, we will be analyzing min-sum belief propagation. The two are equivalent, however, so we will use “max-product” (MP) throughout, and it should be clear from context when we mean “min-sum”.

potential is non-negative, and that the other is 0. This will not change the optimal assignment. We assume that energies are expressed in this form throughout.³ Note that θ_{ij}^{01} and θ_{ji}^{10} refer to the same quantity.

2.1 Graph Cuts

Graph cuts is a well-known algorithm for minimizing graph-structured binary submodular energy functions. It is known to converge to the optimal solution in low-order polynomial time by transformation into a maximum network flow problem. The energy function is converted into a weighted directed graph $\mathcal{GC} = (\mathcal{V}^{GC}, \mathcal{E}^{GC}, C)$, where C is an edge function that maps each directed edge $(i, j) \in \mathcal{E}^{GC}$ to a non-negative real number representing the initial capacity of the edge. One *non-terminal* node $v_i \in \mathcal{V}^{GC}$ is constructed for each variable $x_i \in \mathcal{V}$, and two *terminal* nodes, a source s , and a sink t , are added to \mathcal{V}^{GC} . Undirected edges in \mathcal{E} are mapped to two directed edges in \mathcal{E}^{GC} , one per direction. The initial capacity of the directed edge $(i, j) \in \mathcal{E}^{GC}$ is set to θ_{ij}^{01} , and the initial capacity of the directed edge $(j, i) \in \mathcal{E}^{GC}$ is set to θ_{ij}^{10} . In addition, directed edges are created from the source node to every non-terminal node, and from every non-terminal node to the sink node. The initial capacity of the terminal edge from s to v_i is set to be θ_i^1 , and the initial capacity of the terminal edge from v_i to t is set to be θ_i^0 . We assume that the energy function has been normalized so that one of the initial terminal edge capacities is 0 for every non-terminal node.

Residual graph: Throughout the course of an augmenting paths-based max-flow algorithm, *residual capacities* (or equivalently hereafter, *capacities*) are maintained for each directed edge. The residual capacity is the amount of flow that can be pushed through an edge either by using unused capacity or by reversing flow that has been pushed in the opposite direction. Given a flow of f_{ij} from v_i to v_j via edge (i, j) and a flow of f_{ji} from v_j to v_i via edge (j, i) , the residual capacity is $r_{ij} = \theta_{ij}^{10} - f_{ij} + f_{ji}$. An *augmenting path* is a path from s to t through the residual graph (the graph with edge capacities given by r) that has positive capacity. We call the minimum residual capacity of any edge along an augmenting path the *bottleneck capacity* for the augmenting path.

Two phases of graph cuts: Augmenting path algorithms for graph cuts proceed in two phases. In Phase 1, flow is pushed through augmenting paths until all source-connected nodes (i.e., those with an edge from source to the node with positive capacity) are separated from all sink-connected nodes (i.e., those with

³See Kolmogorov and Zabih (2002) for a more thorough discussion of representational matters.

an edge to the sink with positive capacity). In Phase 2, to determine assignments, a connected components algorithm is run to find all nodes that are reachable from the source and sink, respectively.

Phase 1 – Reparametrization: The first phase can be viewed as reparameterizing the energy function, moving mass from unary and pairwise potentials to other pairwise potentials and from unary potentials to a constant potential (Kohli & Torr, 2007). The constant potential is a lower bound on the optimum.

We begin by rewriting (1) as

$$E(\mathbf{x}; \Theta) = \sum_{i \in \mathcal{V}} \theta_i^0 (1 - x_i) + \sum_{i \in \mathcal{V}} \theta_i^1 x_i + \sum_{ij \in \mathcal{E}} \theta_{ij}^{01} (1 - x_i) x_j + \sum_{ij \in \mathcal{E}} \theta_{ij}^{10} x_i (1 - x_j) + \theta_{const}, \quad (2)$$

where we added a constant term θ_{const} , initially set to 0, to $E(\mathbf{x}; \Theta)$ without changing the energy.

A reparametrization is a change in potentials from Θ to $\tilde{\Theta}$ such that $E(\mathbf{x}; \Theta) = E(\mathbf{x}; \tilde{\Theta})$ for all assignments \mathbf{x} . Pushing flow corresponds to factoring out a constant, f , from some subset of terms and applying the following algebraic identity to terms from (2):

$$f \cdot [x_1 + (1 - x_1)x_2 + \dots + (1 - x_{N-1})x_N + (1 - x_N)] = f \cdot [x_1(1 - x_2) + \dots + x_{N-1}(1 - x_N) + 1].$$

By ensuring that f is positive (choosing paths that can sustain flow), the constant potential can be made to grow at each iteration. When no paths exist with nonzero f , θ_{const} is the optimal energy value (Ford & Fulkerson, 1956).

In terms of the individual coefficients, pushing flow through a path corresponds to reparameterizing entries of the potentials on an augmenting path:

$$\theta_1^1 := \theta_1^1 - f \quad (3)$$

$$\theta_N^0 := \theta_N^0 - f \quad (4)$$

$$\theta_{ij}^{01} := \theta_{ij}^{01} - f \quad \text{for all } ij \text{ on path} \quad (5)$$

$$\theta_{ij}^{10} := \theta_{ij}^{10} + f \quad \text{for all } ij \text{ on path} \quad (6)$$

$$\theta_{const} := \theta_{const} + f. \quad (7)$$

Phase 2 – Connected Components: After no more paths can be found, most nodes will not be directly connected to the source or the sink by an edge that has positive capacity in the residual graph. In order to determine assignments, information must be propagated from nodes that are directly connected to a terminal via positive capacity edges to other non-terminal nodes. A connected components procedure is run, and any node that is (possibly indirectly) connected to the sink is assigned label 0, and any node that is (possibly indirectly) connected to the source is given label

1. Nodes that are not connected to either terminal can be given an arbitrary label without changing the energy of the configuration, so long as within a connected component the labels are consistent. In practice, terminal-disconnected nodes are given label 0.

