CSC373

Algorithm Design, Analysis & Complexity

Deepanshu Kush

Introduction

- Instructor
  - Deepanshu Kush
    - cs.toronto.edu/~deepkush/
    - Email: csc373-2022-05@cs.toronto.edu
    - TAs: Mian, Lily, Yibin, Soroush
- Disclaimer!
  - First time being an instructor, so expect a somewhat bumpy ride at the start, but hopefully, we'll get through it together and have fun!
  - Use any of the feedback mediums (email, Piazza, …) to let me know if you have any suggestions for improvement

Course Information

- Course Page www.cs.toronto.edu/~deepkush/teaching/373s22/
- Discussion Board piazza.com/utoronto.ca/summer2022/csc373h1y

- Main mode of communication will be Piazza – make sure you sign up soon!

- Grading – MarkUs
  - LaTeX preferred, scans are OK!

- All times will be in the Eastern time zone

- Course info sheet: Be sure to go through it! (Find it under 'Course Info' tab)

Lectures & Tutorials

- Lectures for LEC 5101 (Only Section!)
  - Wed 6-9pm, BA 1170
- Tutorials
  - Thu 6-7pm
  - In-person (room details on the course webpage)

Delivery

- All lectures and tutorials in person!

- Lectures will be recorded and posted on the course webpage afterwards

Masking Policy

- Students are required to wear a mask at all times (“intermittent, temporary removal of masks may occur e.g. demonstration of a procedure, drinking water, short break as per EHS/division approval”)
- We have been able to obtain an exemption for the instructor (on account of the being recorded and its somewhat long duration)
- Feel free to write to me if you ever have concerns about health & safety measures in the lecture/tutorial classrooms

Lecture Format

- Delivered by me

- Will start at 10 minutes past the hour
  - 2 10-minute breaks at the hour marks in the 3-hour slot

- In-person: Ask questions by raising your hand/Speak up

Tutorial Format

- Delivered by the TAs
- Think of them as preparation for assignments/exams
  - Some of the tutorial problems may be easier than assignment/exam questions
- Problem sets & solutions
  - Problem sets will be posted to the course webpage in advance of the tutorial
  - Solutions will be posted to the course webpage after the tutorial
- What to do
  - Please attempt the problems before coming to the tutorials
  - During the tutorials, the TAs will go over the solutions and explain key ideas

Tutorial Format

- Further details
  - The class is divided into three parts (A,B,C)
  - Division by birth month: A = Jan-Apr, B = May-Aug, C = Sep-Dec
  - Feel free to attend a different tutorial than the one you're assigned
  - If the tutorial attendance is really low, the number of tutorials per section may be reduced

Office Hours

- Time & Place: Wed 1-3pm, Zoom during second hour
  - If you have a conflict with this slot, feel free to schedule 1-1 office hours by emailing me
- Details
  - I will conduct them
  - Use the “raise hand” feature
  - When I call your name, unmute and ask the question
  - Try to phrase your question without giving away your approach/solution to an assignment problem
    - If this is not possible, we will go to a breakout room

Tests

- 2 term tests, one end-of-term test (final exam/assessment)

- Time & Place
  - First midterm is sometime during June 22-27
  - Exact date to be announced soon (the FAS decides this centrally to avoid clashes)
  - Second midterm on July 27 (Wednesday, during class hours)

- Delivery method: in person
- Extra Office Hours: by TAs in the week prior to a test – remind me!

Assignments

- 4 assignments, best 3 out of 4
- Group work
  - In groups of up to three students
  - Best way to learn is for each member to try each problem
- Questions will be more difficult
  - May need to mull them over for several days; do *not* expect to start and finish the assignment on the same day!
  - May include bonus questions
- Submission on MarkUs, more details on the webpage/course info sheet
  - May need to compress the PDF

Grading Policy

• Best 3/4 homeworks	*	10%	=	30%
• 2 term tests	*	20%	=	40%
• Final exam	*	30%	=	30%

- NOTE: To pass, you must earn at least 40% on the final exam

Approximate Due Dates

- Assignment 1: May 31
- Assignment 2: June 15
- Assignment 3: July 16
- Assignment 4: August 7
- Midterm 1: June 22-27
- Midterm 2: July 27

Textbook

- Primary reference: lecture slides

- Primary textbook (required)
  - [CLRS] Cormen, Leiserson, Rivest, Stein: *Introduction to Algorithms*.

- Supplementary textbooks (optional)
  - [DPV] Dasgupta, Papadimitriou, Vazirani: *Algorithms*.
  - [KT] Kleinberg; Tardos: *Algorithm Design*.

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FACULTY OF ARTS & SCIENCE

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CCR
CO-CURRICULAR RECORD

Other Policies

- Collaboration
  - Free to discuss with classmates or read online material
  - Must write solutions in your own words
    - Easier if you do not take any pictures/notes from discussions

- Citation
  - For each question, must cite the peer (write the name) or the online sources (provide links), if you obtained a significant insight directly pertinent to the question
  - Failing to do this is plagiarism!

Other Policies

- “No Garbage” Policy
  - Borrowed from: Prof. Allan Borodin (citation!)
  - Applies to assignments (except for bonus questions) and tests

1. Partial marks for viable approaches
2. Zero marks if the answer makes no sense
3. 20% marks if you admit to not knowing how to approach the question (“I do not know how to approach this question”)

- 20% > 0% !!

Other Policies

- Late Days
  - 4 total late days across all 4 assignments
  - Managed by MarkUs
  - At most 2 late days can be applied to a single assignment
  - Already covers legitimate reasons such as illness, university activities, etc.
    - Petitions will only be granted for circumstances which cannot be covered by this

How to address me

- “Kush”, “Deepanshu”, “Deep”, “Deeps” – all is good
- Sir/Professor is okay
- But not ideal!

Questions?

Enough with the boring stuff.

What will we study?

Why will we study it?

Muhammad ibn Musa al-Khwarizmi
c. 780 – c. 850

What is this course about?

- Algorithms
  - Ubiquitous in the real world
    - From your smartphone to self-driving cars
    - From graph problems to graphics problems
    - …
  - Important to be able to design and analyze algorithms
  - For some problems, good algorithms are hard to find
    - For some of these problems, we can formally establish complexity results
    - We’ll often find that one problem is easy, but its minor variants are suddenly hard

What is this course about?

- Algorithms
  - Algorithms in specialized environments or using advanced techniques
    - Distributed, parallel, streaming, sublinear time, spectral, genetic…
  - Other concerns with algorithms
    - Fairness, ethics, …
  - …mostly beyond the scope of this course

What is this course about?

- Designing fast algorithms
  - Divide and Conquer
  - Greedy
  - Dynamic programming
  - Network flow
  - Linear programming
- Proving that no fast algorithms are likely possible
  - Reductions & NP-completeness
- What to do if no fast algorithms are likely possible
  - Approximation algorithms (if time permits)
  - Randomized algorithms (if time permits)

What is this course about?

