CSC373

Week 5:
Dynamic Programming (contd)
Network Flow (start)

Recap

- Dynamic Programming Basics
  - Optimal substructure property
  - Bellman equation
  - Top-down (memoization) vs bottom-up implementations
- Dynamic Programming Examples
  - Weighted interval scheduling
  - Knapsack problem
  - Single-source shortest paths
  - Chain matrix product

This Lecture

- Some more DP
  - Traveling salesman problem (TSP)

- Start of network flow
  - Problem statement
  - Ford-Fulkerson algorithm
  - Running time
  - Correctness

Traveling Salesman

- Input
  - Complete directed graph $G = (V, E)$
  - $d_{i,j}$ = distance from node i to node j

- Output
  - Minimum distance which needs to be traveled to start from some node v, visit every other node exactly once, and come back to v
    - That is, the minimum cost of a Hamiltonian cycle

Traveling Salesman

- Approach
  - Let's start at node $v_1 = 1$
    - It's a cycle, so the starting point does not matter
  - Want to visit the other nodes in some order, say $v_2, \dots, v_n$
  - Total distance is $d_{1,v_2} + d_{v_2,v_3} + \cdots + d_{v_{n-1},v_n} + d_{v_n,1}$
    - Want to minimize this distance

- Naïve solution
  - Check all possible orderings
  - $(n-1)! = \Theta\left(\sqrt{n} \cdot \left(\frac{n}{e}\right)^n\right)$ (Stirling's approximation)

Traveling Salesman

- DP Approach
  - Consider v_n (the last node before returning to $v_1 = 1$)
    - If $v_n = c$
      - Find the optimal order of visiting nodes $\{2, \ldots, n\}$ that ends at c
      - Need to keep track of the subset of nodes to be visited and the end node
  - $OPT[S, c]$ = minimum total travel distance when starting at 1, visiting each node in S exactly once, and ending at $c \in S$
  - Answer to the original problem:
    - $\min_{c \in S} OPT[S, c] + d_{c,1}$, where $S = \{2, \ldots, n\}$

Traveling Salesman

- DP Approach
  - To compute $OPT[S, c]$, we can condition over the vertex visited right before c in the optimal trip

- Bellman equation

$$OPT[S, c] = \min_{m \in S \setminus \{c\}} \left(OPT[S \setminus \{c\}, m] + d_{m,c} \right)$$

$$\text{Final solution} = \min_{c \in \{2,\ldots,n\}} \left(OPT[\{2, \ldots, n\}, c] + d_{c,1} \right)$$

- Time: $O(n \cdot 2^n)$ calls, $O(n)$ time per call $\Rightarrow O(n^2 \cdot 2^n)$
  - Much better than the naïve solution which has $\left({}^{n}/_{e}\right)^n$

Traveling Salesman

- Bellman equation

$$OPT[S,c] = \min_{m \in S \setminus \{c\}} \left(OPT[S \setminus \{c\}, m] + d_{m,c} \right)$$

$$\text{Final solution} = \min_{c \in \{2,\dots,n\}} OPT[\{2,\dots,n\}, c] + d_{c,1}$$

- Space complexity: $O(n \cdot 2^n)$
  - But computing the optimal solution with $|S| = k$ only requires storing the optimal solutions with $|S| = k - 1$
- Question:
  - Using this observation, how much can we reduce the space complexity?

DP Concluding Remarks

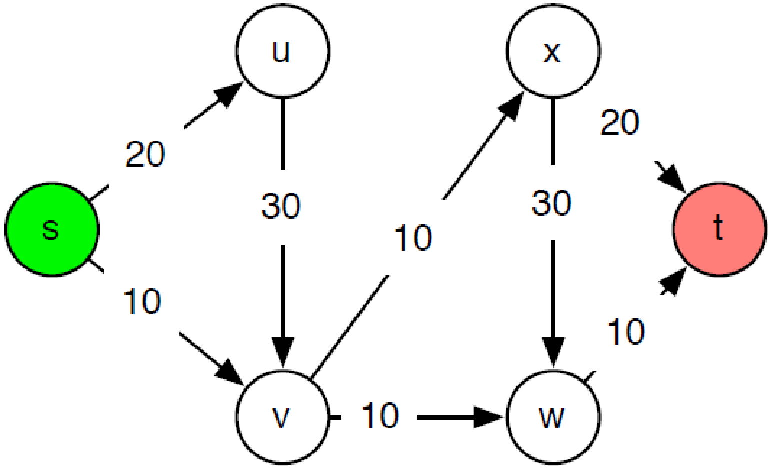
- High-level steps in designing a DP algorithm
  - Focus on a single decision in optimal solution
    - Typically, the first/last decision
  - For each possible way of making that decision…
    - [Optimal substructure] Write the optimal solution of the problem in terms of the optimal solutions to subproblems
  - Generalize the problem…
    - …by looking at the type of subproblems needed
    - E.g., in the weighted interval scheduling problem, we realize that we need to solve the problem for prefixes (i.e. either for jobs $1, \dots, j-1$ or $1, \dots, p[j]$)
  - Write the Bellman equation, cover your base cases
  - Think about optimizing the running time/space using tricks
    - Often easier in the bottom-up implementation

Network Flow

Network Flow

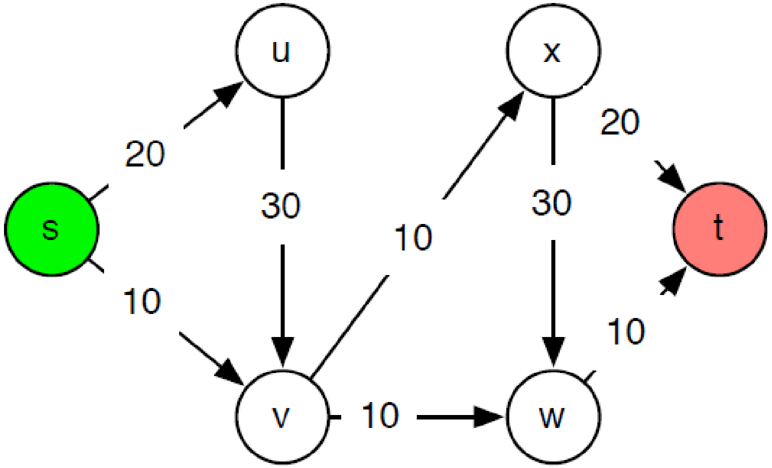
- Input
  - A directed graph $G = (V, E)$
  - Edge capacities $c : E \to \mathbb{R}_{\geq 0}$
  - Source node s, target node t

- Output
  - Maximum "flow" from s to t



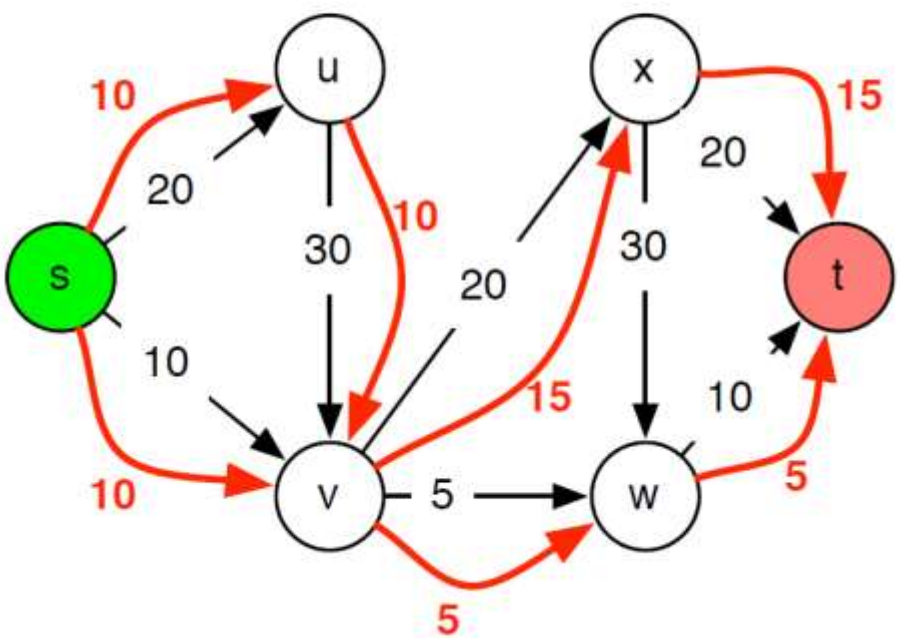
Network Flow

- Assumptions
  - No edges enter s
  - No edges leave t
  - Edge capacity $c(e)$ is a non-negative integer
    - Later, we'll see what happens when $c(e)$ can be a rational or irrational number



