CSC373

Week 7: Linear Programming

Illustration Courtesy:
Kevin Wayne & Denis Pankratov

Recap

- Network flow
  - Ford-Fulkerson algorithm
    - Ways to make the running time polynomial
  - Correctness using max-flow, min-cut
  - Applications:
    - Edge-disjoint paths
    - Multiple sources/sinks
    - Circulation
    - Circulation with lower bounds
    - Survey design
    - Image segmentation
    - Profit maximization

Brewery Example

- A brewery can invest its inventory of corn, hops and malt into producing some amount of ale and some amount of beer
  - Per unit resource requirement and profit of the two items are as given below

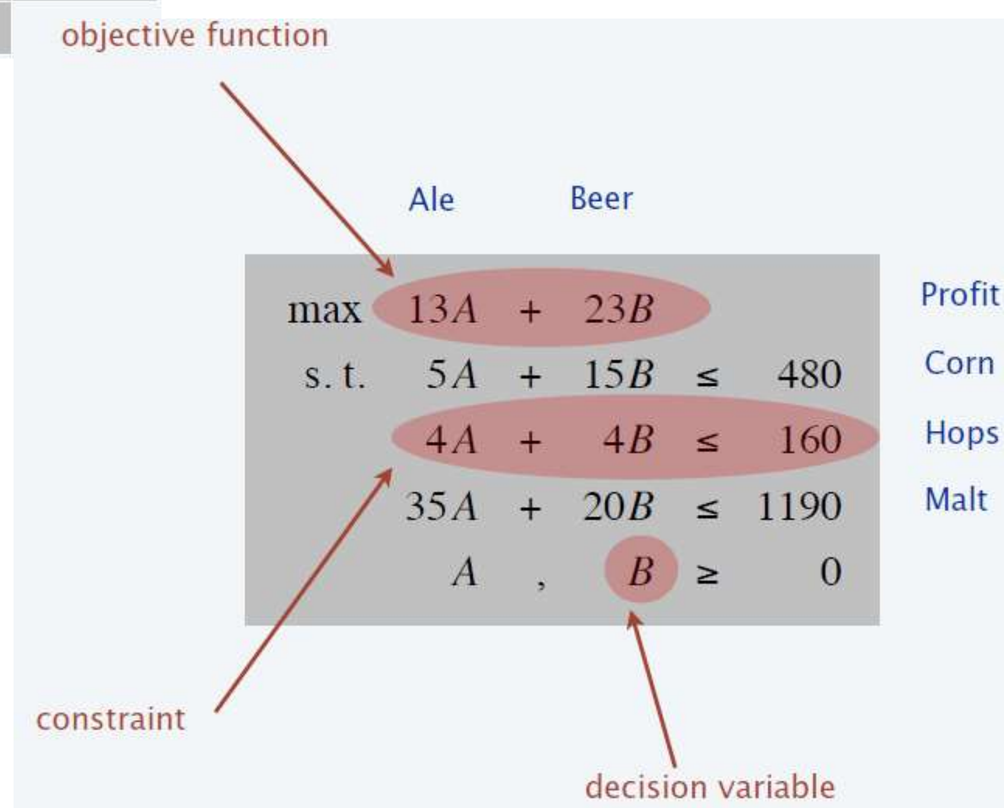
Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

Example Courtesy: Kevin Wayne

Brewery Example

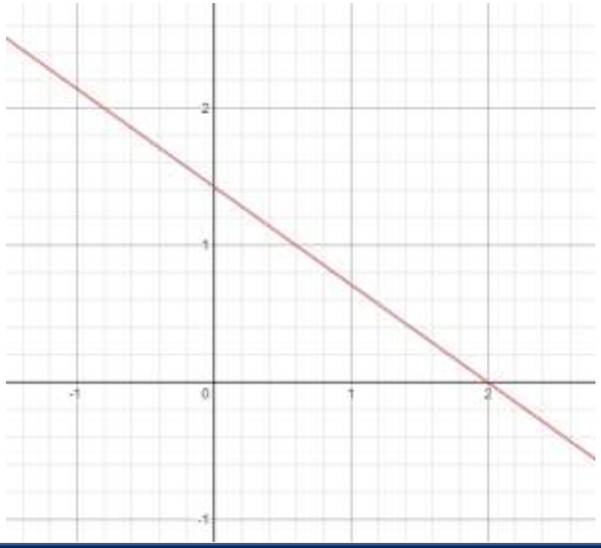
Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

- Suppose it produces A units of ale and B units of beer
- Then we want to solve this program:



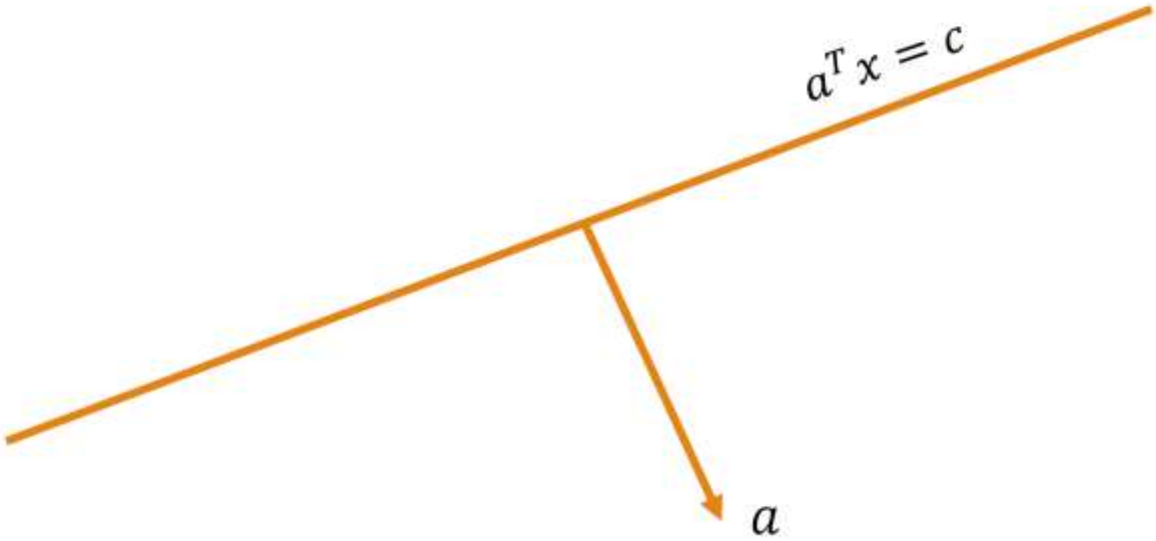
Linear Function

- $f: \mathbb{R}^n \to \mathbb{R}$ is a linear function if $f(x) = a^T x$ for some $a \in \mathbb{R}^n$
  - Example: $f(x_1, x_2) = 3x_1 - 5x_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
- Linear objective: f
- Linear constraints:
  - $g(x) = c$, where $g: \mathbb{R}^n \to \mathbb{R}$ is a linear function and $c \in \mathbb{R}$
  - Line in the plane (or a hyperplane in $\mathbb{R}^n$)
  - Example: $5x_1 + 7x_2 = 10$