2.2 Strict Max-Product Belief Propagation

Strict max-product belief propagation (Strict MP) is an iterative, local, message passing algorithm that can be used to find the optimal MAP configuration of a distribution specified by a tree-structured graphical model. The algorithm can equally be applied to loopy graphs. Employing the energy function notation, the algorithm is usually referred to as min-sum. Using the factor-graph representation (Kschischang et al., 2001), the iterative updates on simple graph-structured energies involve sending messages from factors to variables

$$m_{\Theta_i \rightarrow x_i}(x_i) = \Theta_i(x_i) \quad (8)$$

$$m_{\Theta_{ij} \rightarrow x_j}(x_j) = \min_{x_i} [\Theta_{ij}(x_i, x_j) + m_{x_i \rightarrow \Theta_{ij}}(x_i)] \quad (9)$$

and from variables to factors, $m_{x_i \rightarrow \Theta_{ij}}(x_i) = \sum_{i' \in \mathcal{N}(i) \setminus \{j\}} m_{\Theta_{i' i} \rightarrow x_i}(x_i)$, where $\mathcal{N}(i)$ is the set of neighbor variables of i in \mathcal{G} . In Strict MP, we require that all messages are updated in parallel in each iteration. Assignments are typically decoded from beliefs as $\hat{x}_i = \arg \min_{x_i} b_i(x_i)$, where $b_i(x_i) = \Theta_i(x_i) + \sum_{j \in \mathcal{N}(i)} m_{\Theta_{ji} \rightarrow x_i}(x_i)$. Pairwise beliefs are defined as $b_{ij}(x_i, x_j) = \Theta_{ij}(x_i, x_j) + m_{x_i \rightarrow \Theta_{ij}}(x_i) + m_{x_j \rightarrow \Theta_{ij}}(x_j)$.⁴ We use the notation $\mathcal{FG} = (\mathcal{G}, \Theta, \mathbf{m})$ to contain the factor graph structure, potentials, and messages.

2.3 Max-Product Belief Propagation

In practice, Strict MP does not converge well, so a combination of damping and asynchronous message passing schemes is typically used. Message damping with strength γ defines a new message to be γ times the old message plus $1 - \gamma$ times the newly computed message. Thus, MP is actually a family of algorithms. We formally define the family as follows:

Definition 1 (Max-Product Belief Propagation (MP)). *MP is a message passing algorithm that computes messages as in Section 2.2. Messages may be initialized arbitrarily, scheduled in any (possibly dynamic) ordering, and damped in any (possibly dynamic) manner, so long as the fixed points of the algorithm are the same as the fixed points of Strict MP.*

We believe this definition to be broad enough to contain most algorithms that are considered to be max-

⁴Note that we only need message values to be correct up to a constant, so it is common practice to normalize messages and beliefs so that the minimum entry in a message or belief vector is 0.

product, yet restrictive enough to exclude e.g., fundamentally different linear program-based algorithms like tree-reweighted max-product.

Instances of MP: There has been much work on scheduling messages, notably including recent work on dynamic asynchronous scheduling (Elidan et al., 2006; Sutton & McCallum, 2007), which shows that adaptive schedules can lead to improved convergence. An equally important practical concern is message damping. Dueck (2010), for example, discusses the importance of damping in detail with respect to using MP for exemplar-based clustering (affinity propagation). Our definition of MP includes these examples.

2.4 Augmenting Path = Chain Subgraph

Our scheduling makes use of dynamically chosen chains, which are analogous to augmenting paths. Formally, an augmenting path is a sequence of nodes

$$\mathcal{T} = (s, v_{\mathcal{T}_1}, v_{\mathcal{T}_2}, \dots, v_{\mathcal{T}_{n-1}}, v_{\mathcal{T}_n}, t) \quad (10)$$

where a nonzero amount of flow can be pushed through the path. It will be useful to refer to $\mathcal{E}^{GC}(\mathcal{T})$ as the set of edges encountered along \mathcal{T} .

Let $\mathbf{x}_{\mathcal{T}} \subseteq \mathbf{x}$ be the variables corresponding to non-terminal nodes in \mathcal{T} . The potentials corresponding to the edges $\mathcal{E}^{GC}(\mathcal{T})$ and the entries of these potentials are denoted by $\Theta_{\mathcal{T}}$ and a subset of potential values $\theta_{\mathcal{T}}$. Formally,

$$\begin{aligned} \mathbf{x}_{\mathcal{T}} &= \{x_{\mathcal{T}_1}, \dots, x_{\mathcal{T}_n}\} \\ \Theta_{\mathcal{T}} &= \Theta_{\mathcal{T}_1} \cup \{\Theta_{\mathcal{T}_i, \mathcal{T}_{i+1}}\}_{i=1}^{n-1} \cup \Theta_{\mathcal{T}_n} \\ \theta_{\mathcal{T}} &= \theta_{\mathcal{T}_1}^1 \cup \{\theta_{\mathcal{T}_i, \mathcal{T}_{i+1}}^{01}\}_{i=1}^{n-1} \cup \theta_{\mathcal{T}_n}^0. \end{aligned} \quad (11)$$

Note that there are only two unary potentials on a chain corresponding to an augmenting path, which correspond to terminal edges in $\mathcal{E}^{GC}(\mathcal{T})$. It will be useful to map edges in $\mathcal{E}^{GC}(\mathcal{T})$ to edges in the factor graph representation. We use $\mathcal{E}^{FG}(\mathcal{T})$ to denote all edges in \mathcal{FG} between potentials in $\Theta_{\mathcal{T}}$ and variables in $\mathbf{x}_{\mathcal{T}}$.

As an example, an augmenting path $\mathcal{T} = (s, v_i, v_j, v_k, t)$ would be mapped to $\mathbf{x}_{\mathcal{T}} = \{x_i, x_j, x_k\}$, $\Theta_{\mathcal{T}} = \{\Theta_i, \Theta_{ij}, \Theta_{jk}, \Theta_k\}$, and $\theta_{\mathcal{T}} = \{\theta_i^1, \theta_{ij}^{01}, \theta_{jk}^{01}, \theta_k^0\}$ in \mathcal{FG} .

3 Augmenting Paths Max-Product

In this section, we present Augmenting Paths Max-Product (APMP), a particular scheduling and damping scheme for MP, that—like graph cuts—has two phases. At each iteration of the first phase, the scheduler returns a chain on which to pass messages. Holding all other messages fixed, messages are passed forward and backward on the chain, with standard message normalization applied, to complete the iteration.

Adaptive message damping applied to messages leaving unary factors (described below) ensures that messages propagate across the chain in a particularly structured way. Messages leaving pairwise factors and messages from variables to factors are not damped. Phase 1 terminates when the scheduler indicates there are no more messages to send, then in Phase 2, Strict MP is run until convergence (we guarantee it will converge). The full APMP is given in Algorithm 1.

3.1 Phase 1: Path Scheduling and Damping

For convenience, we use the convention that chains go from “left” to “right,” where the left-most variable on a chain corresponding to an augmenting path is $x_{\mathcal{T}_1}$, and the right-most variable is $x_{\mathcal{T}_n}$. In these terms, a *forward* pass is from left to right, and a *backward* pass is from right to left.