Network Flow

- Flow
  - An s-t flow is a function $f: E \to \mathbb{R}_{\geq 0}$
  - Intuitively, $f(e)$ is the "amount of material" carried on edge e



Network Flow

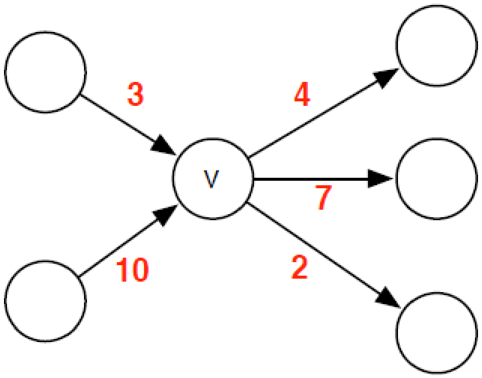
- Constraints on flow f

1. Respecting capacities

   $\forall e \in E : 0 \leq f(e) \leq c(e)$

2. Flow conservation

   $\forall v \in V \setminus \{s, t\} : \sum_{e \text{ entering } v} f(e) = \sum_{e \text{ leaving } v} f(e)$



Flow in = flow out at every node other than s and t

Network Flow

- $f^{in}(v) = \sum_{e \text{ entering } v} f(e)$
- $f^{out}(v) = \sum_{e \text{ leaving } v} f(e)$

- Value of flow f is $v(f) = f^{out}(s) = f^{in}(t)$
  - Q: Why is $f^{out}(s) = f^{in}(t)$?

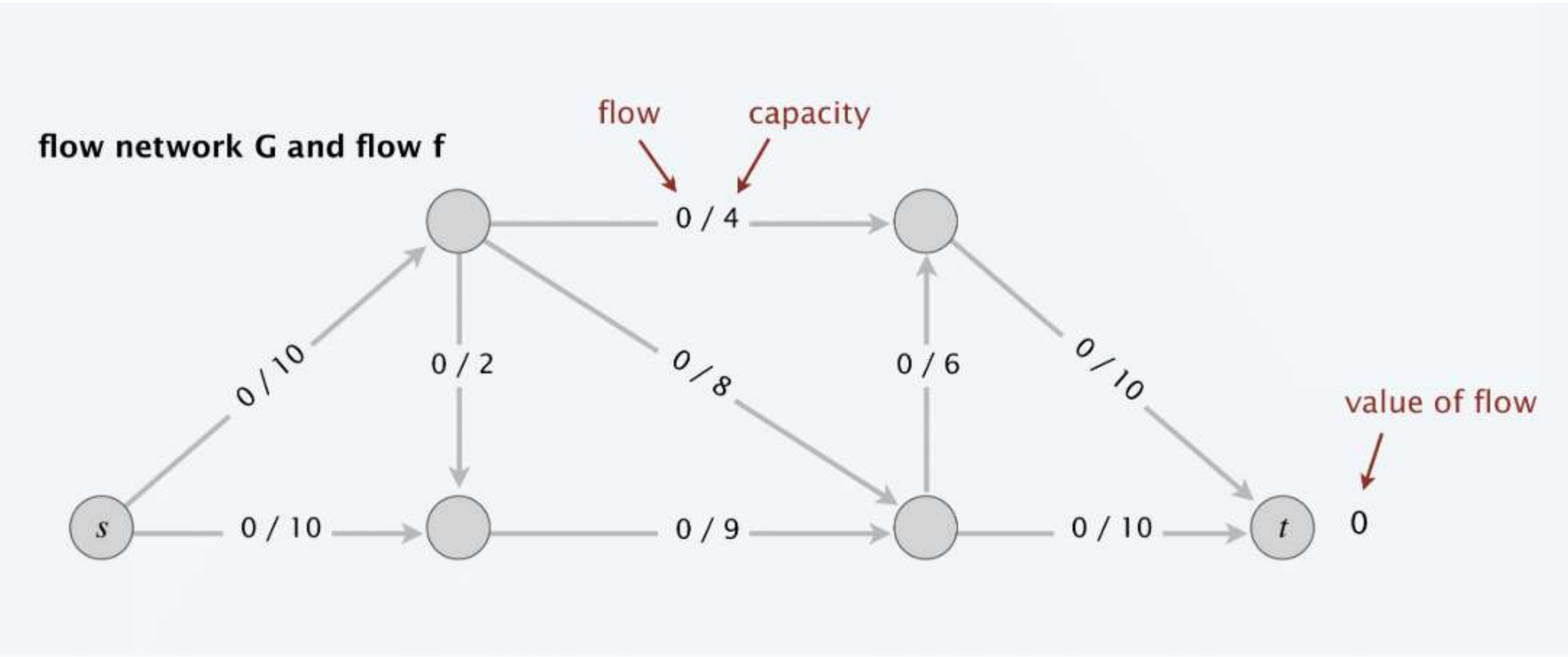
- Restating the problem:
  - Given a directed graph $G = (V, E)$ with edge capacities $c: E \to \mathbb{R}_{\geq 0}$, find a flow f^* with the maximum value.

First Attempt

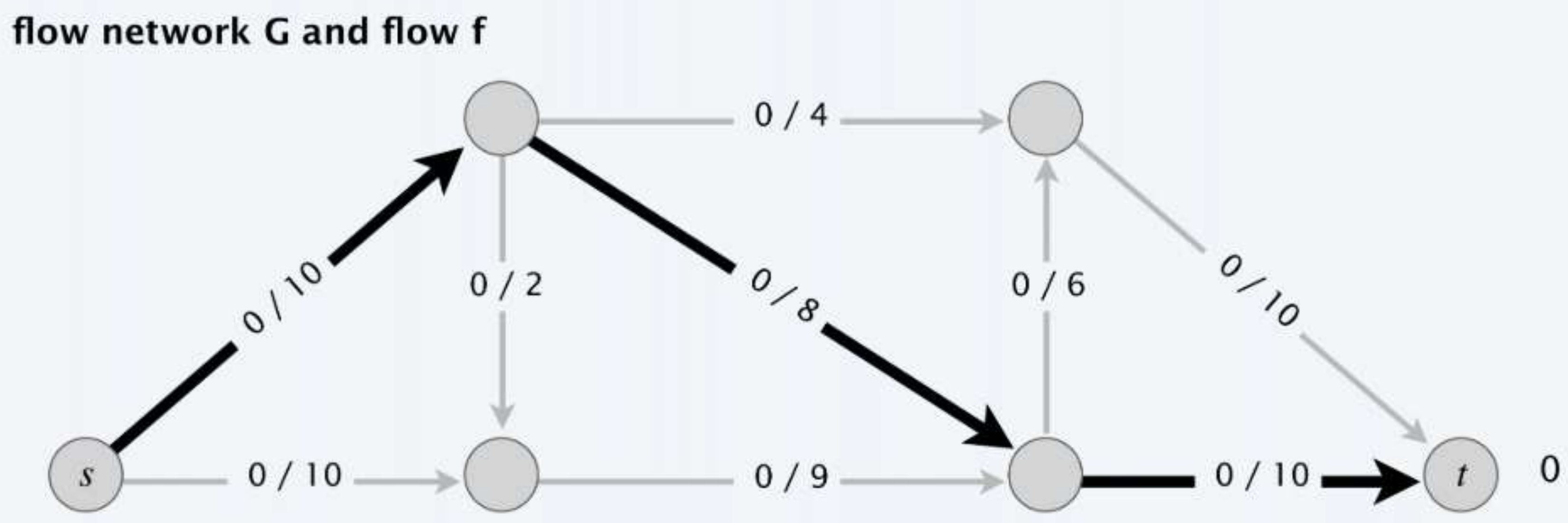
- A natural greedy approach
  1. Start from zero flow ($f(e) = 0$ for each e).
  2. While there exists an s-t path P in G such that $f(e) < c(e)$ for each $e \in P$
     a. Find any such path P
     b. Compute $\Delta = \min_{e \in P} \big(c(e) - f(e)\big)$
     c. Increase the flow on each edge $e \in P$ by Δ