- How do we know which paradigm is right for a given problem?
  - A very interesting question!
  - Subject of much ongoing research…
    - Sometimes, you just know it when you see it…
- How do we analyze an algorithm?
  - Proof of correctness
  - Proof of running time
    - We'll try to prove the algorithm is *efficient* in the *worst case*
    - In practice, average case matters just as much (or even more)

What is this course about?

- What does it mean for an algorithm to be efficient in the worst case?
  - Polynomial time
  - It should use at most poly(n) steps on *any* n-bit input
    - $n, n^2, n^{100}, 100n^6 + 237n^2 + 432, \ldots$
  - If the input to an algorithm is a number x, the number of bits of input is $\log x$
    - This is because it takes $\log x$ bits to represent the input x in binary
    - So the running time should be polynomial in $\log x$, not in x
  - How much is too much?

What is this course about?

Picture-Hanging Puzzles*

Erik D. Demaine† Martin L. Demaine† Yair N. Minsky‡ Joseph S. B. Mitchell§
Ronald L. Rivest† Mihai Pătraşcu¶

Theorem 7 *For any $n \geq k \geq 1$, there is a picture hanging on n nails, of length $n^{c'}$ for a constant c', that falls upon the removal of any k of the nails.*

$n^{6{,}100 \log_2 c}$. Using the $c \leq 1{,}078$ upper bound, we obtain an upper bound of $c' \leq 6{,}575{,}800$. Using

So, while this construction is polynomial, it is a rather large polynomial. For small values of n, we can use known small sorting networks to obtain somewhat reasonable constructions.

What is this course about?

Better Balance by Being Biased:
A 0.8776-Approximation for Max Bisection

Per Austrin*, Siavosh Benabbas*, and Konstantinos Georgiou†

---

has a lot of flexibility, indicating that further improvements may be possible. We remark that, while polynomial, the running time of the algorithm is somewhat abysmal; loose estimates places it somewhere around $O\left(n^{10^{100}}\right)$; the running time of the algorithm of [RT12] is similar.

What is this course about?

- What if we can't find an efficient algorithm for a problem?
  - Try to prove that the problem is hard
  - Formally establish complexity results
  - NP-completeness, NP-hardness, …

- We'll often find that one problem may be easy, but its simple variants may suddenly become hard
  - Minimum spanning tree (MST) vs bounded degree MST
  - 2-colorability vs 3-colorability

I’m not convinced.

Will I really ever need to know how to design abstract algorithms?

At the very least…

This will help you prepare for your technical job interview!

Real Microsoft interview question:

- Given an array a, find indices (i, j) with the largest $j - i$ such that $a[j] > a[i]$
- Greedy? Divide & conquer?

Disclaimer

- The course is theoretical in nature
  - You'll be working with abstract notations, proving correctness of algorithms, analyzing the running time of algorithms, designing new algorithms, and proving complexity results.
  - Think of it as a maths course – you learn only by doing! Try to become a voracious problem solver.

- Something for everyone…
  - If you're somewhat scared going into the course
  - If you're already comfortable with the proofs, and want challenging problems

Related/Follow-up Courses

- Direct follow-up
  - CSC473: Advanced Algorithms
  - CSC438: Computability and Logic
  - CSC463: Computational Complexity and Computability

- Algorithms in other contexts
  - CSC304: Algorithmic Game Theory and Mechanism Design (promoting my buddy Nisarg!)
  - CSC384: Introduction to Artificial Intelligence
  - CSC436: Numerical Algorithms
  - CSC418: Computer Graphics

Divide & Conquer

History?

- Maybe you saw a subset of these algorithms?
  - Mergesort - $O(n \log n)$
  - Karatsuba algorithm for fast multiplication - $O\left(n^{\log_2 3}\right)$ rather than $O(n^2)$
  - Largest subsequence sum in $O(n)$
  - …

- Have you seen some divide & conquer algorithms before?
  - Maybe in CSC236/CSC240 and/or CSC263/CSC265

Divide & Conquer

- General framework
  - Break (a large chunk of) a problem into two smaller subproblems of the same type
  - Solve each subproblem recursively and independently
  - At the end, quickly combine solutions from the two subproblems and/or solve any remaining part of the original problem

- Hard to formally define when a given algorithm is divide-and-conquer…
- Let’s see some examples!

Counting Inversions

- Problem
  - Given an array a of length n, count the number of pairs (i, j) such that $i < j$ but $a[i] > a[j]$
- Applications
  - Voting theory
  - Collaborative filtering
  - Measuring the "sortedness" of an array
  - Sensitivity analysis of Google's ranking function
  - Rank aggregation for meta-searching on the Web
  - Nonparametric statistics (e.g., Kendall's tau distance)

Counting Inversions

- Problem
  - Count (i, j) such that $i < j$ but $a[i] > a[j]$
- Brute force
  - Check all $\Theta(n^2)$ pairs
- Divide & conquer
  - Divide: break array into two equal halves x and y
  - Conquer: count inversions in each half recursively
  - Combine:
    - Solve (we'll see how): count inversions with one entry in x and one in y
    - Merge: add all three counts

Counting Inversions

- From Kevin Wayne's slides

SORT-AND-COUNT (L)

IF list L has one element

  RETURN $(0, L)$.

DIVIDE the list into two halves A and B.

$(r_A , A) \leftarrow$ SORT-AND-COUNT(A).

$(r_B , B) \leftarrow$ SORT-AND-COUNT(B).

$(r_{AB} , L') \leftarrow$ MERGE-AND-COUNT(A, B).

RETURN $(r_A + r_B + r_{AB} ,\ L')$.

Counting Inversions

input

1	5	4	8	10	2	6	9	3	7

count inversions in left half A

1	5	4	8	10

5-4

count inversions in right half B

2	6	9	3	7

6-3 9-3 9-7

count inversions (a, b) with $a \in A$ and $b \in B$

1	5	4	8	10	2	6	9	3	7

4-2 4-3 5-2 5-3 8-2 8-3 8-6 8-7 10-2 10-3 10-6 10-7 10-9

output 1 + 3 + 13 = 17

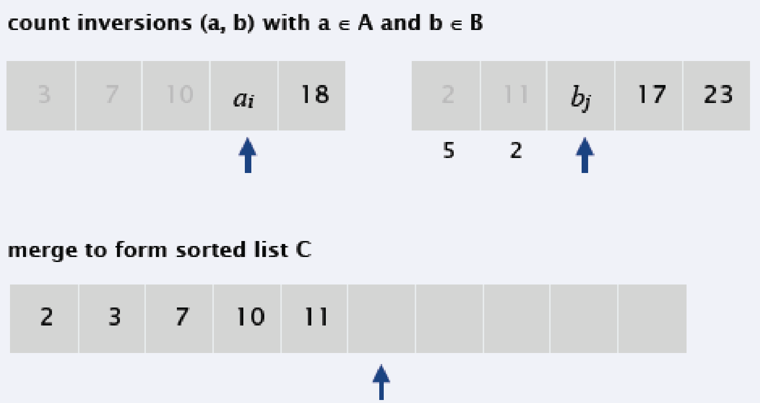
Counting Inversions

Q. How to count inversions (a, b) with $a \in A$ and $b \in B$?

A. Easy if A and B are sorted!

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in B.
- If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.