Suppose that at the end of iteration $t-1$, the outgoing message from the unary factor at the start of the chain used in iteration t , $\mathcal{T} = \mathcal{T}(t)$, is $m_{\Theta_{\mathcal{T}_1} \rightarrow x_{\mathcal{T}_1}}^{(t-1)}(x_{\mathcal{T}_1}) = (0, b)^T$. The unary factor must have potential $(0, d)^T$ for some $d \geq b$ to have previously produced a message of $(0, b)^T$. This implies that passing another message will leave the structure of the potential unchanged and will simply increment the b entry by some amount. If the factor increments its outgoing message in such a way as to guarantee that $b + f \leq \theta_{\beta}^{01}$ for all steps along \mathcal{T} , the messages as shown in Fig. 1 will be computed (see Corollary 1 below). Later analysis will explain why this is desirable. Accounting for message normalization, this can be accomplished by limiting the change $\Delta m(\cdot) = m^{(t)}(\cdot) - m^{(t-1)}(\cdot)$ in outgoing message from the first unary variable on a path to be $\Delta m_{\Theta_{\mathcal{T}_1} \rightarrow x_{\mathcal{T}_1}}(x_{\mathcal{T}_1}) = (0, f)^T$. We also constrain the increment in the backward direction to equal the increment in the forward direction.

Under the constraints, the largest f we can choose is

$$f = \min \left(\theta_{\mathcal{T}_n}^0 - m_{\Theta_{\mathcal{T}_n} \rightarrow x_{\mathcal{T}_n}}^{(t-1)}(0), \theta_{\mathcal{T}_1}^1 - m_{\Theta_{\mathcal{T}_1} \rightarrow x_{\mathcal{T}_1}}^{(t-1)}(1), \min_{ij | \Theta_{ij} \in \Theta_{\mathcal{T}}} \left[\theta_{ij}^{01} - m_{x_i \rightarrow \Theta_{ij}}^{(t-1)}(1) + m_{x_i \rightarrow \Theta_{ij}}^{(t-1)}(0) \right] \right) \quad (12)$$

which is exactly the bottleneck capacity of the corresponding augmenting path. In other words, limiting the change in outgoing message value from unary factors to be the bottleneck capacity of the augmenting path will ensure that message increments propagate through a chain unmodified—that is, when one variable on the path receives an increment of $(0, f)^T$ as x_i does in Fig. 1(b), it will propagate the same increment to the next variable on the path (x_j), as in Fig. 1(c). This will be proved in Lemma 1.

Damping: The key, simple idea to the damping scheme is that we want unary factors to increment

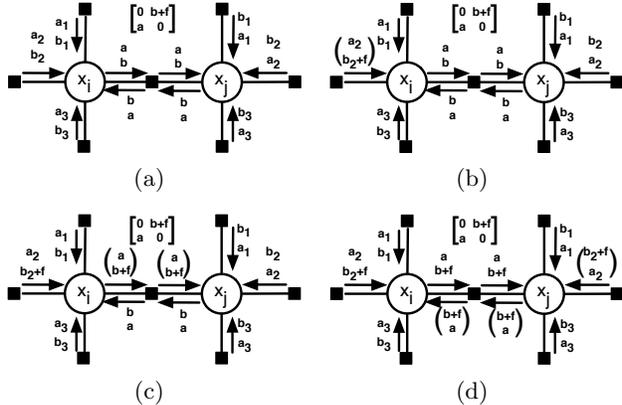


Figure 1: The pairwise potential is in square brackets. Only messages changed relative to previous subfigure are shown in parentheses. Let $a = a_1 + a_2 + a_3$ and $b = b_1 + b_2 + b_3$. (a) Start of iteration. The capacity of the edge ij is f . (b) Inductive assumption that each node on the augmenting path will receive a message increment of $(0, f)^T$ from the left-neighbor. (c) Passing messages completes the inductive step where x_j receives an incremented message. (d) Similarly, receiving an incremented message in the backwards direction then updating messages from j to i completes the iteration.

their messages by the bottleneck capacity of the current chain. The necessary value of f can be achieved by damping the outgoing message from the first and last unary potential in each chain. For the first unary factor, if we previously have message $(0, b)^T$ on the edge, then to produce message $(0, b + f)^T$, we can apply damping $\gamma_{\mathcal{T}_1}(t)$ where $\gamma_{\mathcal{T}_1}(t)$ is chosen by solving the equation:

$$\gamma_{\mathcal{T}_1}(t) \cdot b + (1 - \gamma_{\mathcal{T}_1}(t)) \cdot \theta_{\mathcal{T}_1}^1 = f + b, \quad (13)$$

yielding $\gamma_{\mathcal{T}_1}(t) = \frac{\theta_{\mathcal{T}_1}^1 - f - b}{\theta_{\mathcal{T}_1}^1 - b}$. The algorithm never chooses an augmenting path with 0 capacity, so we will never get a zero denominator.

Analogous damping is applied in the opposite direction. This damping will produce the same message increments in the forward and backward direction, which will be a key property used in later analysis.

SCHEDULE Implementation: The combination of potentials and messages on the edges contain the same information as the residual capacities in the graph cuts residual graph. Using this equivalence, any algorithm for finding augmenting paths in the graph cuts setting can be used to find chains to pass messages on for MP. The terms being minimized over in Eq. (12) are residual capacities, which are defined in terms of messages and potentials. Specifically, at the end of any iteration of the MP algorithm described in the next section, the residual capacities of edges between non-terminal nodes can be constructed from

Algorithm 1 Augmenting Paths Max-Product

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 $f(0) \leftarrow \infty$ 
 $t \leftarrow 0$ 
while  $f(t) > 0$  do {Phase 1}
   $\mathcal{T}(t), f(t) \leftarrow \text{SCHEDULE}(\mathcal{FG}(t))$ 
   $\gamma_{\mathcal{T}_1}(t), \gamma_{\mathcal{T}_N}(t) \leftarrow \text{DAMPING}(\mathcal{FG}(t), \mathcal{T}(t), f(t))$ 
   $\mathcal{FG}(t+1) \leftarrow \text{MP}(\mathcal{FG}(t), \mathcal{E}^{FG}(\mathcal{T}(t)), \gamma_{\mathcal{T}_1}(t), \gamma_{\mathcal{T}_N}(t))$ 
   $t \leftarrow t + 1$ 
end while
while not converged do {Phase 2}
  Run Strict MP( $\mathcal{FG}$ )
end while

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potentials and current messages m as follows:

$$r_{ij} = \theta_{ij}^{01} - m_{x_i \rightarrow \Theta_{ij}}(1) + m_{x_i \rightarrow \Theta_{ij}}(0). \quad (14)$$

The difference in messages $m_{x_i \rightarrow \Theta_{ij}}(1) - m_{x_i \rightarrow \Theta_{ij}}(0)$ is then equivalent to the difference in flows $f_{ij} - f_{ji}$ in the graph cuts formulation. The residual capacities for terminal edges can be constructed from messages and potentials related to unary factors:

$$r_{si} = \theta_i^1 - m_{\Theta_i \rightarrow x_i}(1) \quad (15)$$

$$r_{it} = \theta_i^0 - m_{\Theta_i \rightarrow x_i}(0). \quad (16)$$

3.2 Phase 2: Strict MP

When the scheduler cannot find a positive-capacity path on which to pass messages, it switches to its second phase and passes all messages at all iterations, with no damping i.e., Strict MP. It continues until reaching a fixed point. (We will prove in Section 5 that if potentials are finite, it will always converge). The choice of Strict MP is not essential. The same result holds for any reasonable scheduling of messages.