- Note
  - Capacity and flow conservation constraints remain satisfied

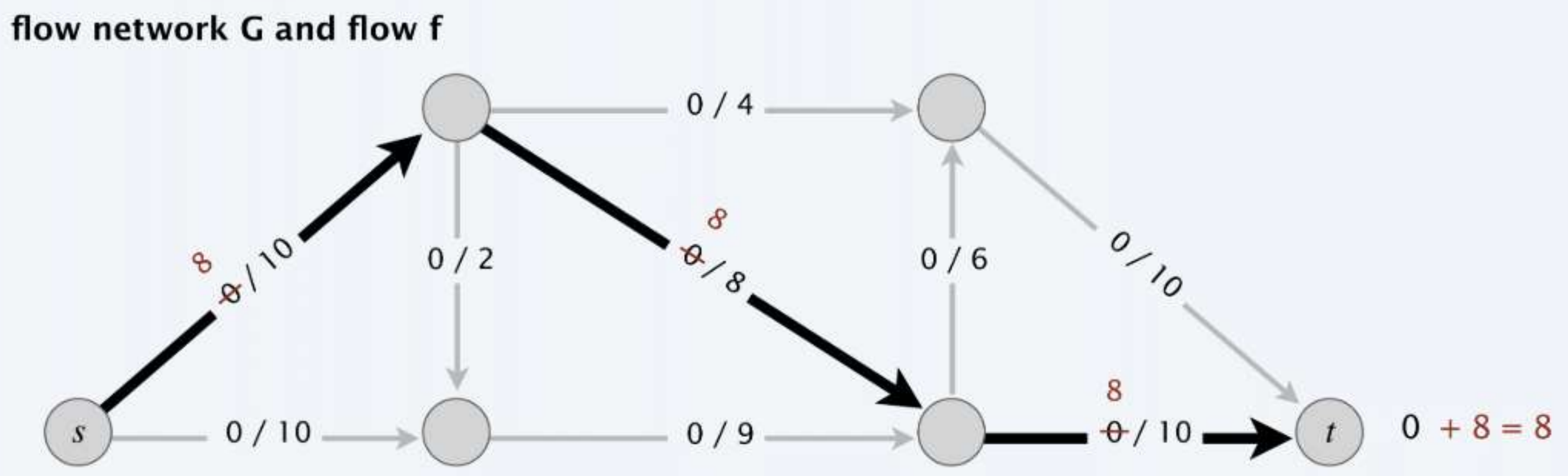
First Attempt



First Attempt



First Attempt



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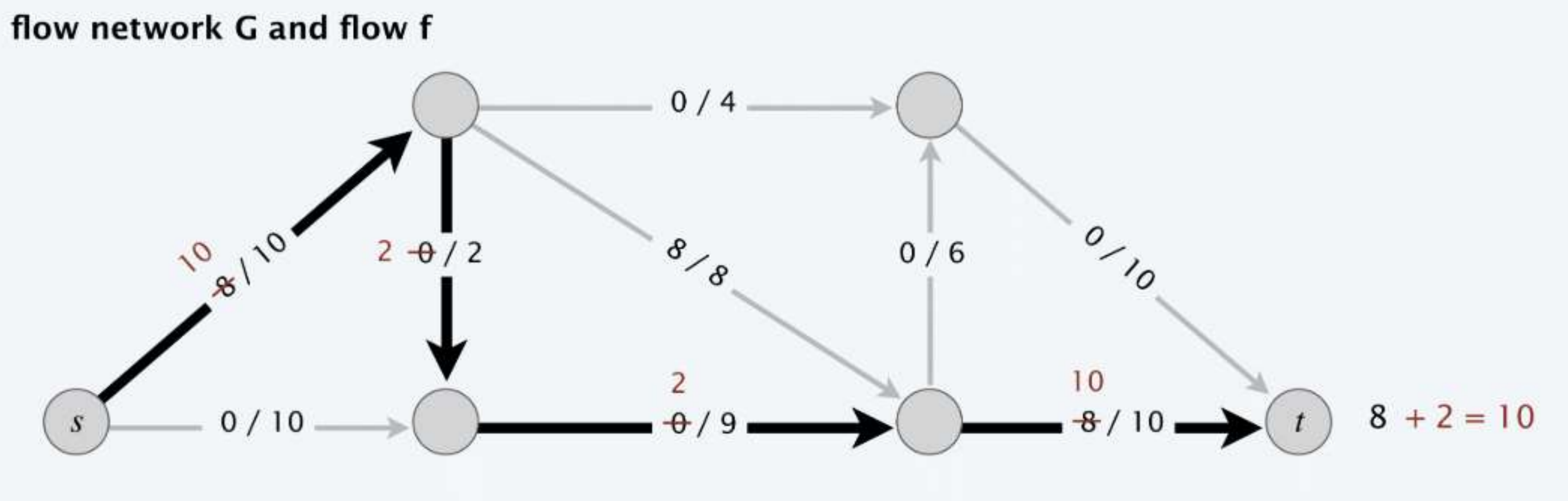
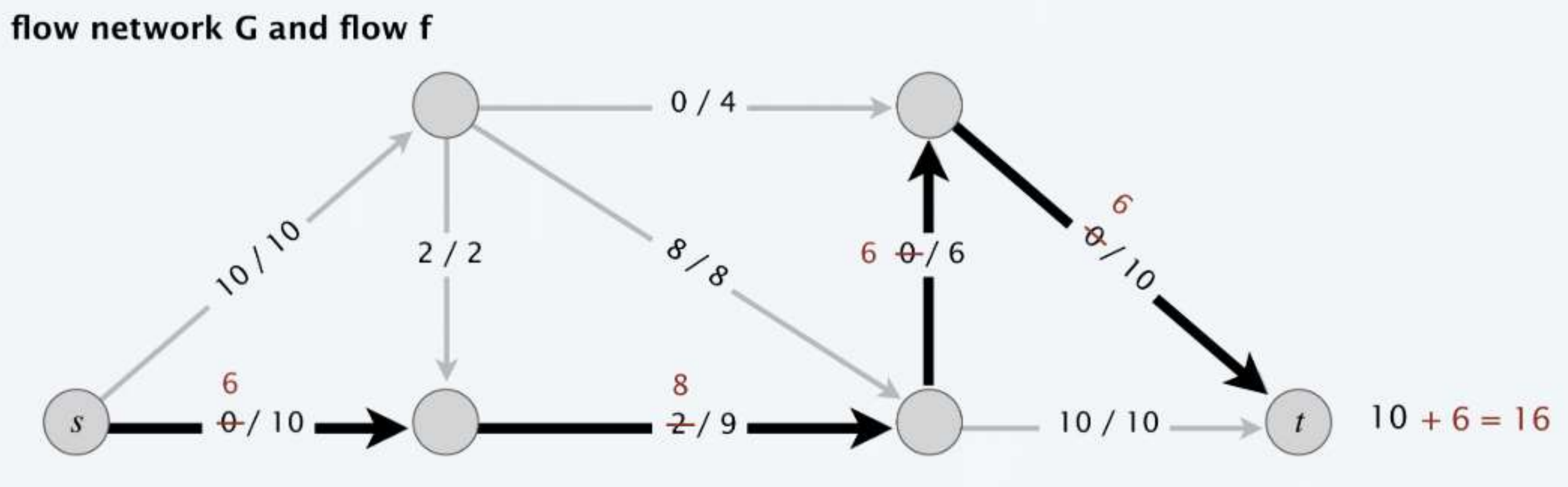
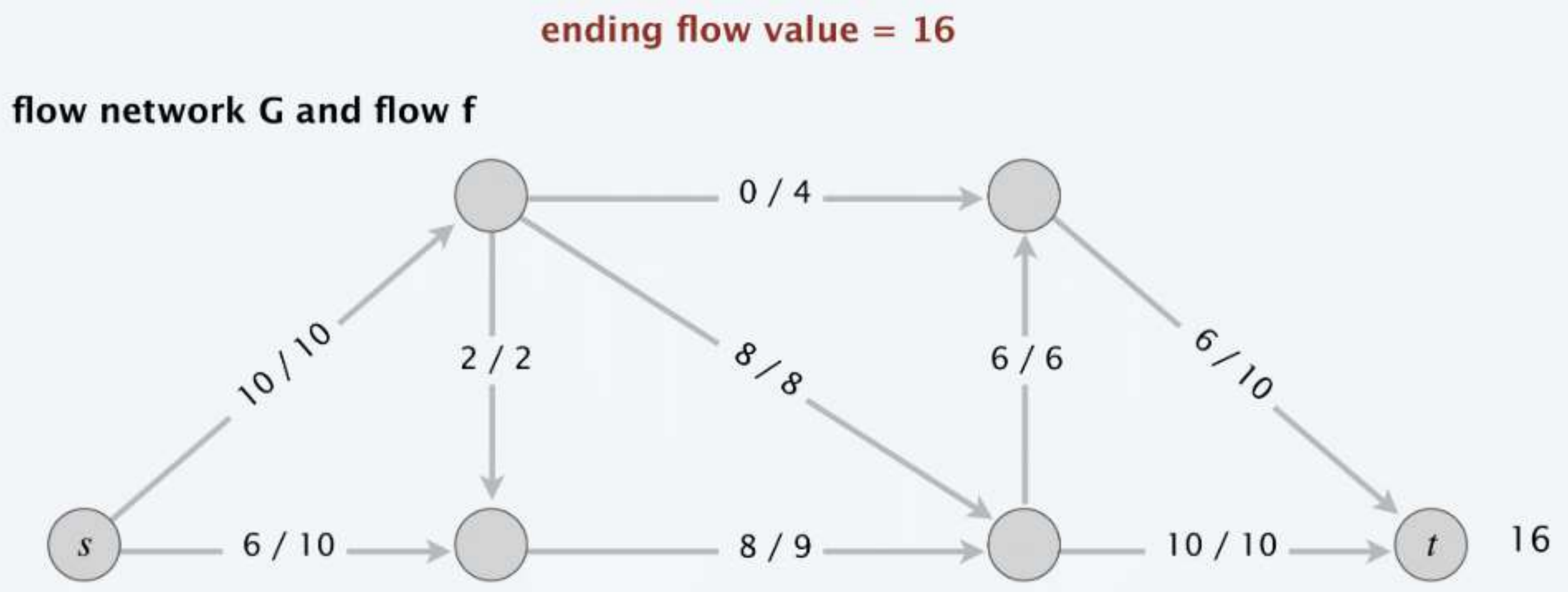