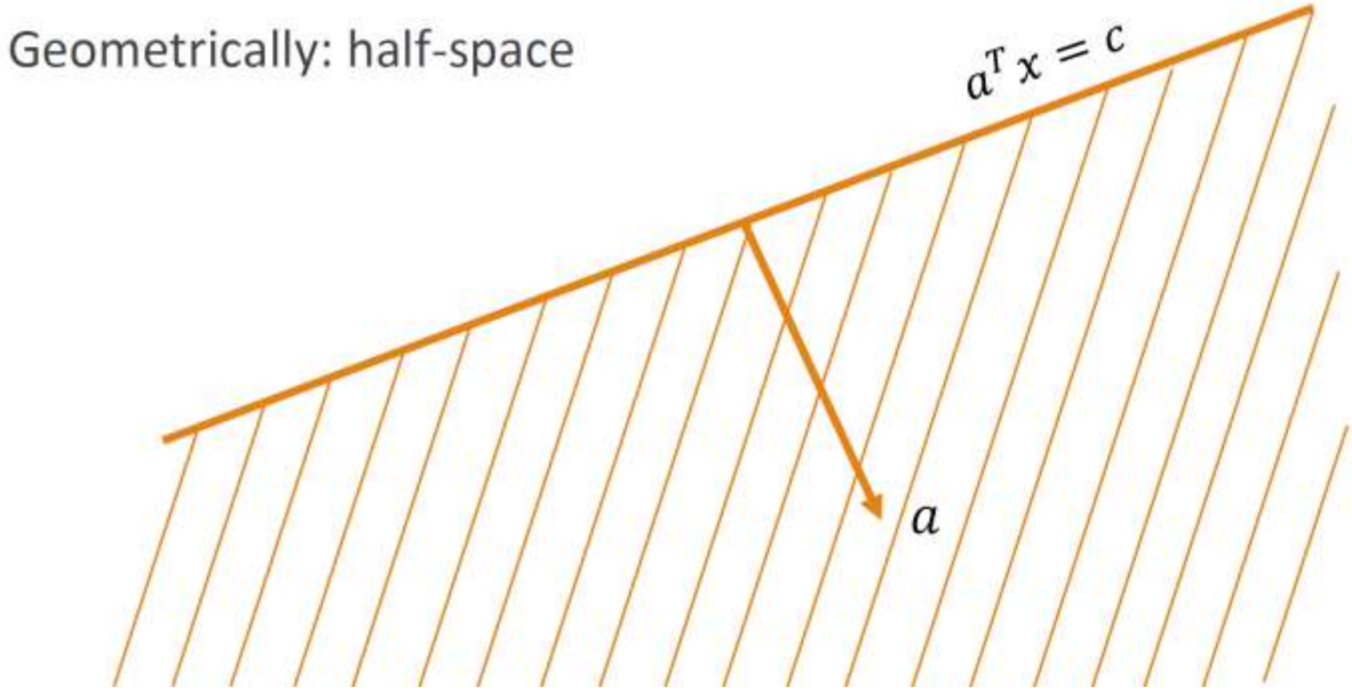
Linear Function

- Geometrically, a is the normal vector of the line(or hyperplane) represented by $a^T x = c$



Linear Inequality

- $a^T x \leq c$ represents a "half-space"



Linear Programming

- Maximize/minimize a linear function subject to linear equality/inequality constraints

Could be min

Linear objective!

Objective function $\max x_1 + 6x_2$

Constraints

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 \leq 400$$

$$x_1, x_2 \geq 0$$

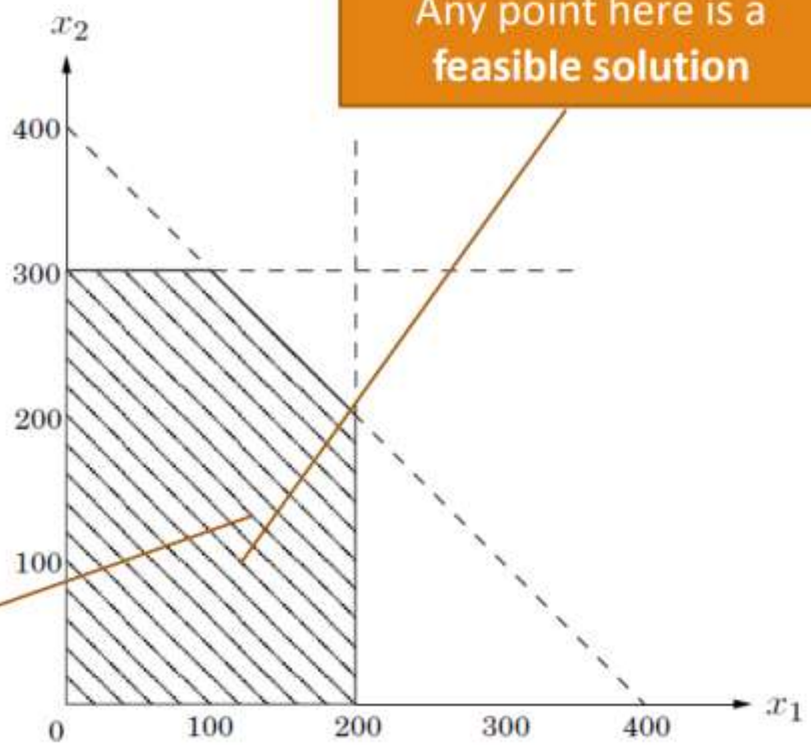
Linear constraints: inequalities/equalities

Geometrically…

Objective function $\max x_1 + 6x_2$

Constraints

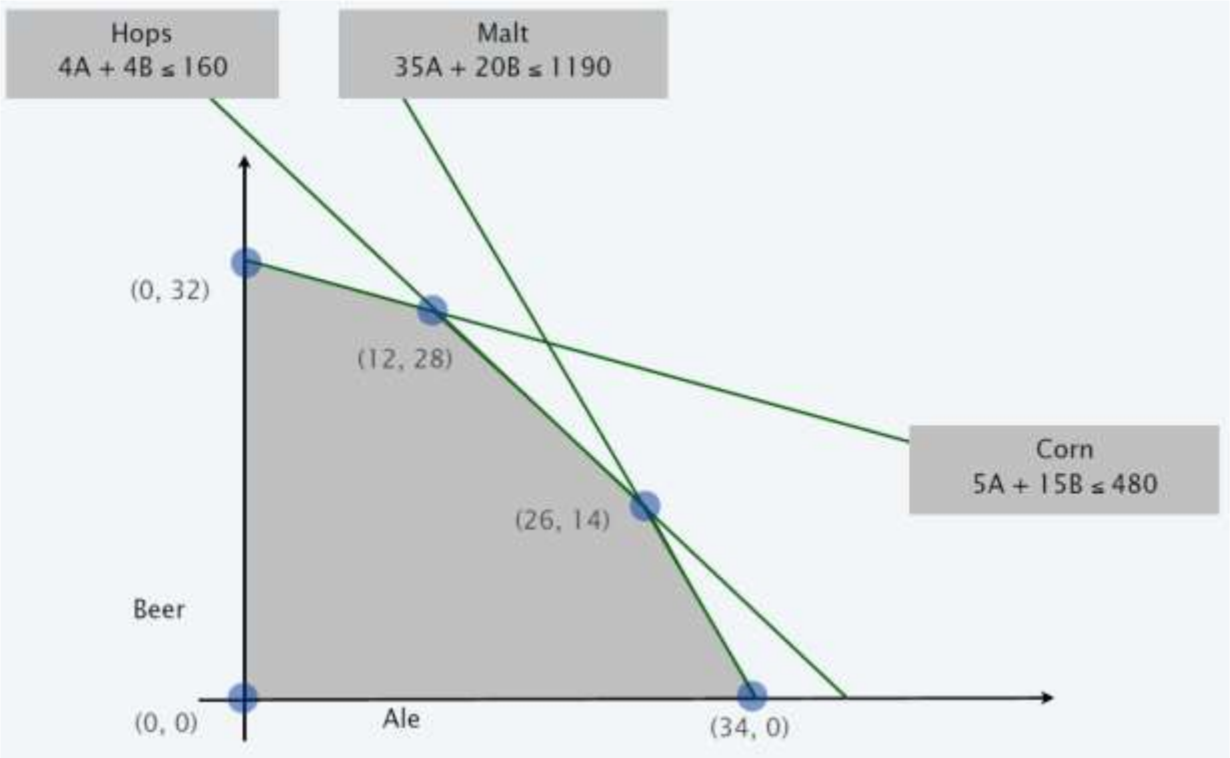
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$