4 APMP Phase 1 Analysis

Assume that at the beginning of iteration t , each variable $x_i \in \mathbf{x}_{\mathcal{T}(t)}$ has received an incoming message from its left-neighboring factor Θ_α , $m_{\Theta_\alpha \rightarrow x_i}(x_i) = (a, b)^T$. We want to show that when each variable receives an incremented message, $(a, b + f)^T$, the increment $(0, f)^T$ —up to a normalizing constant—will be propagated through the variable and the next factor, Θ_{ij} , to the next variable on the path.

The pairwise potential at the next pairwise factor along the chain will be Θ_{ij} . The damping scheme ensures that $\theta_{ij}^{10} \geq a$ and $\theta_{ij}^{01} \geq b + f$. Lemma 1 shows that under these conditions, factors will propagate messages unchanged.

Lemma 1 (Message-Preserving Factors). *When passing standard MP messages with the factors as above,*

$\theta_{ij}^{10} \geq a$, and $\theta_{ij}^{01} \geq b + f$, the outgoing factor-to-variable message is equal to the incoming variable-to-factor message i.e. $m_{\Theta_{ij} \rightarrow x_j} = m_{x_i \rightarrow \Theta_{ij}}$ and $m_{\Theta_{ij} \rightarrow x_i} = m_{x_j \rightarrow \Theta_{ij}}$.

Proof. This follows from inserting values in to the message updates. See supplementary materials.⁵ \square

Lemma 1 allows us to easily compute the values of all messages passed during the execution of Phase 1 of APMP and thus the change in beliefs at each variable.

Corollary 1 (Structured Belief Changes). *Before and after an iteration t of Phase 1 APMP, the change in unary belief at each variable in $\mathbf{x}_{\mathcal{T}(t)}$ will be $(0, 0)^T$, up to a constant normalization.*

Proof. Under the APMP damping scheme, the change in message from the first unary factor in $\mathcal{T}(t)$ will be $(0, f)^T$, and the change in message from the last unary factor in $\mathcal{T}(t)$ will be $(f, 0)^T$ where f is as defined in Eq. (12). Without message normalization, these messages will propagate unchanged through the pairwise factors in $\mathcal{T}(t)$ by Lemma 1. Variable to factor messages will also propagate the change unaltered.

Message normalization subtracts a positive constant $c = \min(a, b + f)$ from both entries in a message vector. Existing message values will only get smaller, so the message-preserving property of factors will be maintained. Thus, each variable will receive a message change of $(-c_L, f - c_L)^T$ from the left and a message change of $(f - c_R, -c_R)^T$ from the right. The total change in belief is then $(f - c_L - c_R, f - c_L - c_R)^T$, which completes the proof. \square

Fig. 1 illustrates the structured message changes.

4.1 Message Free View

Here, using the reparameterization view of max-product from Wainwright et al. (2004), we analyze the equivalent “message-free” version of the first phase of APMP—one that directly modifies potentials rather than sending messages. Corollary 1 shows that all messages in APMP can be analytically computed. We then use these message values to compute the change in parameterization due to the messages at each iteration. The main result in this section is that this change in parameterization is exactly equivalent to that performed by graph cuts.

An important identity, which is a special case of the junction tree representation (Wainwright et al., 2004),

⁵Available in Appendix of (Tarlow et al., 2011).

states that we can equivalently view MP on a tree as reparameterizing Θ according to beliefs b :

$$\begin{aligned} & \sum_{i \in \mathcal{V}} \tilde{\Theta}_i(x_i) + \sum_{ij \in \mathcal{E}} \tilde{\Theta}_{ij}(x_i, x_j) \\ &= \sum_{i \in \mathcal{V}} b_i(x_i) + \sum_{ij \in \mathcal{E}} [b_{ij}(x_i, x_j) - b_i(x_i) - b_j(x_j)] \end{aligned} \quad (17)$$

where $\tilde{\Theta}$ is a reparameterization i.e. $E(\mathbf{x}; \Theta) = E(\mathbf{x}; \tilde{\Theta}) \forall \mathbf{x}$. At any point, we can stop and calculate current beliefs and apply the reparameterization (i.e., replace original potentials with reparameterized potentials and set all messages to 0). This holds even if factor to variable messages are damped.

“Used” and “Remainder” Energies: To analyze reparameterizations, we begin by splitting E into two components: a part that has been used so far, and a remainder part. The used part is defined as the energy function that would have produced the current messages if no damping were used. The remainder is everything else. Since damping is only applied at unary potentials, we assign all pairwise potentials to the used component: $\Theta_{ij}^{(U)}(x_i, x_j) = \Theta_{ij}(x_i, x_j)$. The used component of unary potentials is defined as the current message leaving the factor: $\Theta_i^{(U)}(x_i) = m_{\Theta_i \rightarrow x_i}(x_i)$. Consequently, the remainder pairwise potentials are zero, and the remainder unary potentials are $\Theta_i^{(R)}(x_i) = \Theta_i(x_i) - \Theta_i^{(U)}(x_i)$. We apply the message-free interpretation to get a reparameterized version of $E(\mathbf{x}; \Theta^{(U)})$ then add in the remainder component of the energy unmodified.

Analyzing Beliefs: The parameterization in Eq. (17) depends on unary and pairwise beliefs. We consider the change in beliefs from that defined by messages at the start of an iteration of APMP to that defined by messages at the end of an iteration. There are three cases to consider.

Case 1 Variables and potentials not in or neighboring $\mathbf{x}_{\mathcal{T}(t)}$ will not have any potentials or adjacent beliefs changed, so the reparameterization will not change.

Case 2 Potentials neighboring $x \in \mathbf{x}_{\mathcal{T}(t)}$ but not in $\mathcal{E}^{FG}(\mathcal{T}(t))$ could possibly be affected by the belief at a variable in $\mathbf{x}_{\mathcal{T}(t)}$, since the belief at an edge depends on the beliefs at variables at each of its endpoints. However, by Corollary 1, after applying standard normalization, this belief does not change after a forward and backward pass of messages, so overall they are unaltered.