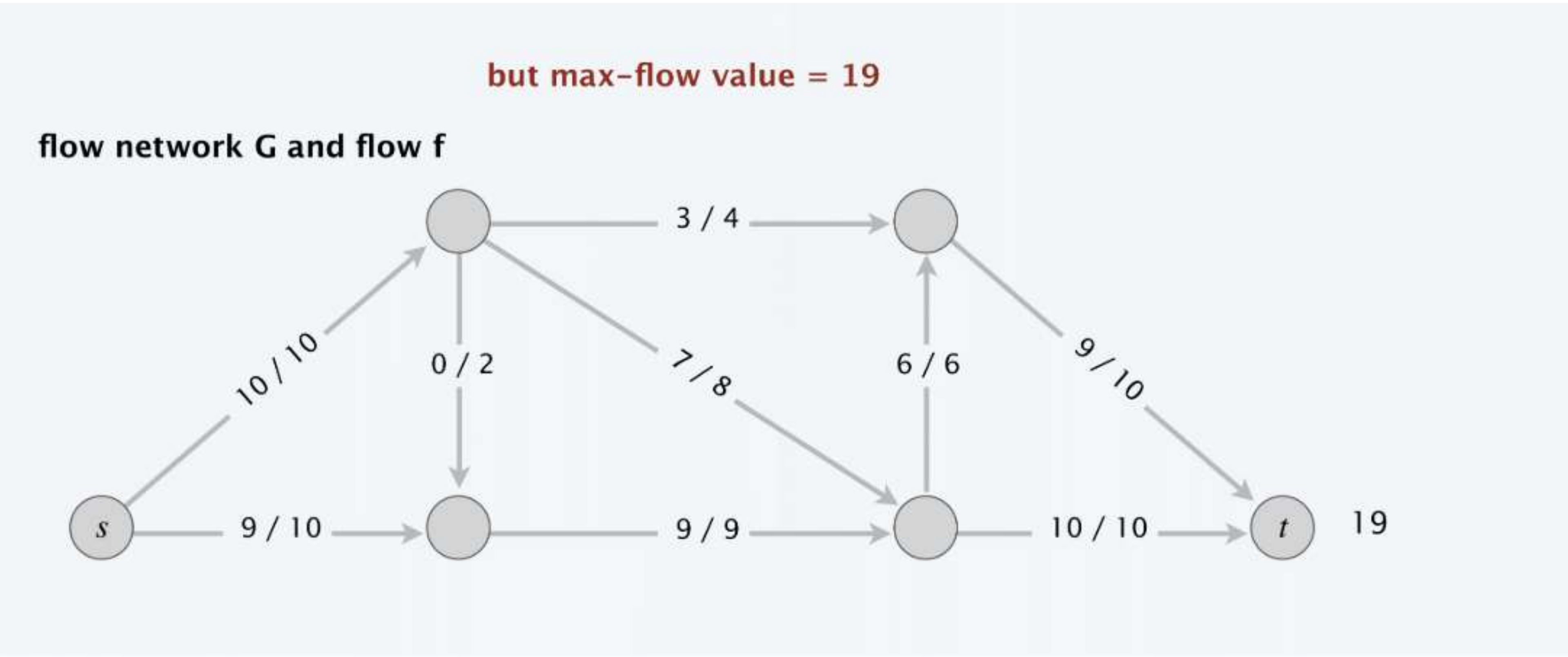
First Attempt



First Attempt



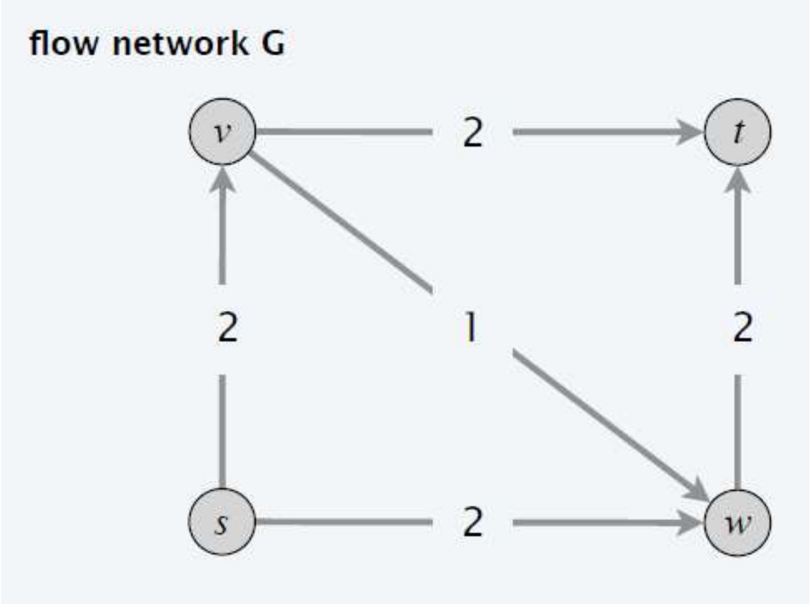
First Attempt



First Attempt

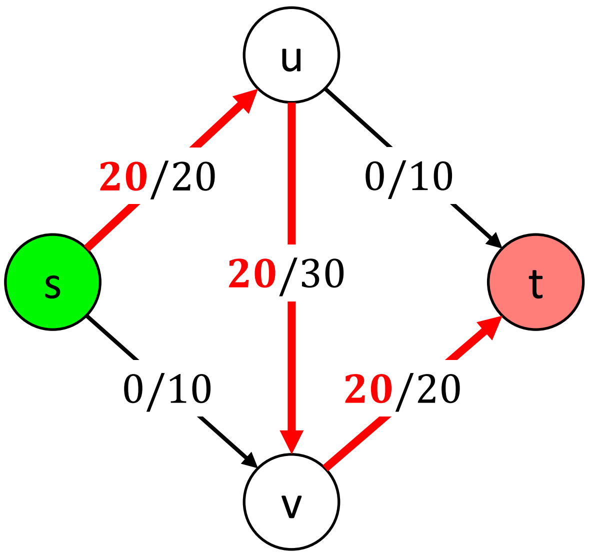
- Q: Why does the simple greedy approach fail?
- A: Because once it increases the flow on an edge, it is not allowed to decrease it ever in the future.