Counting Inversions

- How do we formally prove correctness?
  - Induction on n is usually very helpful
  - Allows you to assume correctness of subproblems

- Running time analysis
  - Suppose $T(n)$ is the worst-case running time for inputs of size n
  - Our algorithm satisfies $T(n) \le 2\,T(^n/_2) + O(n)$
  - Master theorem says this is $T(n) = O(n \log n)$

Master Theorem

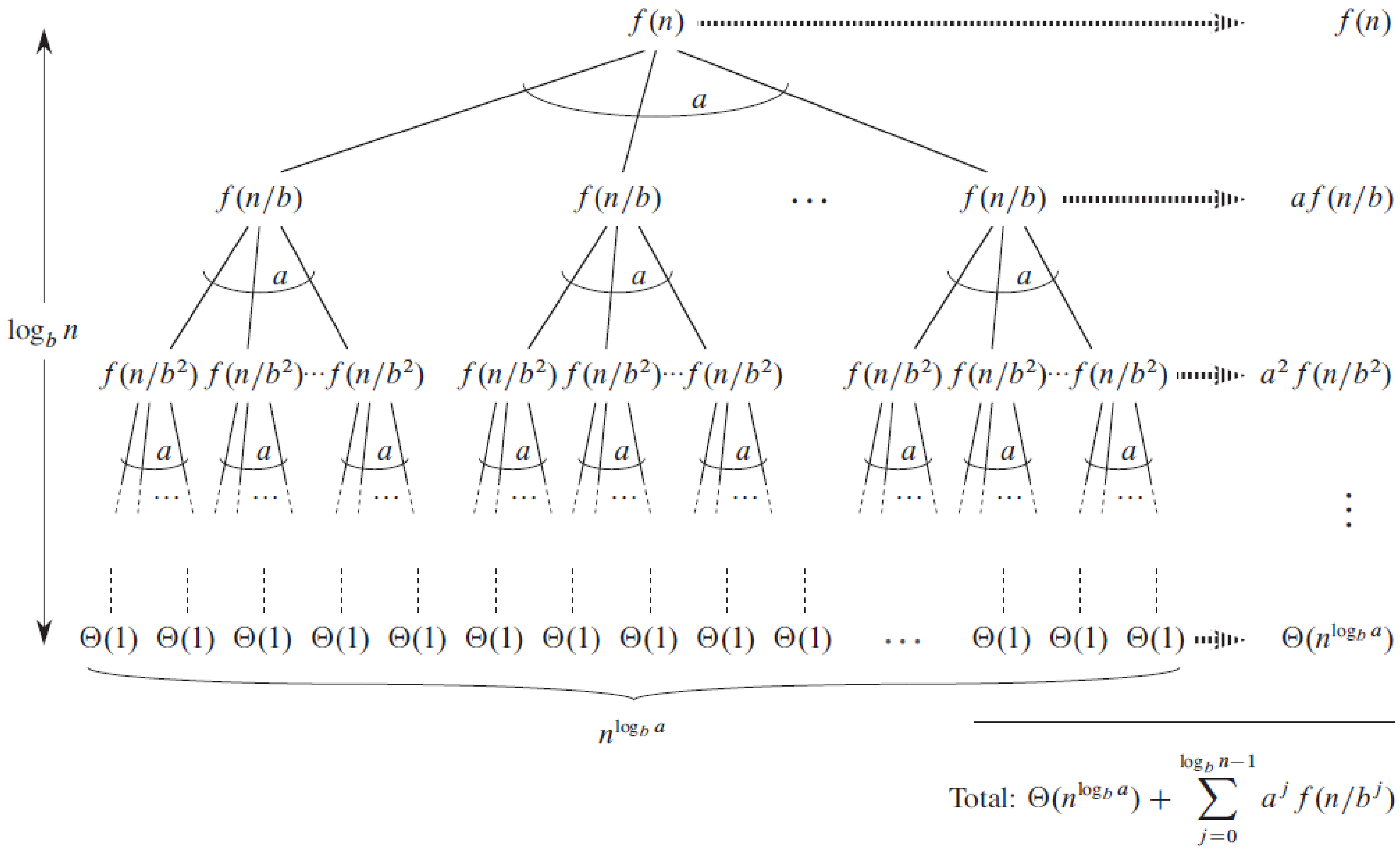
- Here's the master theorem
  - Useful for analyzing divide-and-conquer running time
  - If you haven't already seen it, please spend some time understanding it

  - Theorem: Let $a \geq 1$ and $b > 1$ be constants, $f(n)$ be a function, and $T(n)$ be defined on non-negative integers by the recurrence $T(n) \leq a \cdot T\left(\frac{n}{b}\right) + f(n)$, where n/b can be $\left\lceil\frac{n}{b}\right\rceil$.
    Let $d = \log_b a$. Then:

    - If $f(n) = O\left(n^{d-\epsilon}\right)$ for some constant $\epsilon > 0$, then $T(n) = O\left(n^d\right)$.
    - If $f(n) = O\left(n^d \log^k n\right)$ for some $k \geq 0$, then $T(n) = O\left(n^d \log^{k+1} n\right)$.
    - If $f(n) = O\left(n^{d+\epsilon}\right)$ for some constant $\epsilon > 0$, then $T(n) = O\left(f(n)\right)$.

Master Theorem

Intuition: Compare *f(n)* with $n^{\log_b a}$. The larger determines the recurrence solution.



Closest Pair in $\mathbb{R}^2$

- Problem:
  - Given n points of the form (x_i, y_i) in the plane, find the closest pair of points.
- Applications:
  - Basic primitive in graphics and computer vision
  - Geographic information systems, molecular modeling, air traffic control
  - Special case of nearest neighbor
- Brute force: $\Theta(n^2)$

Intuition from 1D?

- In 1D, the problem would be easily $O(n \log n)$
  - Sort and check!