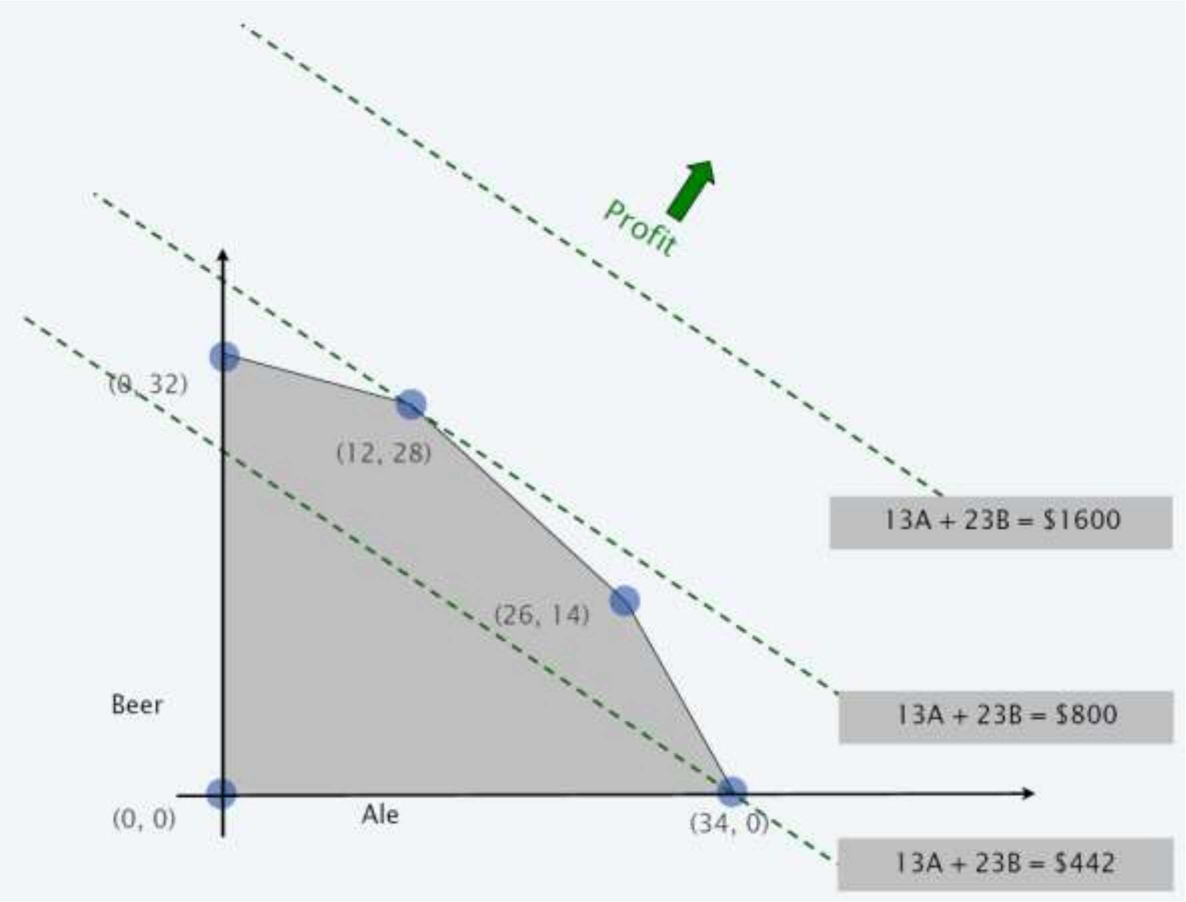
Any point here is a **feasible solution**

Feasible region – polytope, aka intersection of half-spaces!

Back to Brewery Example

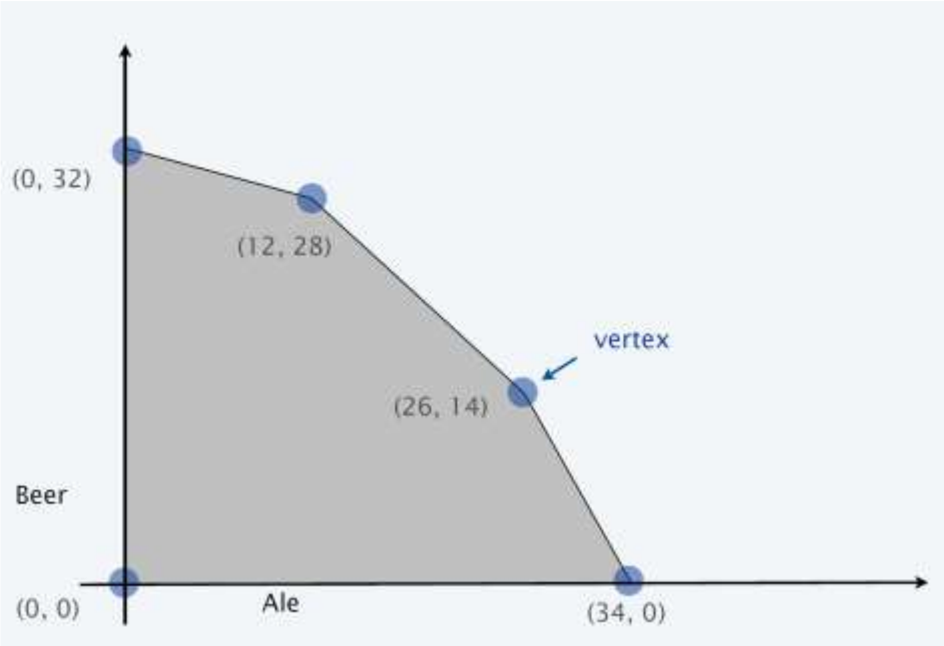


Back to Brewery Example



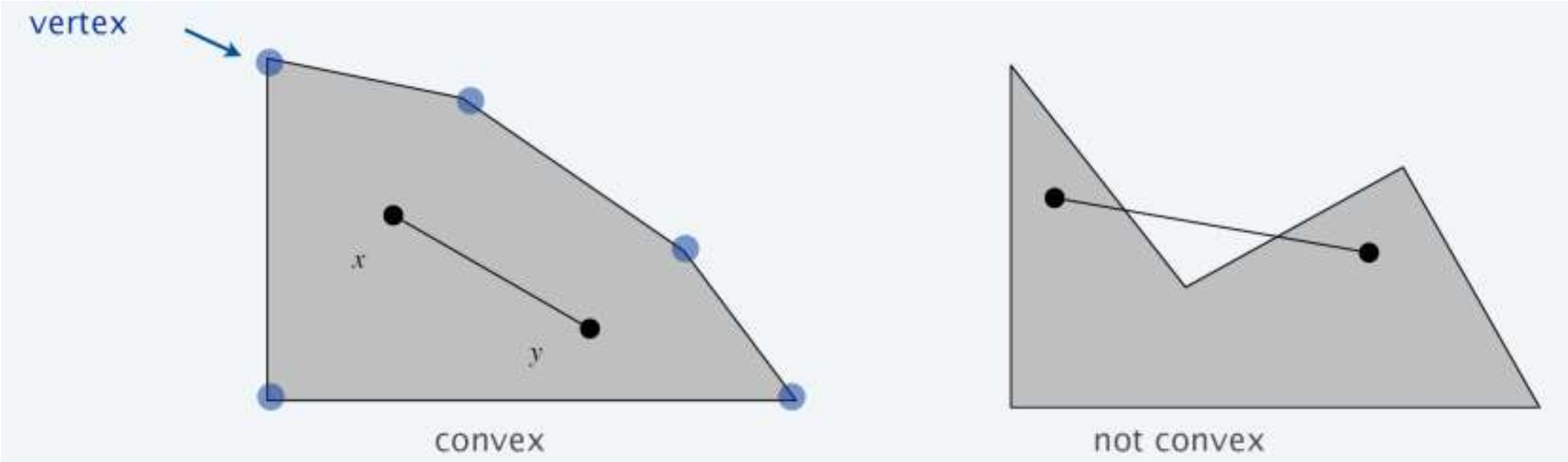
Optimal Solution At A Vertex

- Claim: Regardless of the objective function, there must be a vertex that is an optimal solution