- Need a way to "reverse" bad decisions

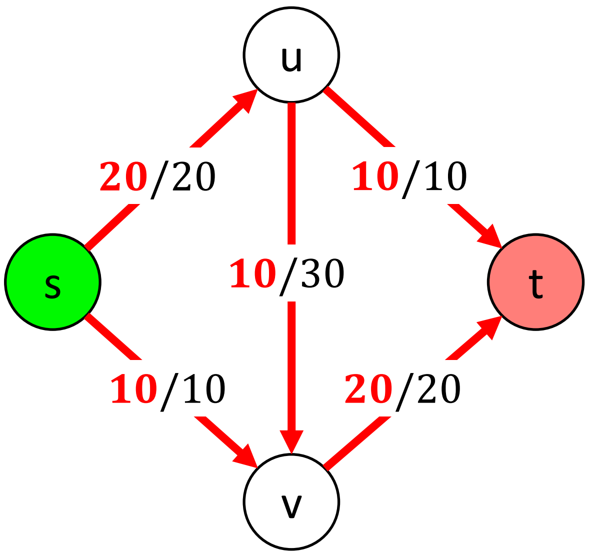


Reversing Bad Decisions

Suppose we start by sending
20 units of flow along this path

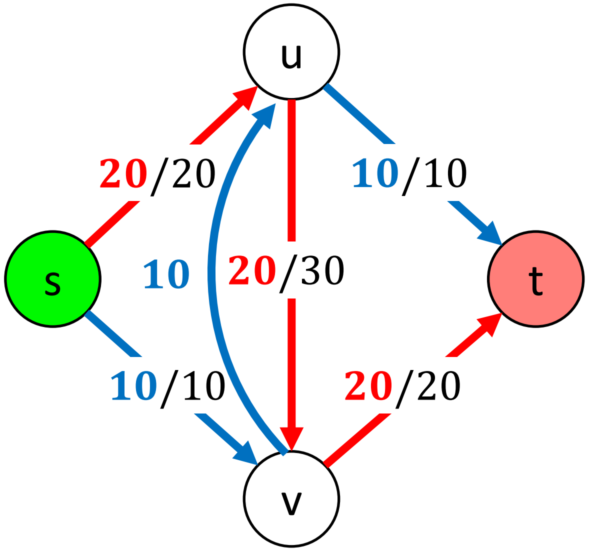


But the optimal configuration requires
10 fewer units of flow on $u \to v$

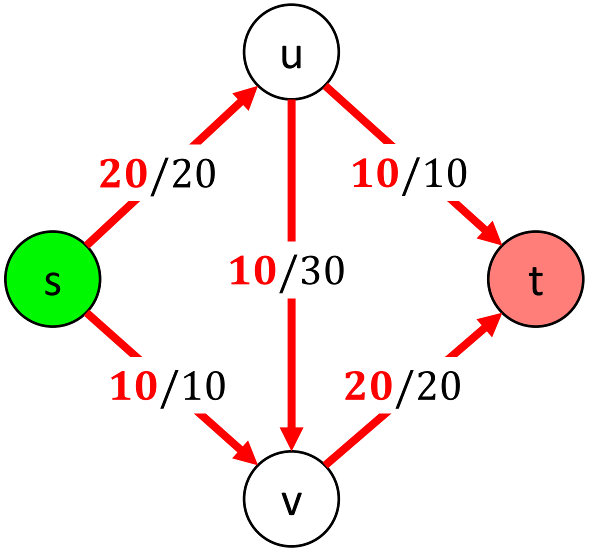


Reversing Bad Decisions

We can essentially send a "reverse" flow of 10 units along $v \to u$



So now we get this optimal flow



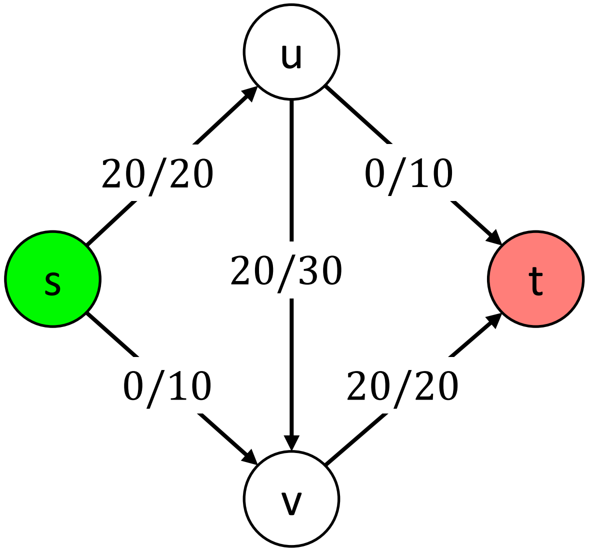
Residual Graph

- Suppose the current flow is f
- Define the residual graph G_f of flow f
  - G_f has the same vertices as G
  - For each edge $\mathrm{e} = (u, v)$ in G, G_f has at most two edges
    - Forward edge $e = (u, v)$ with capacity $c(e) - f(e)$
      - We can send this much additional flow on e
    - Reverse edge $e^{rev} = (v, u)$ with capacity $f(e)$
      - The maximum "reverse" flow we can send is the maximum amount by which we can reduce flow on e, which is $f(e)$
    - We only really add edges of capacity > 0

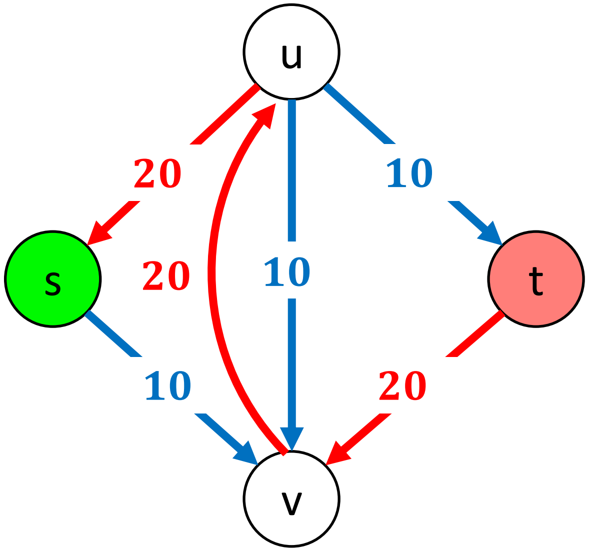
Residual Graph

- Example!

Flow f



Residual graph G_f



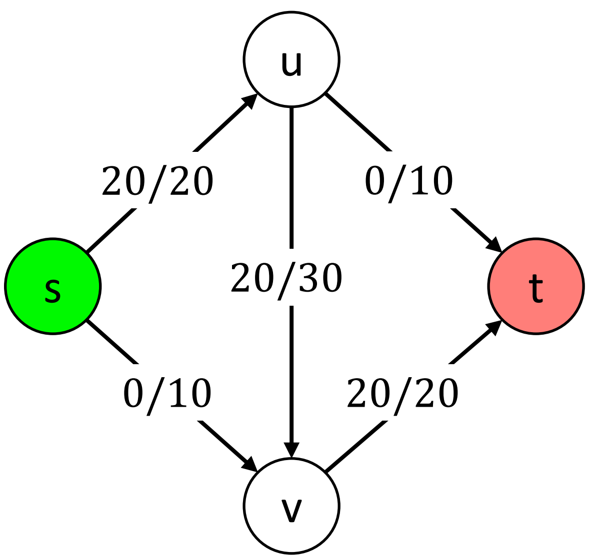
Augmenting Paths

- Let P be an s-t path in the residual graph G_f
- Let $\text{bottleneck}(P, f)$ be the smallest capacity across all edges in P
- "Augment" flow f by "sending" $\text{bottleneck}(P, f)$ units of flow along P
  - What does it mean to send x units of flow along P?
  - For each forward edge $e \in P$, increase the flow on e by x
  - For each reverse edge $e^{rev} \in P$, decrease the flow on e by x

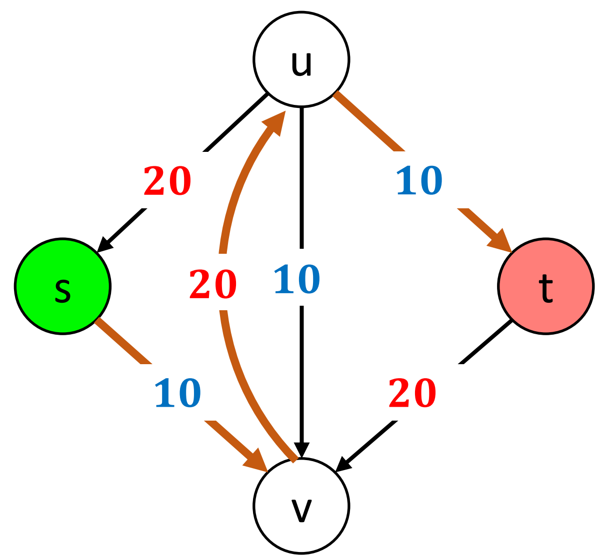
Residual Graph

- Example!