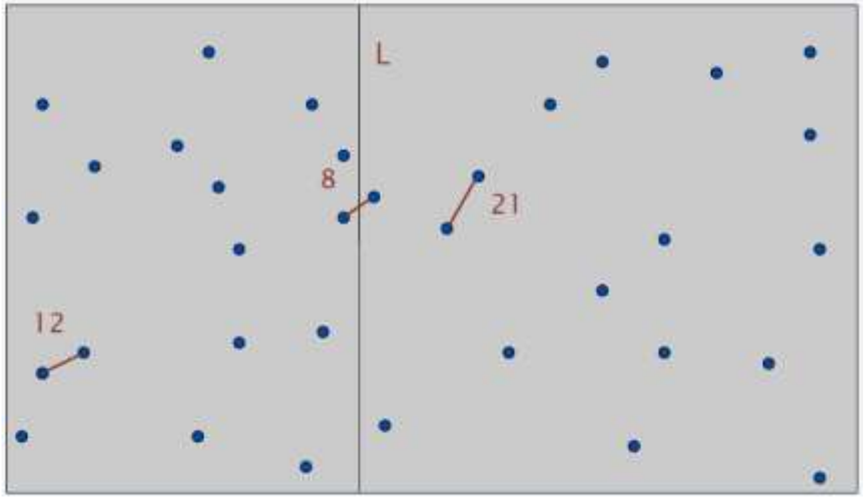
- Sorting attempt in 2D
  - Find closest points by x coordinate
  - Find closest points by y coordinate
  - Doesn't work! (Exercise: come up with a counterexample)

- Non-degeneracy assumption
  - No two points have the same x or y coordinate

Closest Pair in $\mathbb{R}^2$

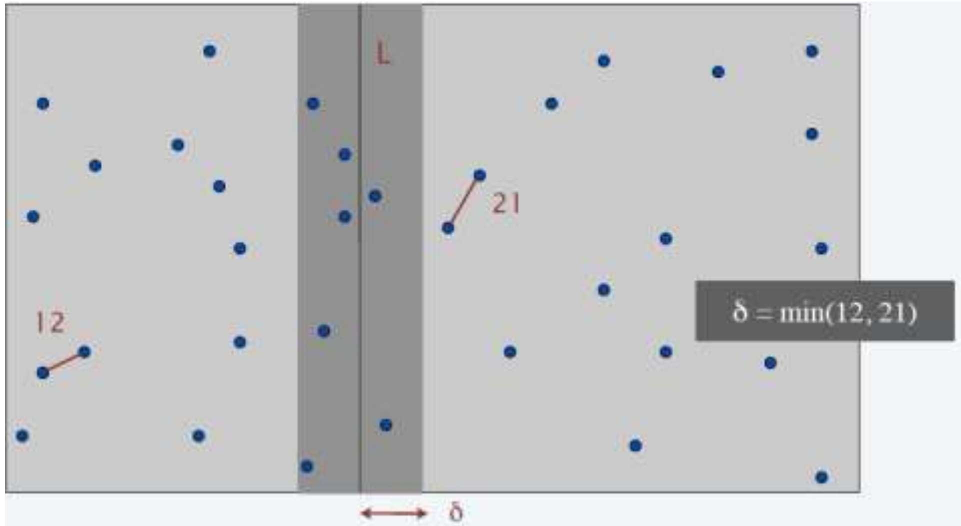
- Let's try divide-and-conquer!
  - Divide: points in equal halves by drawing a vertical line L
  - Conquer: solve each half recursively
  - Combine: find closest pair with one point on each side of L
  - Return the best of 3 solutions

Seems like $\Omega(n^2)$ ☹



Closest Pair in $\mathbb{R}^2$

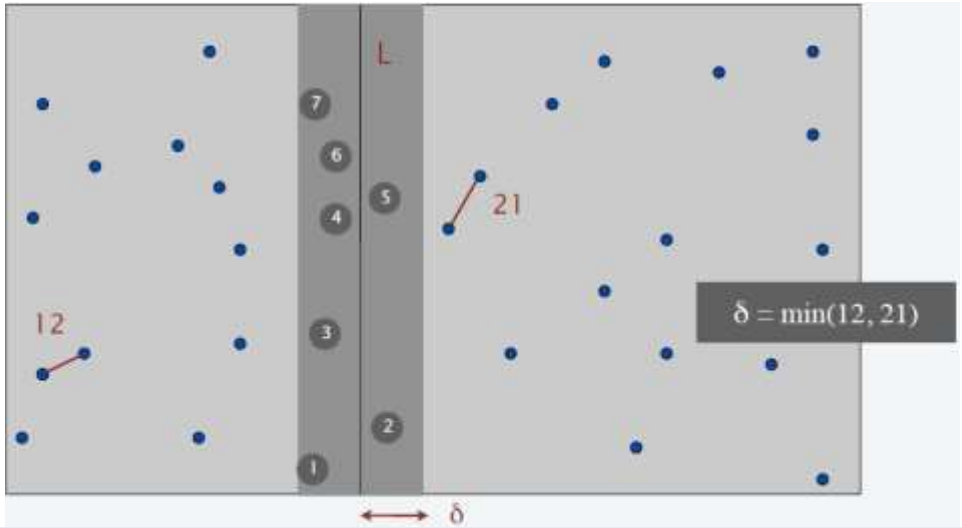
- Combine
  - We can restrict our attention to points within δ of L on each side, where δ = best of the solutions within the two halves



Closest Pair in $\mathbb{R}^2$

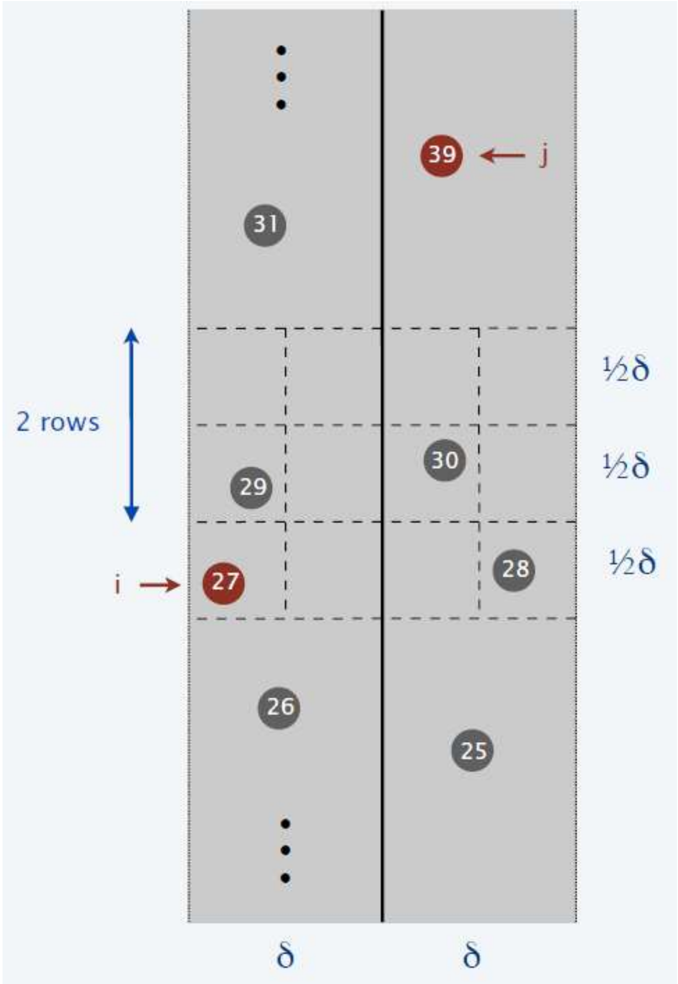
- Combine (let δ = best of solutions in two halves)
  - Only need to look at points within δ of L on each side,
  - Sort points on the strip by y coordinate
  - Only need to check each point with next 11 points in sorted list!