Case 3 We now consider the belief of potentials $\Theta_{ij}^{(U)} \in \Theta_{\mathcal{T}(t)}^{(U)}$. This is the most involved case, where the parameterization does change, but it does so in a very structured way.

Lemma 2. *The change in pairwise belief on the current augmenting path $\mathcal{T}(t)$ from the beginning of an iteration t to the end of an iteration is*

$$\Delta b_{ij}(x_i, x_j) = \begin{bmatrix} 0 & -f \\ +f & 0 \end{bmatrix} + f \quad ij \in \mathcal{T}(t). \quad (18)$$

Proof. This follows from applying the standard reparameterization (17) to messages before and after an iteration of Phase 1 APMP. See supplementary material for details. \square

Unary Reparameterizations: As discussed above, the used part of the energy is grouped with messages and reparameterized as standard, while the remainder part is left unchanged and is added in at the end:

$$\tilde{\Theta}_i(x_i) = b_i(x_i; \Theta^{(U)}) + \Theta_i^{(R)}(x_i). \quad (19)$$

Parameterizations defined in this way are proper reparameterizations of the original energy function.

Lemma 3. *The changes in parameterization during iteration t of Phase 1 APMP at variables $x_{\mathcal{T}_1}$ and $x_{\mathcal{T}_N}$ respectively are $(0, -f)^T$ and $(-f, 0)^T$. The change in all other unary potentials is $(0, 0)^T$.*

Proof. The Phase 1 damping scheme ensures that the message leaving the first factor on $\mathcal{T} = \mathcal{T}(t)$ is incremented by $(0, f)^T$. This means that $\Theta_{\mathcal{T}_1}^{(U)}(x_{\mathcal{T}_1})$ is incremented by $(0, f)^T$, so $\Theta_{\mathcal{T}_1}^{(R)}(x_{\mathcal{T}_1})$ is decremented by $(0, f)^T$ to maintain the decomposition constraint. Unary beliefs do not change, so the new parameterization is then $\Delta\Theta_{\mathcal{T}_1}(x_{\mathcal{T}_1}) = \Delta b_{x_{\mathcal{T}_1}}(x_{\mathcal{T}_1}) + \Delta\Theta_{x_{\mathcal{T}_1}}^{(R)}(x_{\mathcal{T}_1}) = (0, -f)^T$. A similar argument holds for $\Delta\Theta_{x_{\mathcal{T}_N}}$.

The only unary potentials involved in an iteration of APMP are endpoints of $\mathcal{T}(t)$, so no other $\Theta^{(R)}$ values will change. The total change in parameterization at non-endpoint unary potentials is then $(0, 0)^T$. \square

Full Reparameterizations: Finally, we are ready to prove our first main result.

Theorem 1. *The difference between two reparameterizations induced by the messages in Phase 1 APMP, before and after passing messages on the chain corresponding to augmenting path $\mathcal{T}(t)$, is equal to the difference between reparameterizations of graph cuts before and after pushing flow through the equivalent augmenting path.*

Proof. The change in unary parameterization is given by Lemma 3. The change in pairwise parameterization is $\Delta\Theta_{ij}(x_i, x_j) = \Delta b_{ij}(x_i, x_j) - \Delta b_i(x_i) - \Delta b_j(x_j) = \Delta b_{ij}(x_i, x_j)$, where $\Delta b_{ij}(x_i, x_j)$ is given by Lemma 2.

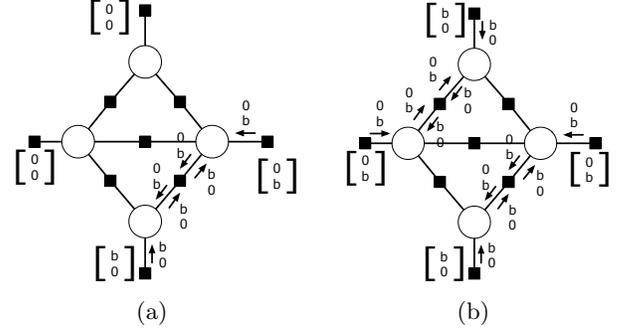


Figure 2: Illustration of first two “used” energies and associated fixed points constructed by APMP on the problem from Fig. 4. Potentials $\Theta^{(U)}$ are given in square brackets. Messages have no parentheses. Edges with messages equal to $(0, 0)$, and pairwise potentials, which are assumed strong, are not drawn to reduce clutter. (a) First energy. (b) Second energy. Note that both sets of messages give a max-product fixed point for the respective energy.

Putting the two together, we see that the changes in potential entries are exactly the same as those performed by graph cuts in (3) - (7):

$$\Delta\Theta_{\mathcal{T}_1}(x_{\mathcal{T}_1}) = \begin{bmatrix} 0 \\ -f \end{bmatrix} \quad (20)$$

$$\Delta\Theta_{\mathcal{T}_N}(x_{\mathcal{T}_N}) = \begin{bmatrix} -f \\ 0 \end{bmatrix} \quad (21)$$

$$\Delta\Theta_{\mathcal{T}_i}(x_{\mathcal{T}_i}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad i \neq 1, N \quad (22)$$

$$\Delta\Theta_{\mathcal{T}_i, \mathcal{T}_j}(x_{\mathcal{T}_i}, x_{\mathcal{T}_j}) = \begin{bmatrix} 0 & -f \\ +f & 0 \end{bmatrix} \quad \Theta_{\mathcal{T}_i, \mathcal{T}_j} \in \Theta_{\mathcal{T}(t)}. \quad (23)$$

This completes the proof of equivalence between Phase 1 APMP and Phase 1 of graph cuts. \square

Fig. 2 shows $\Theta^{(U)}$ and Phase 1 APMP messages from running two iterations on the example from Fig. 4.

5 APMP Phase 2 Analysis

We now consider the second phase of APMP. Throughout this section, we will work with the reparameterized energy that results from applying the equivalent reparameterization view of MP at the end of APMP Phase 1—that is, we have applied the reparameterization to potentials, and reset messages to 0. All results could equivalently be shown by working with original potentials and messages at the end of Phase 1, but the chosen presentation is simpler.

At this point, there are no paths between a unary potential of the form $(0, a)^T, a > 0$ and a unary potential of the form $(b, 0)^T, b > 0$ with nonzero capacity. Practically, as in graph cuts, breadth first search could be used at this point to find an optimal assignment. However, we will show that running Strict MP leads to

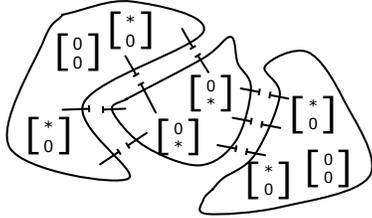


Figure 3: A decomposition into three homogeneous islands. The left-most and right-most islands have beliefs of the form $(*, 0)^T$, while the middle has beliefs of the form $(0, *)^T$. Non-touching cross-island lines indicate that messages passed from one island to another will be identically 0 after any number of internal iterations of message passing within an island.

convergence to an optimal fixed point. This proves the existence of an optimal MP fixed point for any binary submodular energy and gives a constructive algorithm (APMP) for finding it.