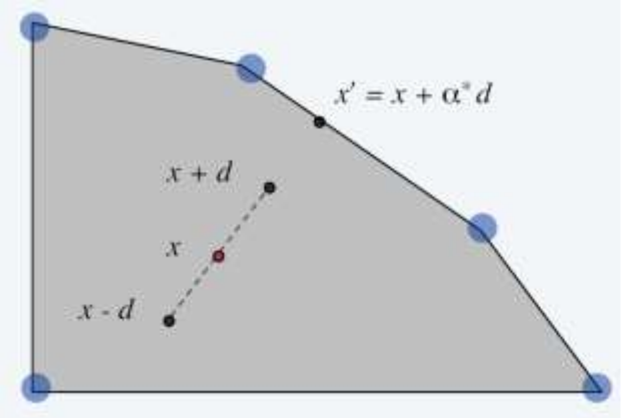
Convexity

- Convex set: S is convex if
$$x, y \in S, \lambda \in [0,1] \Rightarrow \lambda x + (1-\lambda) y \in S$$
- Vertex: A point which cannot be written as a strict convex combination of any two points in the set
- Observation: Feasible region of an LP is a convex set



Optimal Solution At A Vertex

- Intuitive proof of the claim:
  - Start at some point x in the feasible region
  - If x is not a vertex:
    - Find a direction d such that points within a positive distance of ϵ from x in both d and $-d$ directions are within the feasible region
    - Objective must *not decrease* in at least one of the two directions
    - Follow that direction until you reach a new point x for which at least one more constraint is "tight"
  - Repeat until we are at a vertex



LP, Standard Formulation

- Input: $c, a_1, a_2, \ldots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^m$
  - There are n variables and m constraints
- Goal:

$$
\begin{aligned}
\text{Maximize } & c^T x \\
\text{Subject to } & a_1^T x \le b_1 \\
& a_2^T x \le b_2 \\
& \vdots \\
& a_m^T x \le b_m \\
& x \ge 0
\end{aligned}
$$

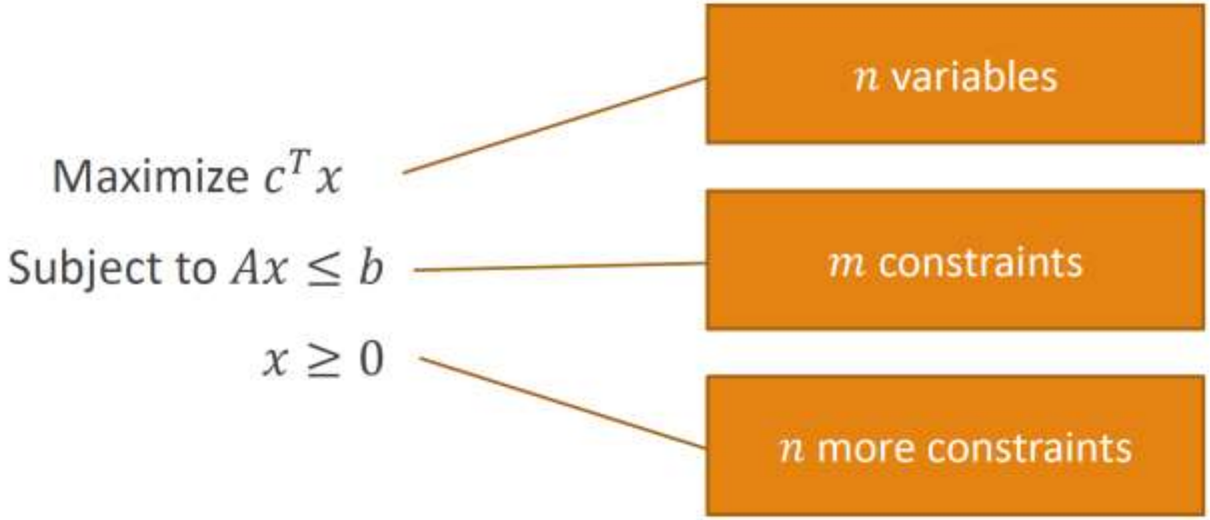
n variables

m constraints

n more constraints

LP, Standard Matrix Form

- Input: $c, a_1, a_2, \dots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^m$
  - There are n variables and m constraints
- Goal:



Convert to Standard Form

- What if the LP is not in standard form?
  - Constraints that use $\geq$
    - $a^T x \geq b \;\Leftrightarrow\; -a^T x \leq -b$
  - Constraints that use equality
    - $a^T x = b \;\Leftrightarrow\; a^T x \leq b,\;\; a^T x \geq b$
  - Objective function is a minimization
    - Minimize $c^T x$ $\Leftrightarrow$ Maximize $-c^T x$
  - Variable is unconstrained
    - x with no constraint $\Leftrightarrow$ Replace x by two variables $x'$and x'', replace every occurrence of x with $x' - x''$, and add constraints $x' \geq 0,\;\; x'' \geq 0$

LP Transformation Example

$$\begin{array}{llcccl} \text{minimize} & -2x_1 & + & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0\ , \end{array}$$

$$\begin{array}{llcccl} \text{maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0\ . \end{array}$$

$$\begin{array}{llcccccl} \text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2' & - & x_2'' & = & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & & x_1, x_2', x_2'' & & & & \geq & 0\ . \end{array}$$

Optimal Solution

- Does an LP always have an optimal solution?

- No! The LP can "fail" for two reasons:
  1. It is *infeasible*, i.e., $\{x \mid Ax \leq b\} = \emptyset$
     - E.g., the set of constraints is $\{x_1 \leq 1, -x_1 \leq -2\}$
  2. It is *unbounded*, i.e., the objective function can be made arbitrarily large (for maximization) or small (for minimization)
     - E.g., "maximize x_1 subject to $x_1 \geq 0$"

- But if the LP has an optimal solution, we know that there must be a vertex which is optimal

Simplex Algorithm

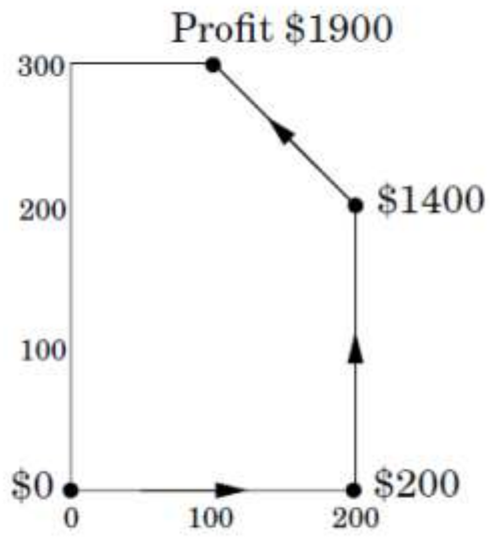
```
let v be any vertex of the feasible region
while there is a neighbor v' of v with better objective value:
    set v = v'
```

- Simple algorithm, easy to specify geometrically
- Worst-case running time is exponential
- Excellent performance in practice