Wait, what? Why 11?



Why 11?

- Claim:
  - If two points are at least 12 positions apart in the sorted list, their distance is at least δ
- Proof:
  - No two points lie in the same $\delta/2 \times \delta/2$ box
  - Two points that are more than two rows apart are at distance at least δ



Running Time Analysis

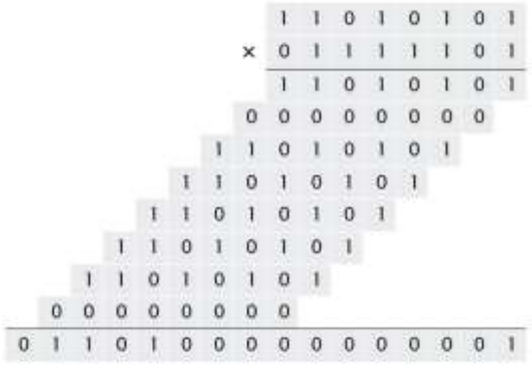
- Running time for the combine operation
  - Finding points on the strip: $O(n)$
  - Sorting points on the strip by their y-coordinate: $O(n \log n)$
  - Testing each point against 11 points: $O(n)$

- Total running time: $T(n) \leq 2T\left(\frac{n}{2}\right) + O(n \log n)$

- By the Master theorem, this yields $T(n) = O(n \log^2 n)$
  - Can be improved to $O(n \log n)$ by doing a single global sort by y-coordinate at the beginning

Recap: Karatsuba’s Algorithm

- Fast way to multiply two n digit integers x and y
- Brute force: $O(n^2)$ operations
- Karatsuba’s observation:
  - ➢ Divide each integer into two parts
    - ○ $x = x_1 * 10^{n/2} + x_2, y = y_1 * 10^{n/2} + y_2$
    - ○ $xy = (x_1 y_1) * 10^n + (x_1 y_2 + x_2 y_1) * 10^{n/2} + (x_2 y_2)$
  - ➢ Four $n/2$-digit multiplications can be replaced by three
    - ○ $x_1 y_2 + x_2 y_1 = (x_1 + x_2)(y_1 + y_2) - x_1 y_1 - x_2 y_2$
  - ➢ Running time
    - ○ $T(n) \leq 3\,T(n/2) + O(n) \Rightarrow T(n) = O\left(n^{\log_2 3}\right)$



Strassen's Algorithm

- Generalizes Karatsuba's insight to design a fast algorithm for multiplying two $n \times n$ matrices
  - Call n the "size" of the problem

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

  - Naively, this requires 8 multiplications of size $n/2$
    - $A_{11} * B_{11}, A_{12} * B_{21}, A_{11} * B_{12}, A_{12} * B_{22}, \ldots$

  - Strassen's insight: replace 8 multiplications by 7
    - Running time: $T(n) \leq 7\, T(n/2) + O(n^2) \Rightarrow T(n) = O\left(n^{\log_2 7}\right)$

Strassen's Algorithm

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

STRASSEN(n, A, B)

IF $(n = 1)$ RETURN $A \times B$.

assume n is a power of 2

Partition A and B into 2-by-2 block matrices.

keep track of indices of submatrices (don't copy matrix entries)

$P_1 \leftarrow$ STRASSEN$(n / 2, A_{11}, (B_{12} - B_{22}))$.

$P_2 \leftarrow$ STRASSEN$(n / 2, (A_{11} + A_{12}), B_{22})$.

$P_3 \leftarrow$ STRASSEN$(n / 2, (A_{21} + A_{22}), B_{11})$.

$P_4 \leftarrow$ STRASSEN$(n / 2, A_{22}, (B_{21} - B_{11}))$.

$P_5 \leftarrow$ STRASSEN$(n / 2, (A_{11} + A_{22}) \times (B_{11} + B_{22}))$.

$P_6 \leftarrow$ STRASSEN$(n / 2, (A_{12} - A_{22}) \times (B_{21} + B_{22}))$.

$P_7 \leftarrow$ STRASSEN$(n / 2, (A_{11} - A_{21}) \times (B_{11} + B_{12}))$.

$C_{11} = P_5 + P_4 - P_2 + P_6$.

$C_{12} = P_1 + P_2$.

$C_{21} = P_3 + P_4$.

$C_{22} = P_1 + P_5 - P_3 - P_7$.

RETURN C.

Median & Selection

- Selection:
  - Given array A of n comparable elements, find kth smallest
  - $k = 1$ is min, $k = n$ is max, $k = \lfloor (n+1)/2 \rfloor$ is median
  - $O(n)$ is easy for min/max

- What about k-selection?
  - $O(nk)$ by modifying bubble sort
  - $O(n \log n)$ by sorting
  - $O(n + k \log n)$ using min-heap
  - $O(k + n \log k)$ using max-heap

- Q: What about just $O(n)$?
- A: Yes! Selection is easier than sorting.