Our analysis relies upon the reparameterization at the end of Phase 1 defining what we term *homogeneous islands* of variables.

Definition 2. A homogeneous island is a set of variables \mathbf{x}_H connected by positive capacity edges such that each variable $x_i \in \mathbf{x}_H$ has normalized beliefs $(\alpha_i, \beta_i)^T$ where either $\forall i. \alpha_i = 0$ or $\forall i. \beta_i = 0$. Further, after any number of rounds of message passing amongst variables within the island, any message $m_{\Theta_{i_j} \rightarrow x_j}(x_j)$ from a variable inside the island x_i to a variable outside the island x_j is identically 0, and vice versa.

Define variables inside a homogeneous island with nonzero unary potentials to be the *seeds* of the island. Fig. 3 shows an illustration of homogeneous islands. Homogeneous islands allow us to analyze messages independently within each island, without considering cross-island messages.

Lemma 4. At the end of Phase 1, the messages of APMP define a collection of homogeneous islands.

Proof. This is essentially equivalent to how the max-flow min-cut theorem proves that the Ford-Fulkerson algorithm has found a minimum cut when no more augmenting paths can be found. The boundaries between islands are the locations of the cuts. See supplementary material. \square

Lemma 4 lets us analyze Strict MP independently within each homogeneous island, because it shows that no non-zero messages will cross island boundaries. Thus, we can prove that internally, each island will reach a MP fixed point:

Lemma 5 (Internal Convergence). *Iterating Strict MP inside a homogeneous island of the form $(\alpha, 0)^T$ (or $(0, \beta)^T$) will lead to a fixed point where beliefs are*

of the form $(\alpha'_i, 0)^T, \alpha'_i \geq 0$ (or $(0, \beta'_i)^T, \beta'_i \geq 0$) at each variable in the island.

Proof. (Sketch) We prove the case where the unary potentials inside the island have form $(\alpha_i, 0)^T$. The case where they have form $(0, \beta_i)^T$ is entirely analogous.

At the beginning of Phase 2, all unary potentials will be of the form $(\alpha, 0)^T, \alpha \geq 0$. By the positive-capacity edge connectivity of homogeneous islands property, messages of the form $(\alpha, 0)^T, \alpha > 0$ will eventually be propagated to all variables in the island by Strict MP. In addition, messages can only reinforce (and not cancel) each other. For example, in a single loop homogeneous island, messages will cycle around the loop, getting larger as unary potentials are added to incoming messages and passed around the loop. Messages will only stop growing when the the variable-to-factor messages become stronger than the pairwise potential.

On acyclic island structures, Strict MP will obviously converge. On loopy graphs, messages will be monotonically increasing until they are capped by the pairwise potentials (i.e., the pairwise potential is *saturated*). The rate of message increase is lower bounded by some constant (that depends on the strength of unary potentials and size of loops in the island graph, which are fixed), so the sequence will converge when all pairwise potentials are saturated. \square

We can now prove our second main result:

Theorem 2 (Convergence and Optimality of APMP Fixed Point). *APMP converges to an optimal fixed point on binary submodular energy functions.*

Proof. After running Phase 2 of APMP, Lemma 5 shows that each homogeneous island will converge to a fixed point where beliefs at all variables in the island can be decoded to give the same assignment as the initial seed of the island. This is the same assignment as the optimal graph cuts-style connected components decoding would yield. Cross-island messages are all zero, and if a variable is not in an island, it has zero potential, sends and receives all zero messages, and can be assigned arbitrarily. Thus, we are globally at a MP fixed point, and beliefs can be decoded at each variable to give the optimal assignment. \square

Finally, we return to the canonical example used to show the suboptimality of MP on binary submodular energies. The potentials and messages defining a suboptimal fixed point, which is reached by certain suboptimal scheduling and damping schemes, are illustrated in Fig. 4 (a). If, however, we run APMP, Phase 1 ends with the messages shown in Fig. 2(b) and Phase 2 converges to the fixed point shown in Fig. 4 (b). Decoding

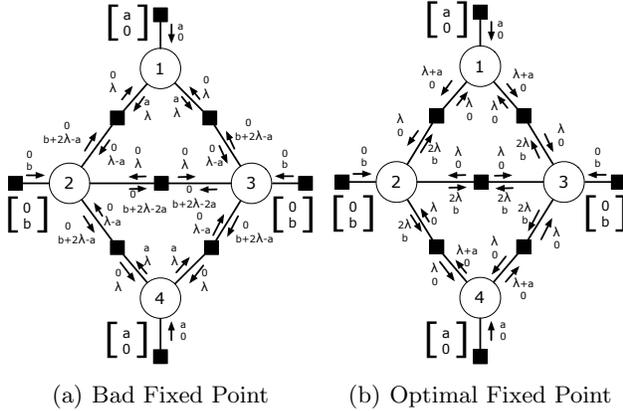


Figure 4: The canonical counterexample used to show that MP is suboptimal on binary submodular energy functions. Potentials are given in square brackets. Messages have no parentheses. Pairwise potentials are symmetric with strength λ , and $\lambda > 2a > 2b$, making the optimal assignment $(1, 1, 1, 1)$. (a) The previously analyzed fixed point. Beliefs at 1 and 4 are $(a, 2\lambda)^T$, and at 2 and 3 are $(0, b + 3\lambda - 2a)^T$, which gives a suboptimal assignment. (b) We introduce a second fixed point. Beliefs at 1 and 4 are $(2\lambda + a, 0)^T$, and at 2 and 3 are $(3\lambda, b)^T$, which gives the optimal assignment. Our new scheduling and damping scheme guarantees MP will find an optimal fixed point like this for any binary submodular energy function.

beliefs from the messages in Fig. 4 (b) indeed gives the optimal assignment of $(1, 1, 1, 1)$.

6 Convergence Guarantees

Previously several variants of message passing algorithms for MAP inference have been theoretically analyzed. There are generally two classes of results: (a) guarantees about the optimality or partial optimality of solutions, assuming that the algorithm has converged to a fixed point; and (b) guarantees about the monotonicity of the updates with respect to some bound and whether the algorithm will converge.

Notable optimality guarantees exist for TRW algorithms (Kolmogorov & Wainwright, 2005) and MPLP (Globerson & Jaakkola, 2008). Kolmogorov and Wainwright (2005) prove that fixed points of TRW satisfying a *weak tree agreement (WTA)* condition yield optimal solutions to binary submodular problems. Globerson and Jaakkola (2008) show that if MPLP converges to beliefs with unique optimizing values, then the solution is optimal.