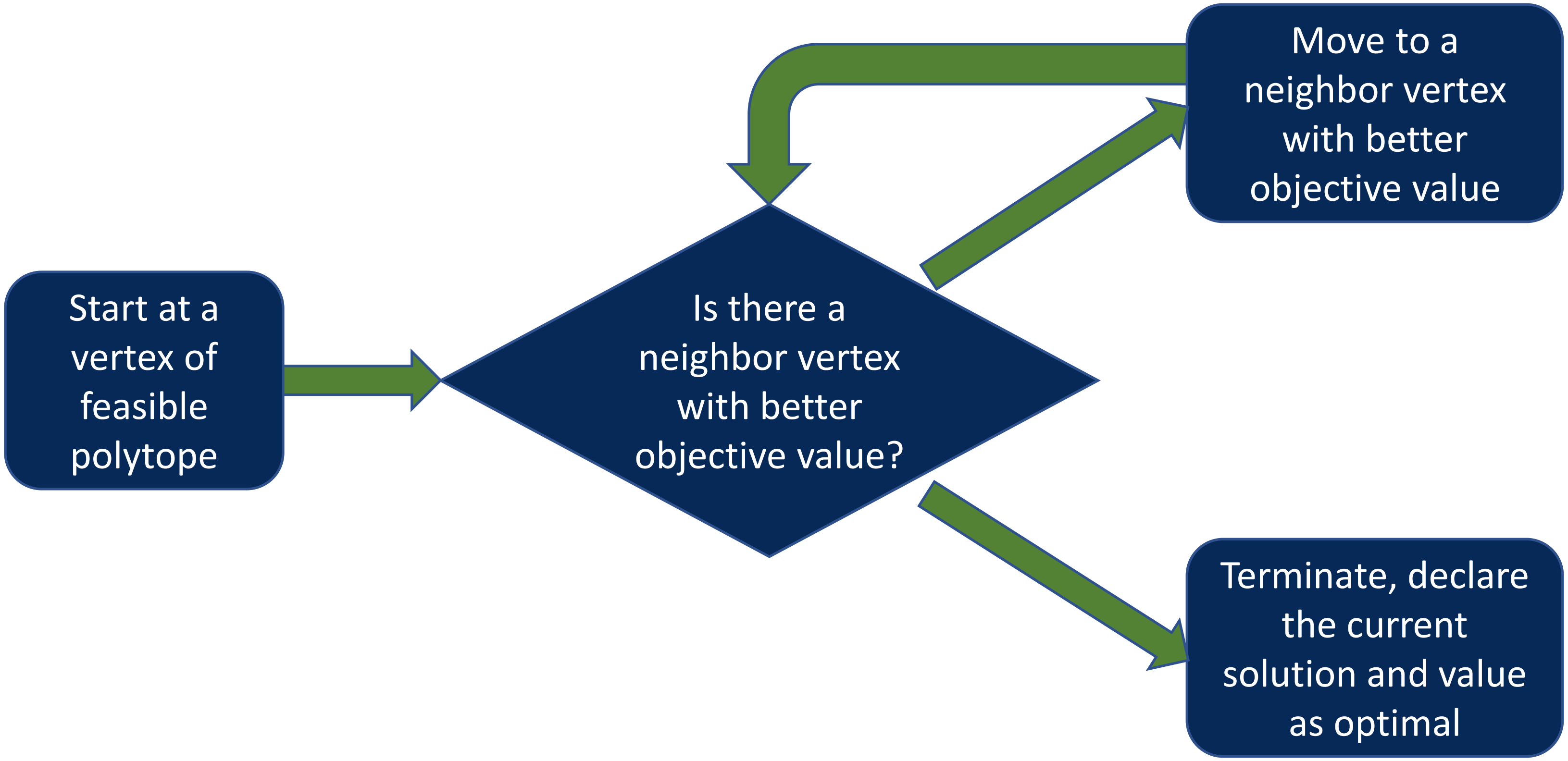
Simplex: Geometric View

```
let v be any vertex of the feasible region
while there is a neighbor v' of v with better objective value:
    set v = v'
```

$$\max\ x_1 + 6x_2$$
$$x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 \le 400$$
$$x_1, x_2 \ge 0$$



Algorithmic Implementation



How Do We Implement This?

- We'll work with the slack form of LP
  - Convenient for implementing simplex operations
  - We want to maximize z in the slack form, but for now, forget about the maximization objective

Standard form:

$$\begin{aligned} \text{Maximize } & c^T x \\ \text{Subject to } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Slack form:

$$\begin{aligned} \mathrm{z} &= c^T x \\ s &= b - Ax \\ s, x &\geq 0 \end{aligned}$$

Slack Form

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0 \; .
\end{array}
$$

Nonbasic Variables

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& x_1, x_2, x_3, x_4, x_5, x_6 & & & \geq & 0 \; . & & &
\end{array}
$$

Basic Variables (x_4, x_5, x_6)

Slack Form

$$
\begin{aligned}
z &=   \quad 2x_1 - 3x_2 + 3x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -7 + x_1 + x_2 - x_3 \\
x_6 &= 4 - x_1 + 2x_2 - 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Nonbasic Variables

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 - 3x_2 + 3x_3 \\
\text{subject to} \quad & \\
& x_4 = 7 - x_1 - x_2 + x_3 \\
& x_5 = -7 + x_1 + x_2 - x_3 \\
& x_6 = 4 - x_1 + 2x_2 - 2x_3 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \, .
\end{aligned}
$$

Basic Variables

Simplex: Step 1

- Start at a feasible vertex
  - How do we find a feasible vertex?
  - For now, assume $b \geq 0$ (that is, each $b_i \geq 0$)
    - In this case, $x = 0$ is a feasible vertex.
    - In the slack form, this means setting the nonbasic variables to 0
  - We'll later see what to do in the general case

Standard form:

$$\text{Maximize } c^T x$$
$$\text{Subject to } Ax \leq b$$
$$x \geq 0$$

Slack form:

$$z = c^T x$$
$$s = b - Ax$$
$$s, x \geq 0$$

Simplex: Step 2

- What next? Let's look at an example

$$
\begin{aligned}
z &= && 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 &-& x_1 - x_2 - 3x_3 \\
x_5 &= 24 &-& 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 &-& 4x_1 - x_2 - 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

- To increase the value of z:
  - Find a nonbasic variable with a positive coefficient
    - This is called an *entering variable*
  - See how much you can increase its value without violating any constraints

Simplex: Step 2

Try to increase!

Obstacles!

$$
\begin{aligned}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

$x_1 \leq 30$

$x_1 \leq 24/2 = 12$

$x_1 \leq 36/4 = 9$

Tightest obstacle!

This is because the current values of x_2 and x_3 are 0, and we need $x_4, x_5, x_6 \geq 0$

Simplex: Step 2

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \\
\multicolumn{5}{l}{x_1, x_2, x_3, x_4, x_5, x_6 \quad \geq} & & 0 & &
\end{array}
$$

← Tightest obstacle

- Solve the tightest obstacle for the nonbasic variable

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

  - Substitute the entering variable (called pivot) in other equations
  - Now x_1 becomes basic and x_6 becomes non-basic
  - x_6 is called the *leaving variable*

Simplex: Step 2

$$\begin{aligned} z &= 3x_1 + x_2 + 2x_3 \\ x_4 &= 30 - x_1 - x_2 - 3x_3 \\ x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\ x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

$$\begin{aligned} z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\ x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\ x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\ x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

- After one iteration of this step:
  - The basic feasible solution (i.e., substituting 0 for all nonbasic variables) improves from $z = 0$ to $z = 27$
- Repeat!

Simplex: Step 2

Entering variable
Try to increase!

$$\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} . \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}$$

Leaving variable
Tightest obstacle!

Pivot!

$$\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}$$

Simplex: Step 2

Entering variable
Try to increase!

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} . \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Leaving variable
Tightest obstacle!

Pivot!