Flow f



Residual graph G_f

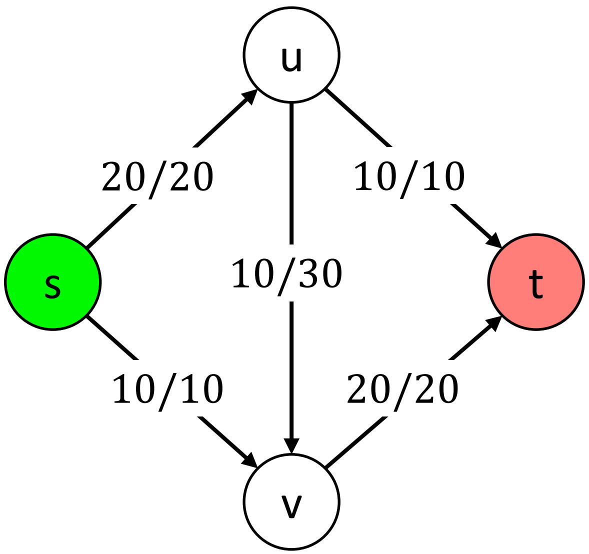


Path P → send flow = bottleneck = 10

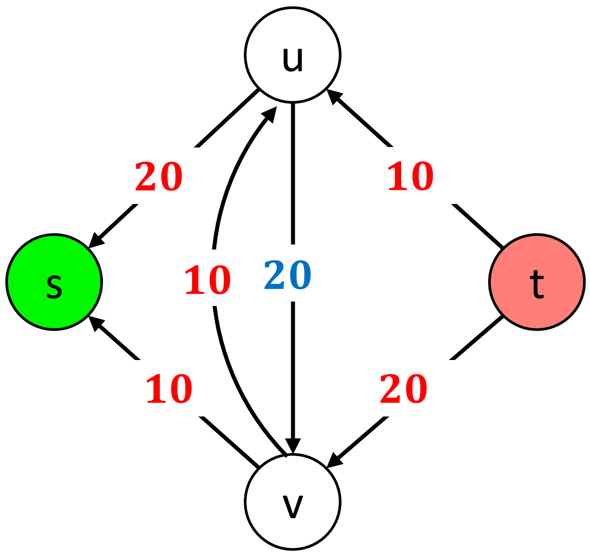
Residual Graph

- Example!

New flow f



New residual graph G_f



No s-t path because no outgoing edge from s

Augmenting Paths

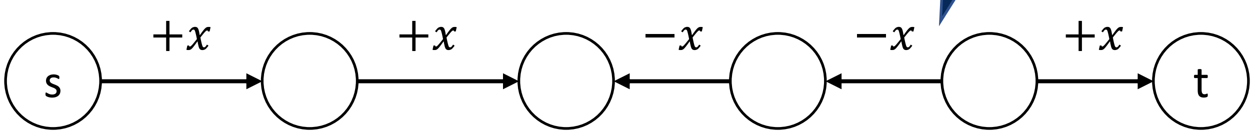
- Let's argue that the new flow is a valid flow
- Capacity constraints (easy):
  - If we increase flow on e, we can do so by at most the capacity of forward edge e in G_f, which is $c(e) - f(e)$
    - So, the new flow can be at most $f(e) + \big(c(e) - f(e)\big) = c(e)$
  - If we decrease flow on e, we can do so by at most the capacity of reverse edge e^{rev} in G_f, which is $f(e)$
    - So, the new flow is at least $f(e) - f(e) = 0$

Augmenting Paths

- Let's argue that the new flow is a valid flow

- Flow conservation (a bit trickier):
  - Each node on the path (except s and t) has exactly two incident edges
    - Both forward / both reverse $\Rightarrow$ one is incoming, one is outgoing
      - Flow increased on both or decreased on both
    - One forward, one reverse $\Rightarrow$ both incoming / both outgoing
      - Flow increased on one but decreased on the other
    - In each case, net flow remains 0

Edge directions as in G



Ford-Fulkerson Algorithm

```
MaxFlow(G):

  // initialize:
  Set f(e) = 0 for all e in G

  // while there is an s-t path in G_f:
  While P = FindPath(s,t,Residual(G,f))!=None:
    f = Augment(f,P)
    UpdateResidual(G,f)
  EndWhile

  Return f
```

Ford-Fulkerson Algorithm

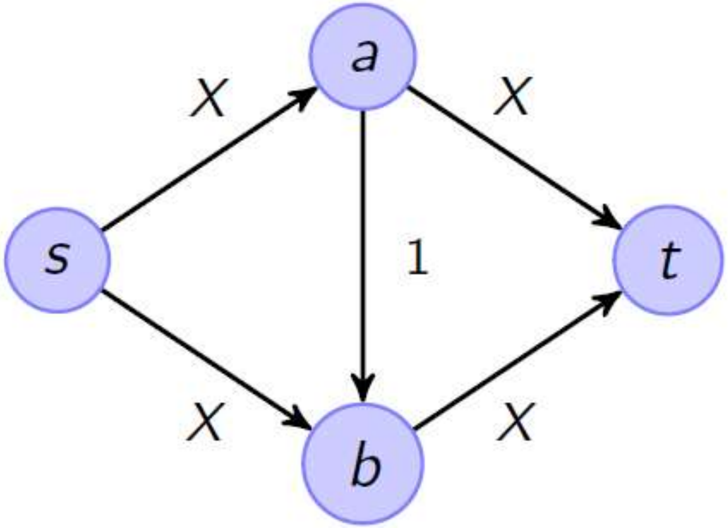
- Running time:
  - #Augmentations:
    - At every step, flow and capacities remain integers
    - For path P in G_f, $\text{bottleneck}(P, f) > 0$ implies $\text{bottleneck}(P, f) \geq 1$
    - Each augmentation increases flow by at least 1
    - Max flow (hence max #augmentations) is at most $C = \sum_{e \text{ leaving } s} c(e)$
  - Time to perform an augmentation:
    - G_f has n vertices and at most $2m$ edges
    - Finding P, computing $\text{bottleneck}(P, f)$, updating G_f
      - $O(m + n)$ time
  - Total time: $O((m + n) \cdot C)$

Ford-Fulkerson Algorithm

- Total time: $O((m+n)\cdot C)$
  - This is pseudo-polynomial time, but NOT polynomial time
  - The value of C can be exponentially large in the input length (the number of bits required to write down the edge capacities)
- Q: Can we convert this to polynomial time?

Ford-Fulkerson Algorithm

- Q: Can we convert this to polynomial time?
  - Not if we choose an *arbitrary* path in G_f at each step
  - In the graph below, we might end up repeatedly sending 1 unit of flow across $a \rightarrow b$ and then reversing it
    - Takes X steps, which can be exponential in the input length



Ford-Fulkerson Algorithm

- Ways to achieve polynomial time
  - Find the maximum bottleneck capacity augmenting path
    - Runs in $O(m^2 \cdot \log C)$ operations
      - "Weakly polynomial time"
  - Find the shortest augmenting path using BFS
    - Edmonds-Karp algorithm
    - Runs in $O(nm^2)$ operations
      - "Strongly polynomial time"
    - Can be found in CLRS
  - …

Max Flow Problem

- Race to reduce the running time
  - 1972: $O(n\,m^2)$ Edmonds-Karp
  - 1980: $O(n\,m\log^2 n)$ Galil-Namaad
  - 1983: $O(n\,m\log n)$ Sleator-Tarjan
  - 1986: $O\left(n\,m\log\left({n^2}/{m}\right)\right)$ Goldberg-Tarjan
  - 1992: $O(n\,m + n^{2+\epsilon})$ King-Rao-Tarjan
  - 1996: $O\left(n\,m\dfrac{\log n}{\log {m}/{n\log n}}\right)$ King-Rao-Tarjan
    - Note: These are $O(n\,m)$ when $m = \omega(n)$
  - 2013: $O(n\,m)$ Orlin
    - Breakthrough!
  - 2021: $O((m + n^{1.5})\cdot\log X)$, where X = max edge capacity
    - Breakthrough based on very heavy techniques!