Convergence guarantees for message passing algorithms are generally significantly weaker. MPLP is a coordinate ascent algorithm so is guaranteed to converge; however, in general it can get stuck at suboptimal points where no improvement is possible via updating the blocks used by the algorithm. Somewhat

similarly, TRW-S is guaranteed not to decrease a lower bound. In the limit where the temperature goes to 0, convexified *sum-product* is guaranteed to converge to a solution of the standard linear program relaxation, but this is not numerically practical to implement (Weiss et al., 2007). However, even for binary submodular energies, we are unaware of results that guarantee convergence for convexified belief propagation, MPLP, or TRW-S in polynomial time.

Our analysis reveals schedules and message passing updates that guarantee convergence in low order polynomial time to a state where an optimal assignment can be decoded for binary submodular problems. This follows directly from analysis of max-flow algorithms. By using shortest augmenting paths, the Edmonds-Karp algorithm converges in $O(|\mathcal{V}||\mathcal{E}|^2)$ time (Edmonds & Karp, 1972). Analysis of the convergence time of Phase 2 is slightly more involved. Given an island with a large single loop of M variables, with strong pairwise potentials (say strength λ) and only one small nonzero unary potential, say $(\alpha, 0)^T$, convergence will take on the order of $\frac{M \cdot \lambda}{\alpha}$ time, which could be large. In practice, though, we can reach the same fixed point by modifying nonzero unary potentials to be $(\lambda, 0)^T$, in which case convergence will take just order M time. Interestingly, this modification causes Strict MP to become equivalent to the connected components algorithm used by graph cuts to decode solutions.

7 Related Work

Close relationships exist among many MAP inference algorithms. Here we discuss the relationships between some of the more notable and similar algorithms. APMP is closely related to dual block coordinate ascent algorithms discussed in (Sontag & Jaakkola, 2009)—Phase 1 of APMP can be seen as block coordinate ascent in the same dual. Interestingly, even though both are optimal ascent steps, APMP reparameterizations are not identical to those of the sequential tree-block coordinate ascent algorithm in (Sontag & Jaakkola, 2009) when applied to the same chain.

Graph cuts is also highly related to the Augmenting DAG algorithm (Werner, 2007). Augmenting DAGs are more general constructs than augmenting paths, so with a proper choice of schedule, the Augmenting DAG algorithm could also implement graph cuts. Our work follows in the spirit of RBP (Elidan et al., 2006), in that we consider dynamic schedules for MP. RBP is more general; our analysis is much stronger.

Finally, our work is also related to the COMPOSE framework of Duchi et al. (2007). In COMPOSE, special purpose algorithms are used to compute MP messages for certain combinatorial-structured subgraphs,

including binary submodular ones. We show here that special purpose algorithms are not needed: the internal graph cuts algorithm can be implemented purely in terms of max-product. Given a problem that contains a graph cut subproblem but also has other high order or nonsubmodular potentials, our work shows how to interleave solving the graph cuts problem and passing messages elsewhere in the graph.

8 Conclusions

While the proof of equivalence to graph cuts is moderately involved, the APMP algorithm is a simple special case of MP. The analysis technique is novel: rather than relying on the computation tree model for analysis, we directly mapped the operations being performed by the algorithm to a known combinatorial algorithm. It would be interesting to consider whether there are other cases where the MP execution might be mapped directly to a combinatorial algorithm.

We have proven strong statements about MP fixed points on binary submodular energies. The analysis has a similar flavor to that of Weiss (2000), in that we construct fixed points where optimal assignments can be decoded, but where the magnitudes of the beliefs do not (generally) correspond to meaningful quantities. The strategy of isolating subgraphs might apply more broadly. For example, if we could isolate single loop structures as we isolate homogeneous islands in Phase 1, a second phase might then be used to find optimal solutions in non-homogeneous, loopy regions. An alternate view of Phase 1 is that it is an intelligent initialization of messages for Strict MP in Phase 2. In this light, our results show that initialization can provably determine whether MP is suboptimal or optimal in the case of binary submodular energy functions.

The connection to graph cuts simplifies the space of MAP algorithms. There are now precise mappings between ideas from graph cuts and ideas from belief propagation (e.g., augmenting paths to scheduling).

A broad, interesting direction of future work is to further investigate how insights related to graph cuts can be used to improve inference in the more general settings of multilabel, nonsubmodular, and high order energy functions. At a high level, APMP separates the concerns of improving the dual objective (Phase 1) from concerns regarding decoding solutions (Phase 2). In loopy MP, this delays overcounting of messages until it is safe to do so. We believe that this and other concepts presented here will generalize. We are currently exploring the non-binary, non-submodular case.

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A Supplementary Material Accompanying “Graph Cuts is a Max-Product Algorithm”

We provide additional details omitted from the main paper due to space limitations.

Lemma 1 (Message-Preserving Factors) *When passing standard MP messages with the factors as above, $\theta_{ij}^{10} \geq a$, and $\theta_{ij}^{01} \geq b + f$, the outgoing factor-to-variable message is equal to the incoming variable-to-factor message i.e. $m_{\Theta_{ij} \rightarrow x_j} = m_{x_i \rightarrow \Theta_{ij}}$ and $m_{\Theta_{ij} \rightarrow x_i} = m_{x_j \rightarrow \Theta_{ij}}$.*

Proof. This follows simply from plugging in values to the message updates. We show the i to j direction.

$$\begin{aligned} m_{x_i \rightarrow \Theta_{ij}}(x_i) &= \begin{pmatrix} a \\ b + f \end{pmatrix} \\ m_{\Theta_{ij} \rightarrow x_j}(x_j) &= \min_{x_i} [\Theta_{ij}(x_i, x_j) + m_{x_i \rightarrow \Theta_{ij}}(x_i)] \\ &= \min_{x_i} [\theta_{ij}^{10} \cdot x_i(1 - x_j) + \theta_{ij}^{01} \cdot (1 - x_i)x_j \\ &\quad + a \cdot (1 - x_i) + (b + f) \cdot x_i] \\ m_{\Theta_{ij} \rightarrow x_j}(0) &= \min(\theta_{ij}^{10} + b + f, a) = a \\ m_{\Theta_{ij} \rightarrow x_j}(1) &= \min(a + \theta_{ij}^{01}, b + f) = b + f \\ m_{\Theta_{ij} \rightarrow x_j}(x_j) &= \begin{pmatrix} a \\ b + f \end{pmatrix} \end{aligned}$$

where the final evaluation of the min functions used the assumptions that $\theta_{ij}^{10} \geq a$ and $\theta_{ij}^{01} \geq b + f$. \square