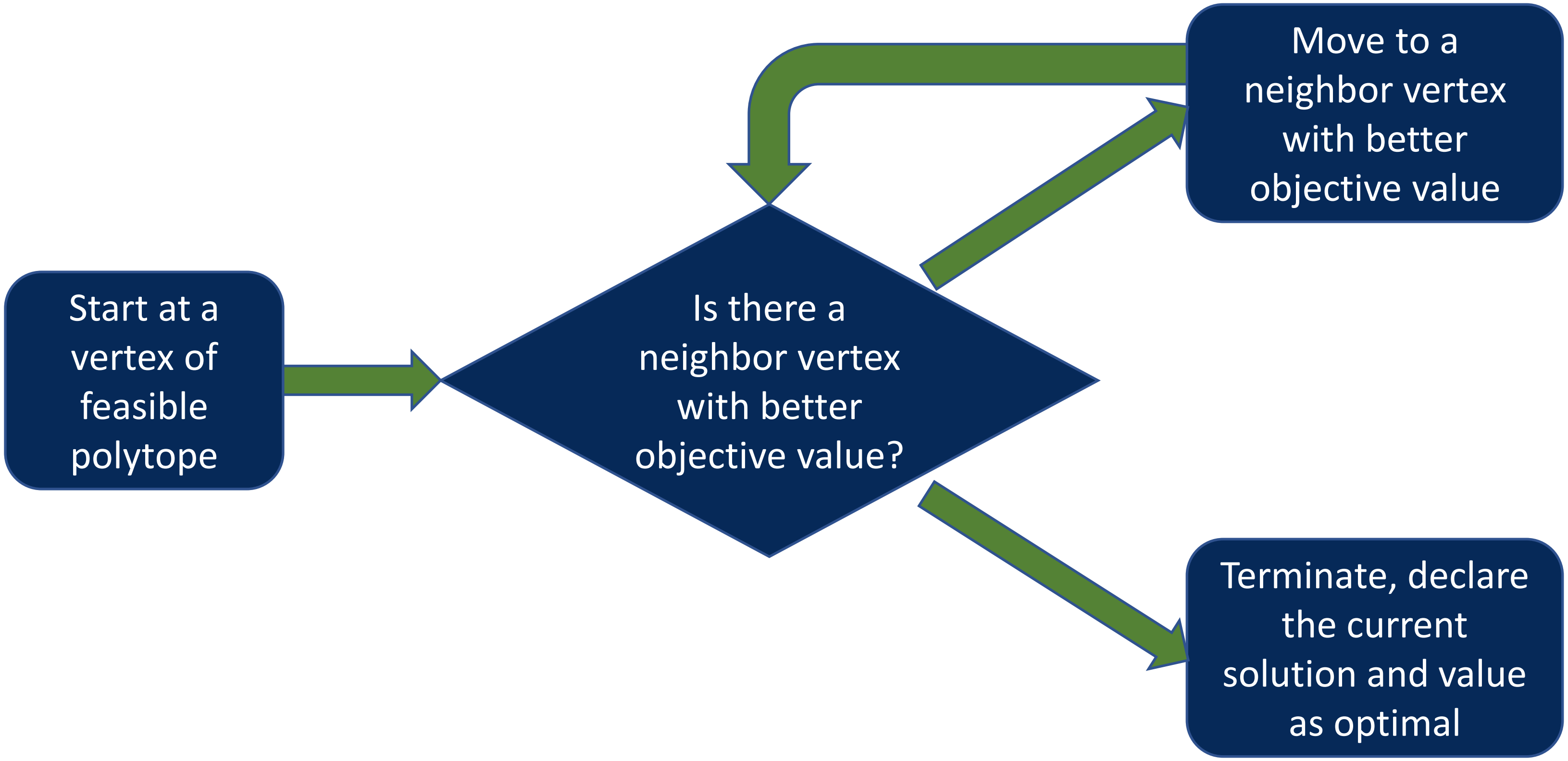
$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} . \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Simplex: Step 2

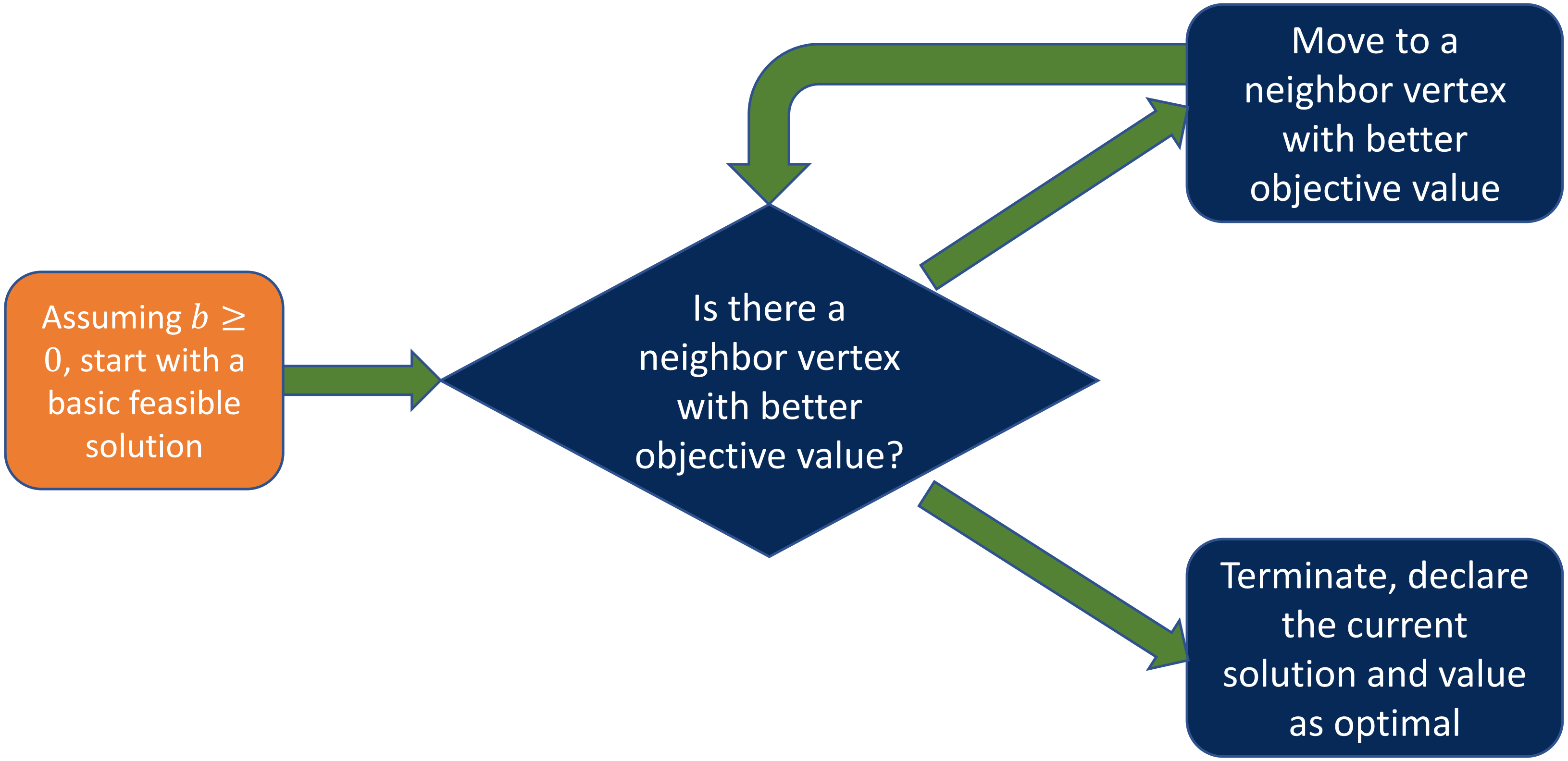
$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} . \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