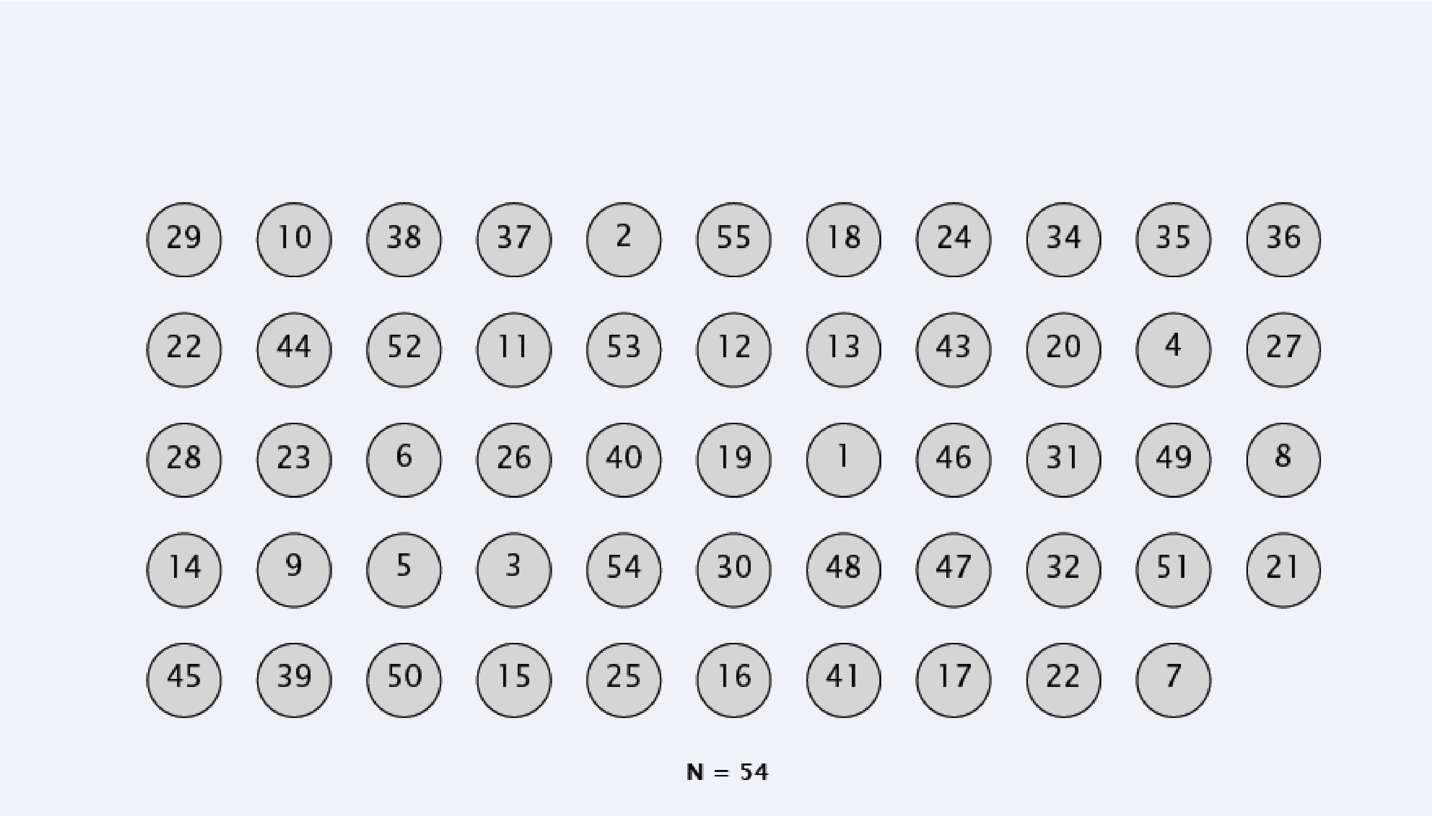
QuickSelect

- Find a pivot p
- Divide A into two sub-arrays
  - A_{less} = elements $\leq p$, A_{more} = elements $> p$
  - If $|A_{less}| \geq k$, return k-th smallest in A_{less}, otherwise return $(k - |A_{less}|)$-th smallest element in A_{more}

- Problem?
  - If pivot is close to the min or the max, then we basically get $T(n) \leq T(n-1) + O(n)$, which only gives us $T(n) = O(n^2)$
  - We want to reduce $n - 1$ to a fraction of n (e.g., $n/2$, $5n/6$, etc)