Lemma 2 *The change in pairwise belief on the current augmenting path $\mathcal{T}(t)$ from the beginning of an iteration t to the end of an iteration is*

$$\Delta b_{ij}(x_i, x_j) = \begin{bmatrix} 0 & -f \\ +f & 0 \end{bmatrix} + f \quad ij \in \mathcal{T}(t). \quad (24)$$

Proof. At the start of the iteration, message $m_{x_i \rightarrow \Theta_{ij}}(x_i) = (a, b)^T$ for some a, b . As mentioned in the proof of Corollary 1, during APMP, $m_{x_j \rightarrow \Theta_{ij}}(x_j)$ will be incremented by exactly the same values as $m_{x_i \rightarrow \Theta_{ij}}(x_i)$, except in opposite positions. All messages are initialized to 0, so $m_{x_j \rightarrow \Theta_{ij}}(x_j) = (b, a)^T$. The initial belief is then

$$b_{ij}^{\text{init}}(x_i, x_j) = \begin{bmatrix} a + b & \theta_{ij}^{01} + 2a \\ \theta_{ij}^{10} + 2b & a + b \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} 0 & \theta_{ij}^{01} + a - b \\ \theta_{ij}^{10} + b - a & 0 \end{bmatrix} + \kappa_1. \quad (26)$$

After passing messages on $\mathcal{T}(t)$, $m_{x_i \rightarrow \Theta_{ij}}(x_i) = (a, b + f)^T$ and $m_{x_j \rightarrow \Theta_{ij}}(x_j) = (b + f, a)^T$. The new belief is

$$b_{ij}^{\text{final}}(x_i, x_j) = \begin{bmatrix} a + b + f & \theta_{ij}^{01} + 2a \\ \theta_{ij}^{10} + 2b + 2f & a + b + f \end{bmatrix} \quad (27)$$

$$= \begin{bmatrix} 0 & \theta_{ij}^{01} + a - b - f \\ \theta_{ij}^{10} + b - a + f & 0 \end{bmatrix} + \kappa_2. \quad (28)$$

Here $\kappa_1 = a + b$ and $\kappa_2 = a + b + f$. Subtracting the initial belief from the final belief finishes the proof:

$$\Delta b_{ij}(x_i, x_j) = \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} + f. \quad (29)$$

\square

Messages at the end of Phase 1 define homogeneous islands:

We prove that messages at the end of Phase 1 define homogeneous islands in two parts:

Lemma 6 (Binary Mask Property). *If a pairwise factor Θ_{ij} computes outgoing message $m_{\Theta_{ij} \rightarrow j}(x_j) = (0, 0)^T$ given incoming message $m_{i \rightarrow \Theta_{ij}}(x_i) = (\alpha, 0)^T$ for some $\alpha > 0$, then it will compute the same $(0, 0)^T$ outgoing message given any incoming message of the form, $m_{i \rightarrow \Theta_{ij}}(x_i) = (\alpha', 0)^T$, $\alpha' \geq 0$. (The same is true of messages with a zero in the opposite position.)*

Proof. This essentially follows from plugging in values to message update equations. Suppose $m_{i \rightarrow \Theta_{ij}}(x_i) = (\alpha, 0)^T$ and $m_{\Theta_{ij} \rightarrow j}(x_j) = (0, 0)^T$. Plugging into the message update equation, we see that,

$$\begin{aligned} m_{\Theta_{ij} \rightarrow x_j}(x_j) &= \min_{x_i} [\Theta_{ij}(x_i, x_j) + m_{x_i \rightarrow \Theta_{ij}}(x_i)] \\ &= \min_{x_i} [\theta_{ij}^{10} \cdot x_i(1 - x_j) + \theta_{ij}^{01} \cdot (1 - x_i)x_j \\ &\quad + \alpha \cdot (1 - x_i)] \\ m_{\Theta_{ij} \rightarrow x_j}(0) &= \min(\theta_{ij}^{10}, \alpha) \\ m_{\Theta_{ij} \rightarrow x_j}(1) &= \min(\alpha + \theta_{ij}^{01}, 0) = 0 \\ m_{\Theta_{ij} \rightarrow x_j}(x_j) &= \begin{pmatrix} \min(\theta_{ij}^{10}, \alpha) \\ 0 \end{pmatrix}. \end{aligned}$$

In order for this to evaluate to $(0, 0)^T$ when $\alpha > 0$, θ_{ij}^{01} must be 0. Since $\theta_{ij}^{01} = 0$, no matter what value of $\alpha' \geq 0$ we are given, it is clear that $\min(\theta_{ij}^{10}, \alpha') = 0$. \square

Lemma 7 (Iterated Homogeneity). *Homogeneous islands of type $(\alpha, 0)$ (or $(0, \beta)$) are closed under passing Strict MP messages between variables in the island. That is, a variable that starts with belief $(\alpha, 0)^T$, $\alpha \geq 0$ will have belief $(\alpha', 0)^T$, $\alpha' \geq 0$ after any number of rounds of message passing.*

Proof. Initially, all beliefs have the form $(\alpha_i, 0)^T$, $\alpha_i \geq 0$ by definition. Given an incoming message of the form $(\alpha, 0)^T$, $\alpha \geq 0$, a submodular pairwise factor will compute outgoing message $(\min(\alpha, \theta_{ij}^{10}), 0)^T$, where $\theta_{ij}^{10} \geq 0$. The minimum of two non-negative quantities is positive. Variable to factor messages will sum messages of this same form, and the sum of two non-negative quantities is non-negative. Thus, all messages passed within the island will be of the form $(\alpha, 0)^T$, $\alpha \geq 0$, which beliefs will be of the proper form. Lemma 6 shows that edges previously defining the boundary of the island will still define the boundary of the island. The case of incoming message $(0, \beta)^T$ is analogous. \square

Lemma 4. *At the end of Phase 1, the messages of APMP define a collection of homogeneous islands.*

Proof. (Sketch) This is essentially equivalent to the max-flow min-cut theorem, which proves the optimality of the Ford-Fulkerson algorithm when no more augmenting paths can be found. In our formulation, at the end of Phase 1, there are by definition no paths with nonzero capacity, which implies that along any path between a variable i with belief $(\alpha, 0)^T, \alpha > 0$ and a variable k with belief $(0, \beta)^T, \beta > 0$, there must be a factor-to-variable message that given incoming message $(\alpha, 0)^T, \alpha > 0$ would produce outgoing message $(0, 0)^T$. (This is similarly true of opposite direction messages.)

Thus, to define the islands, start at each variable with nonzero belief, say of the form $(\alpha, 0)^T$, and search outwards by traversing each edge iff it would pass a nonzero message given incoming message $(\alpha, 0)^T$. Merge all variables encountered along the search into a single homogeneous island. \square