- There is no entering variable (nonbasic variable with positive coefficient)
- What now? Nothing! We are done.
- Take the basic feasible solution ($x_3 = x_5 = x_6 = 0$).
- Gives the optimal value $z = 28$
- In the optimal solution, $x_1 = 8$, $x_2 = 4$, $x_3 = 0$

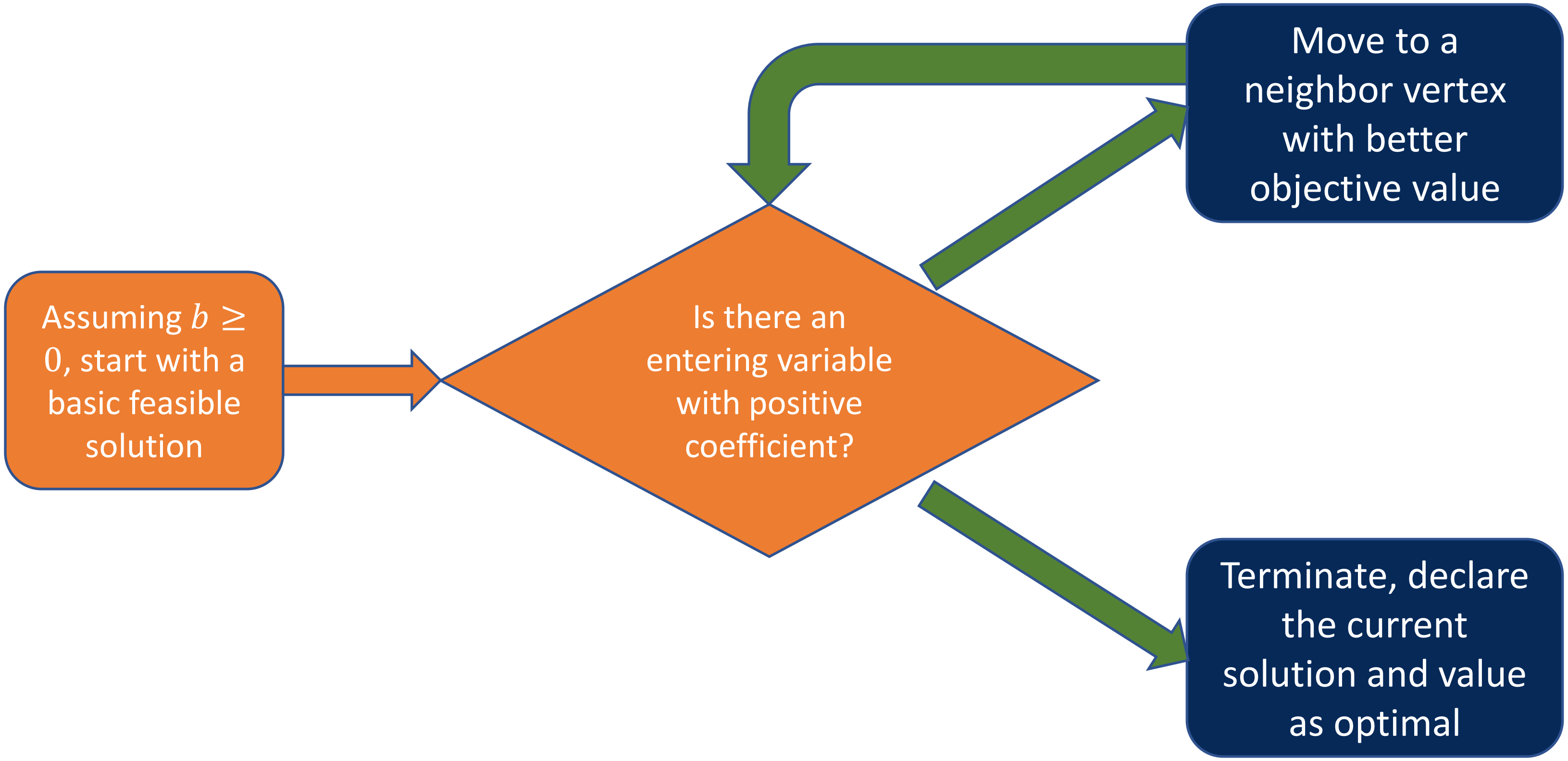
Simplex Overview



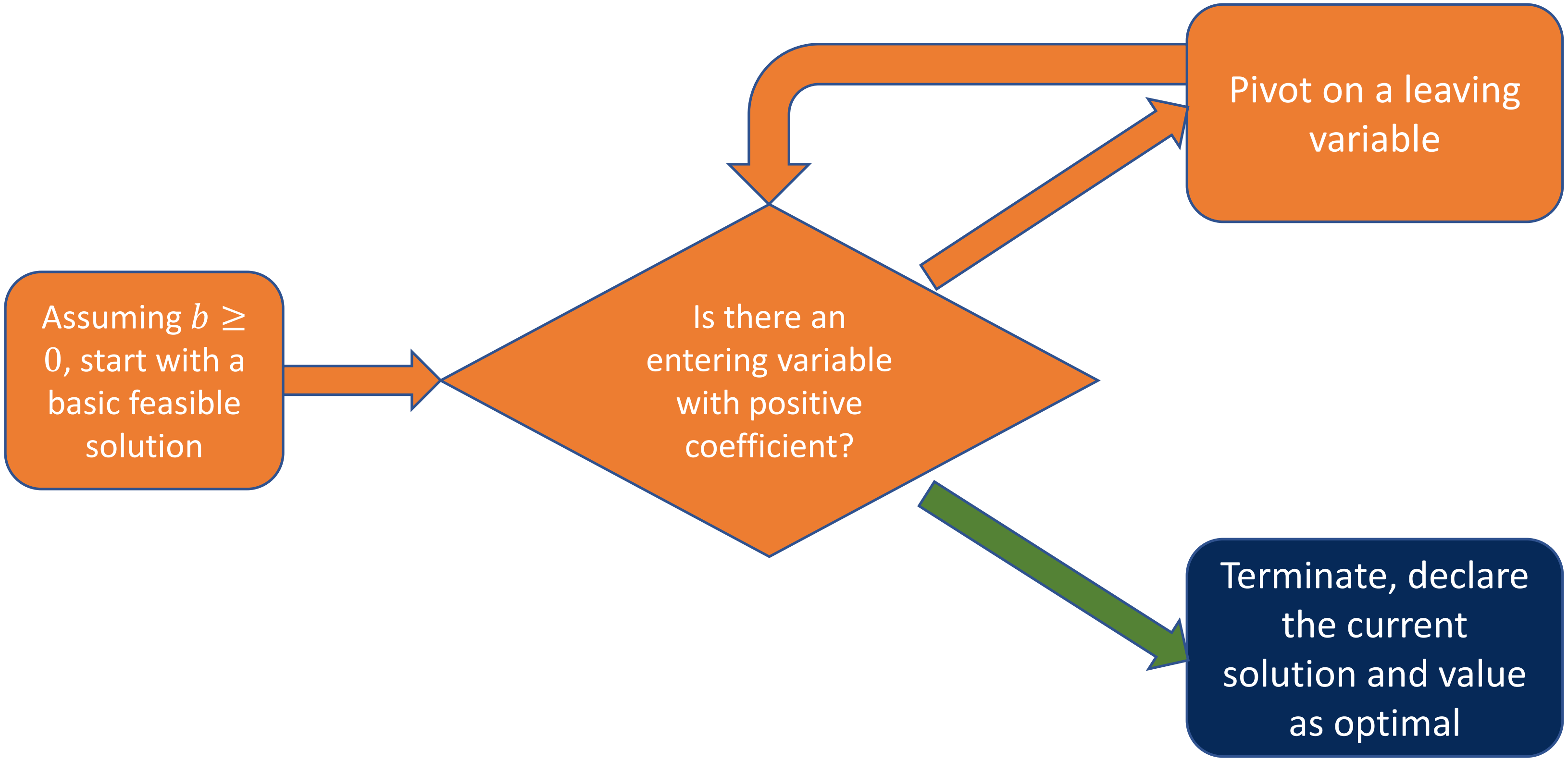
Simplex Overview



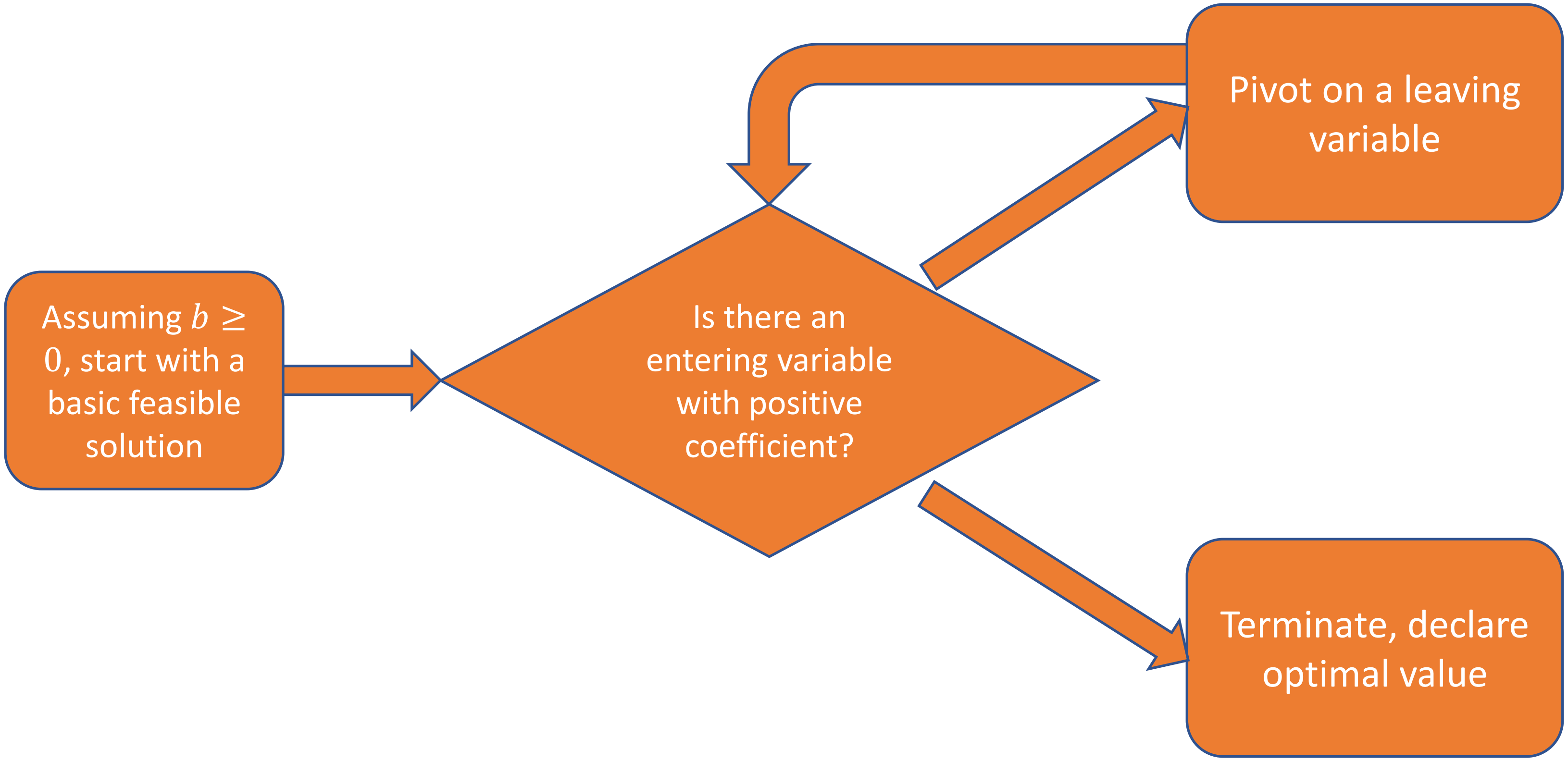
Simplex Overview



Simplex Overview



Simplex Overview



Some Outstanding Issues

- What if the entering variable has no upper bound?
  - If it doesn't appear in any constraints, or only appears in constraints where it can go to ∞
  - Then z can also go to ∞, so declare that LP is unbounded
- What if pivoting doesn't change the constant in z?
  - Known as *degeneracy*, and can lead to infinite loops
  - Can be prevented by "perturbing" b by a small random amount in each coordinate
  - Or by carefully breaking ties among entering and leaving variables, e.g., by smallest index (known as *Bland's rule*)

Some Outstanding Issues

- We assumed $b \geq 0$, and then started with the vertex $x = 0$
- What if this assumption does not hold?

LP_1		LP_2		LP_3
Max $c^T x$		Max $c^T x$		Max $c^T x$
s.t. $a_1^T x \leq b_1$	→	s.t. $a_1^T x + s_1 = b_1$	→	s.t. $a_1^T x + s_1 = b_1$
$a_2^T x \leq b_2$		$a_2^T x + s_2 = b_2$		$-a_2^T x - s_2 = -b_2$
$\vdots$		$\vdots$		$\vdots$
$a_m^T x \leq b_m$		$a_m^T x + s_m = b_m$		$-a_m^T x - s_m = -b_m$
$x \geq 0$		$x, s \geq 0$		$x, s \geq 0$

Multiply every constraint with negative b_i by -1 so RHS is now positive

Some Outstanding Issues

- We assumed $b \geq 0$, and then started with the vertex $x = 0$
- What if this assumption does not hold?

LP_3

Max $c^T x$