Back to Ford-Fulkerson

- We argued that the algorithm must terminate, and must terminate in $O\big((m+n)\cdot C\big)$ time

- But we didn't argue correctness yet, i.e., the algorithm must terminate with the optimal flow

Recall: Ford-Fulkerson

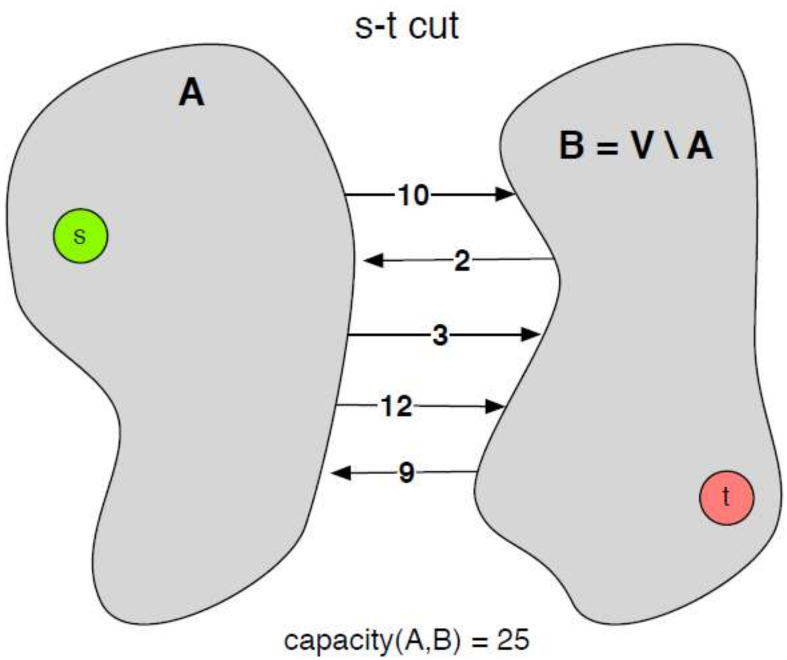
```
MaxFlow(G):

  // initialize:
  Set f(e) = 0 for all e in G

  // while there is an s-t path in G_f:
  While P = FindPath(s,t,Residual(G,f))!=None:
    f = Augment(f,P)
    UpdateResidual(G,f)
  EndWhile

  Return f
```

Recall: Notation

- f = flow, s = source, t = target
- f^{out}, f^{in}
  - For a node u: $f^{out}(u)$, $f^{in}(u)$ = total flow out of and into u
  - For a set of nodes X: $f^{out}(X)$, $f^{in}(X)$ defined similarly
- Constraints
  - Capacity: $0 \leq f(e) \leq c(e)$
  - Flow conservation: $f^{out}(u) = f^{in}(u)$ for all $u \neq s, t$
- $v(f)$ = $f^{out}(s)$ = $f^{in}(t)$ = value of the flow

Cuts and Cut Capacities

- (A, B) is an s-t cut if it is a partition of vertex set V (i.e., $A \cup B = V$, $A \cap B = \emptyset$) with $s \in A$ and $t \in B$
- Its capacity, denoted $cap(A, B)$, is the sum of capacities of edges *leaving* A

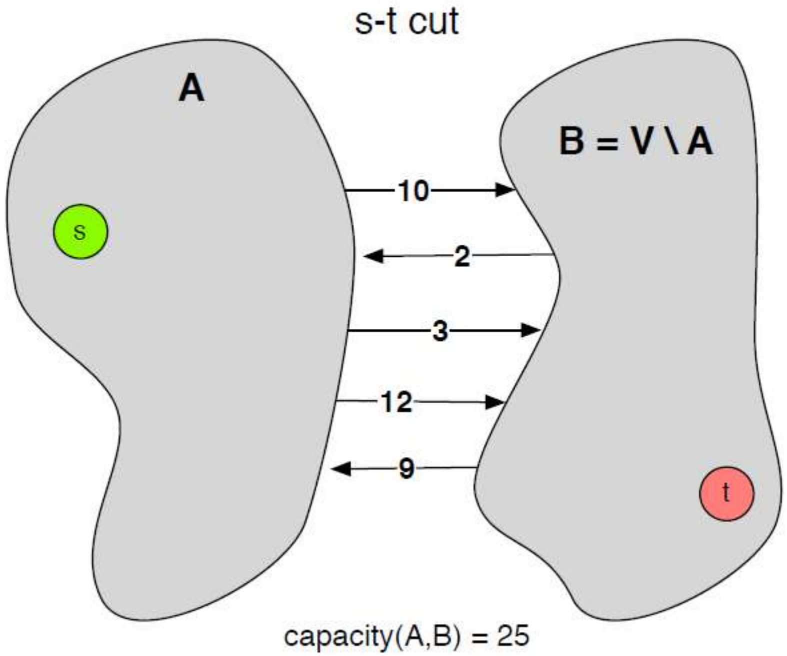


Cuts and Flows

- Theorem: For any flow f and any s-t cut (A, B),

$$v(f) = f^{out}(A) - f^{in}(A)$$

- Proof (on the board): Just take a sum of the flow conservation constraint over all nodes in A



Cuts and Flows

- Theorem: For any flow f and any s-t cut (A, B),

$$v(f) \le cap(A, B)$$

- Proof:

$$\begin{aligned} v(f) &= f^{out}(A) - f^{in}(A) \\ &\le f^{out}(A) \\ &= \textstyle\sum_{e \text{ leaving } A} f(e) \\ &\le \textstyle\sum_{e \text{ leaving } A} c(e) \\ &= cap(A, B) \end{aligned}$$

Cuts and Flows

- Theorem: For any flow f and any s-t cut (A, B),

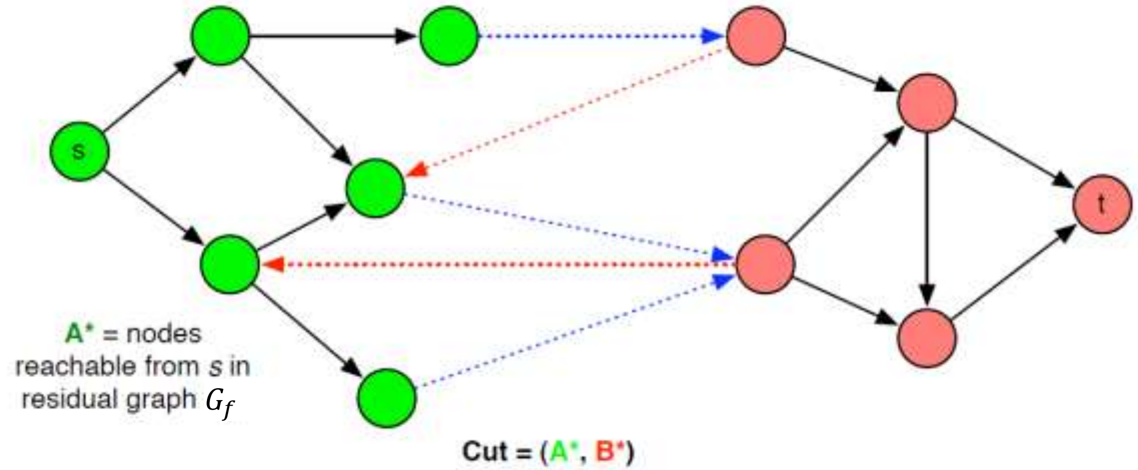
$$v(f) \leq cap(A, B)$$

- Hence, $\max_f v(f) \leq \min_{(A,B)} cap(A, B)$
  - Max value of any flow $\leq$ min capacity of any s-t cut
- We will now prove:
  - Value of flow generated by Ford-Fulkerson = capacity of *some* cut
- Implications
  - 1) Max flow = min cut
  - 2) Ford-Fulkerson generates max flow.

Cuts and Flows

- Theorem: Ford-Fulkerson finds maximum flow.
- Proof:
  - f = flow returned by Ford-Fulkerson
  - A^* = nodes reachable from s in G_f
  - B^* = remaining nodes $V \setminus A^*$
  - Note: We look at the residual graph G_f, but define the cut in G

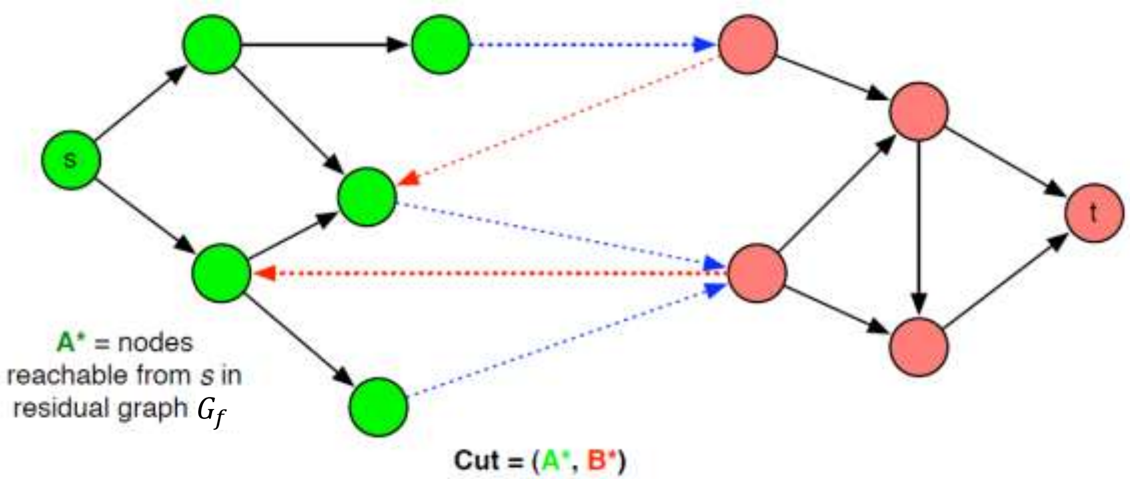
Graph G



Cuts and Flows

- Theorem: Ford-Fulkerson finds maximum flow.
- Proof:
  - Claim: (A^*, B^*) is a valid cut
    - $s \in A^*$ by definition
    - $t \in B^*$ because when Ford-Fulkerson terminates, there are no s-t paths in G_f, so $t \notin A^*$