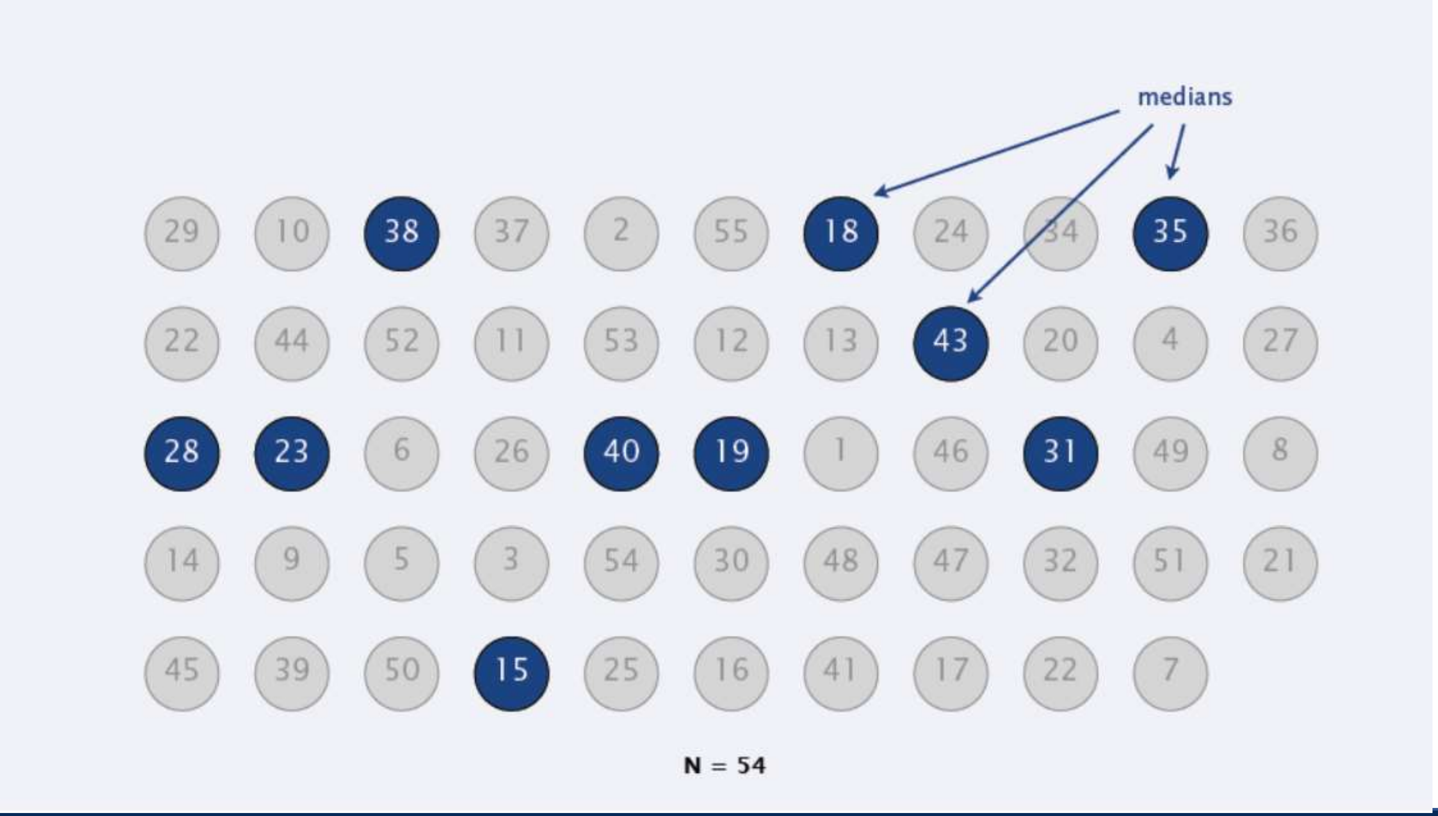
Finding a Good Pivot

- Divide n elements into $^{n}/_{5}$ groups of 5 each



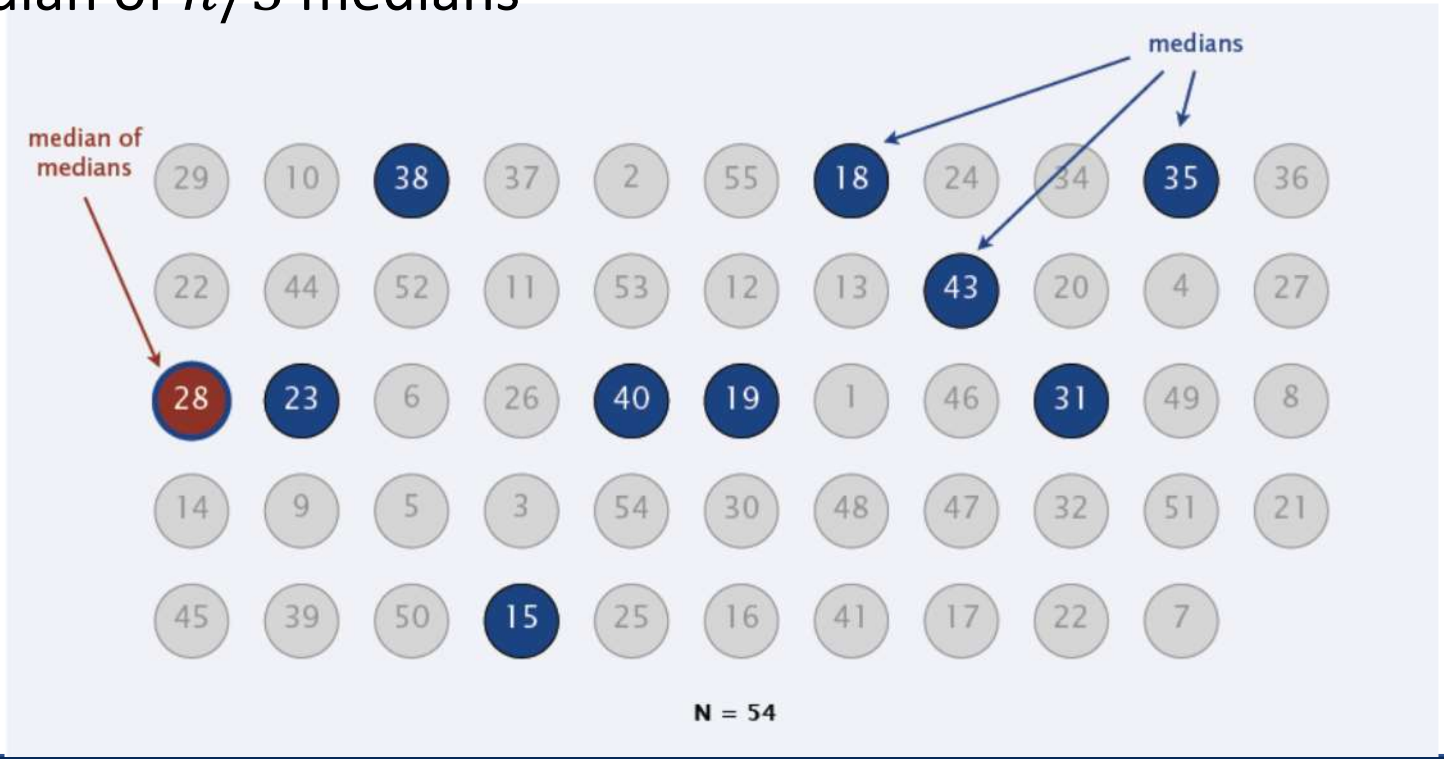
Finding a Good Pivot

- Divide n elements into ${}^{n}/_{5}$ groups of 5 each
- Find the median of each group



Finding a Good Pivot

- Divide n elements into ${}^{n}/_{5}$ groups of 5 each
- Find the median of each group
- Find the median of $n/5$ medians



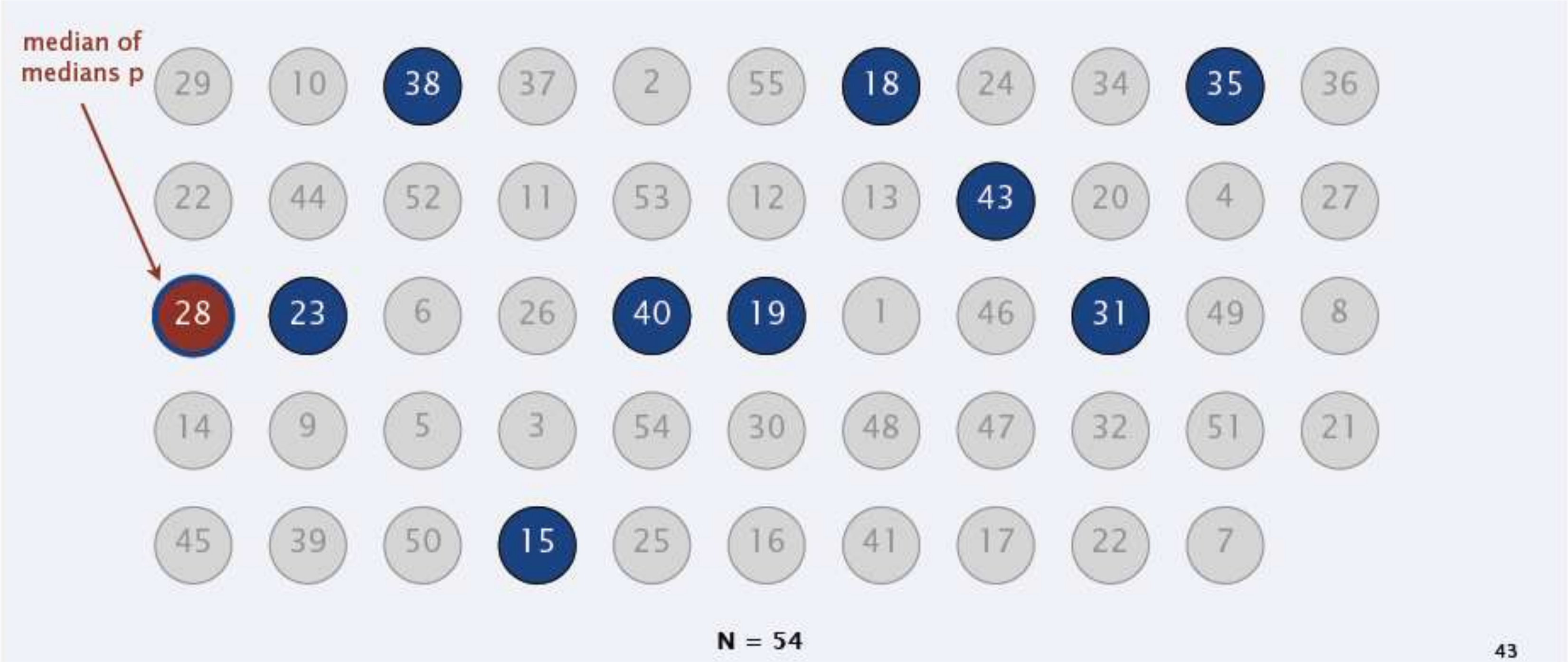
Finding a Good Pivot

- Divide n elements into ${}^{n}/_{5}$ groups of 5 each
- Find the median of each group
- Find the median of $n/5$ medians
- Use this median of medians as the pivot in quickselect

- Q: Why does this work?