s.t. $a_1^T x + s_1 = b_1$

$-a_2^T x - s_2 = -b_2$

$\vdots$

$-a_m^T x - s_m = -b_m$

$x, s \geq 0$

LP_4

Min $\sum_i z_i$

s.t. $a_1^T x + s_1 + z_1 = b_1$

$-a_2^T x - s_2 + z_2 = -b_2$

$\vdots$

$-a_m^T x - s_m + z_m = -b_m$

$x, s, z \geq 0$

Remember: we only want to find a basic feasible solution to LP_1

Remember: RHS is now positive

Some Outstanding Issues

- We assumed $b \geq 0$, and then started with the vertex $x = 0$
- What if this assumption does not hold?

LP_4

Min $\sum_i z_i$

s.t. $a_1^T x + s_1 + z_1 = b_1$

$-a_2^T x - s_2 + z_2 = -b_2$

$\vdots$

$-a_m^T x - s_m + z_m = -b_m$

$x, s, z \geq 0$

Remember: the RHS is now positive

What now?

- Solve LP_4 using simplex with the initial basic solution being $x = s = 0, z = |b|$
- If its optimum value is 0, extract a basic feasible solution x^* from it, use it to solve LP_1 using simplex
- If optimum value for LP_4 is greater than 0, then LP_1 is infeasible

Some Outstanding Issues

- Curious about pseudocode? Proof of correctness? Running time analysis?

- See textbook for details, but this is <u>NOT</u> in syllabus!

Running Time

- Notes
  - #vertices of a polytope can be exponential in the #constraints
    - There are examples where simplex takes exponential time if you choose your pivots arbitrarily
    - No pivot rule known which guarantees polynomial running time
  - Other algorithms known which run in polynomial time
    - Ellipsoid method, interior point method, …
    - Ellipsoid uses $O(mn^3L)$ arithmetic operations
      - L = length of input in binary
    - But no known *strongly polynomial time* algorithm
      - Number of arithmetic operations = poly(m,n)
      - We know how to avoid dependence on length(b), but not for length(A)