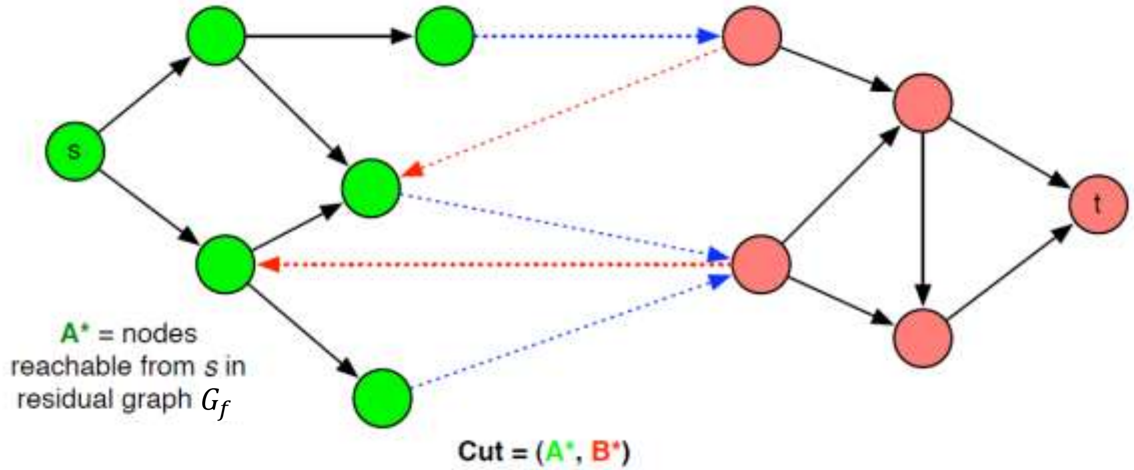
Graph G



Cuts and Flows

- Theorem: Ford-Fulkerson finds maximum flow.
- Proof:
  - Blue edges = edges going out of A^* $\boxed{\text{in } G}$
  - Red edges = edges coming into A^* $\boxed{\text{in } G}$

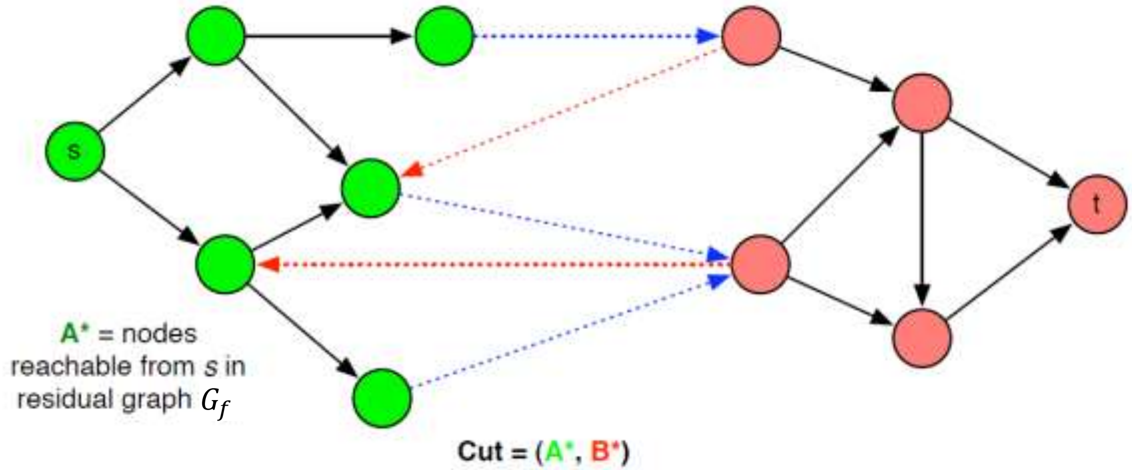
Graph G



Cuts and Flows

- Theorem: Ford-Fulkerson finds maximum flow.
- Proof:
  - Each blue edge (u, v) must be saturated
    - Otherwise G_f would have its forward edge (u, v) and then $v \in A^*$
  - Each red edge (v, u) must have zero flow
    - Otherwise G_f would have its reverse edge (u, v) and then $v \in A^*$

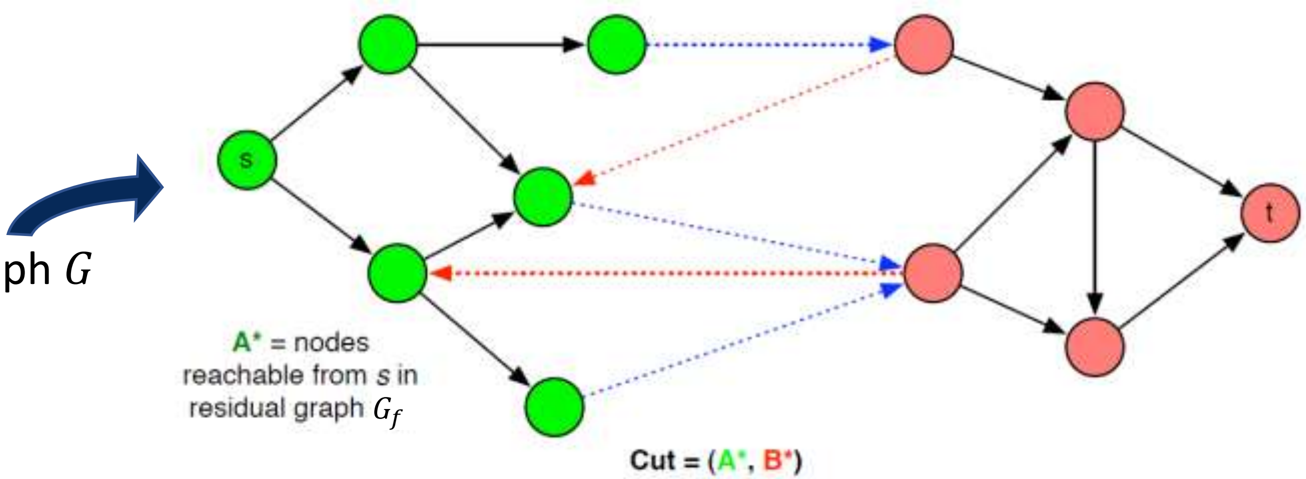
Graph G



Cuts and Flows

- Theorem: Ford-Fulkerson finds maximum flow.
- Proof:
  - Each blue edge (u, v) must be saturated $\Rightarrow f^{out}(A^*) = cap(A^*, B^*)$
  - Each red edge (v, u) must have zero flow $\Rightarrow f^{in}(A^*) = 0$
  - So $v(f) = f^{out}(A^*) - f^{in}(A^*) = cap(A^*, B^*)$ ∎

Graph G



Max Flow - Min Cut

- Max Flow-Min Cut Theorem:
  In any graph, the value of the maximum flow is equal to the capacity of the minimum cut.

- Our proof already gives an algorithm to find a min cut
  - Run Ford-Fulkerson to find a max flow f
  - Construct its residual graph G_f
  - Let $A^* =$ set of all nodes reachable from s in G_f
    - Easy to compute using BFS
  - Then $(A^*, V \setminus A^*)$ is a min cut

Poll

Question

- There is a network G with positive integer edge capacities.
- You run Ford-Fulkerson.
- It finds an augmenting path with bottleneck capacity 1, and after that iteration, it terminates with a final flow value of 1.
- Which of the following statement(s) must be correct about G?

(a) G has a single s-t path.

(b) G has an edge e such that all s-t paths go through e.

(c) The minimum cut capacity in G is greater than 1.

(d) The minimum cut capacity in G is less than 1.

Why Study Flow Networks?

- Unlike divide-and-conquer, greedy, or DP, this doesn't seem like an algorithmic framework
  - It seems more like a single problem

- Turns out that many problems can be reduced to this versatile single problem

- Next lecture!