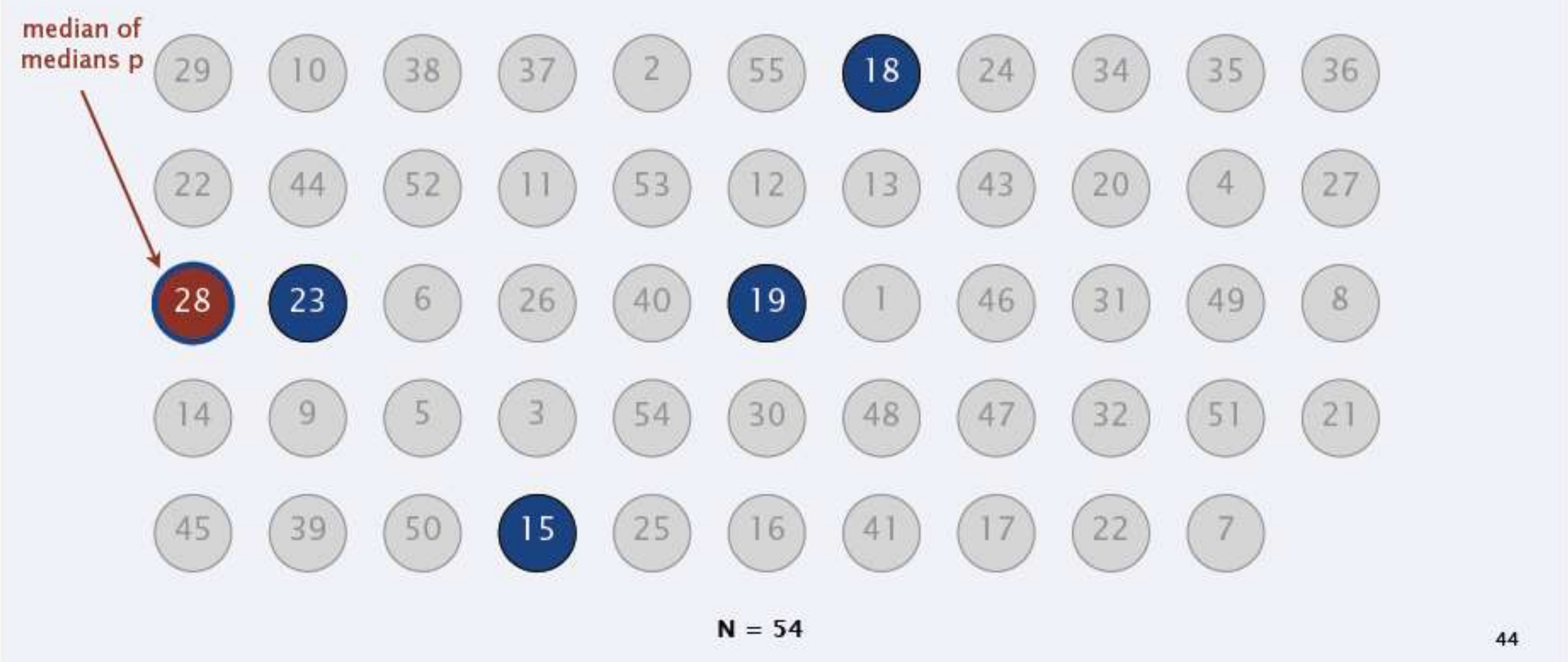
Analysis

- How many elements can be $\leq p^*$?
  - Out of $n/5$ medians, $n/10$ are $> p^*$



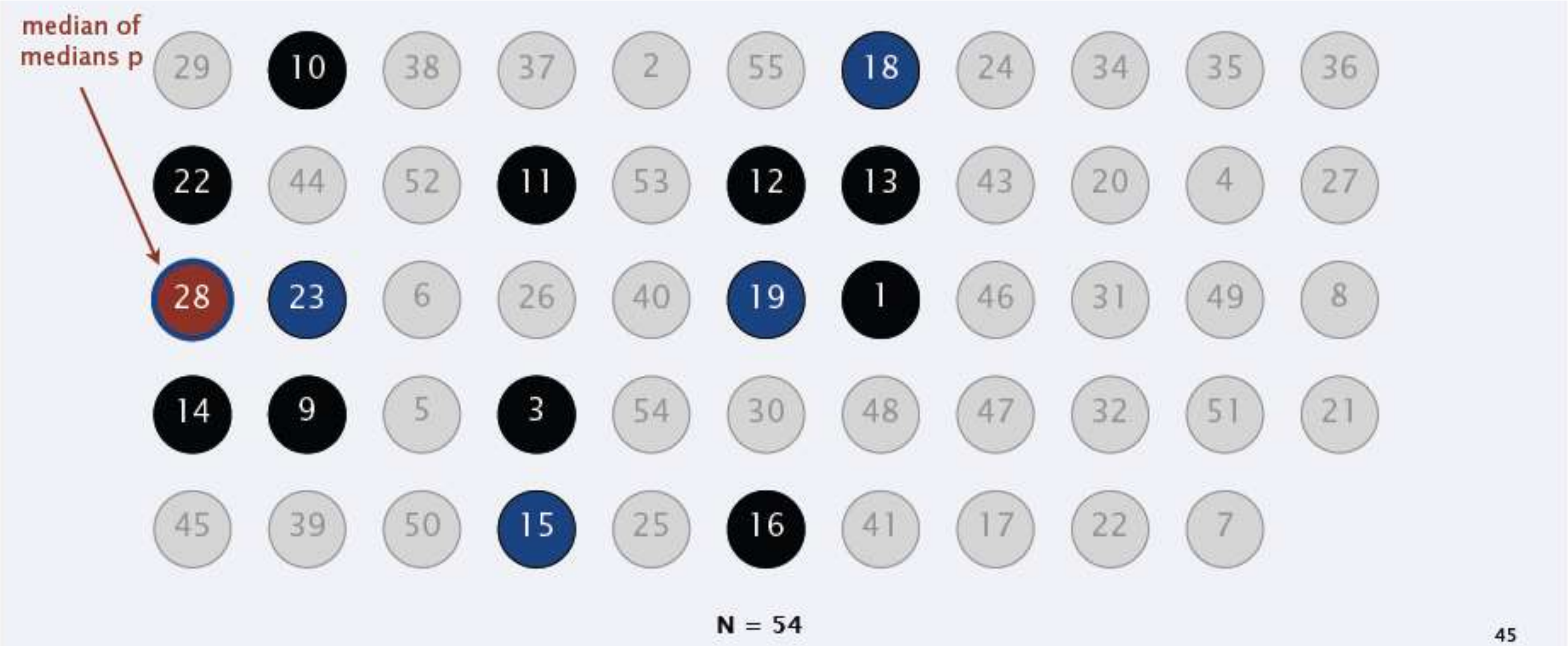
Analysis

- How many elements can be $\leq p^*$?
  - Out of $n/5$ medians, $n/10$ are $> p^*$



Analysis

- $^n/_{10}$ of the $^n/_5$ medians are $\leq p^*$
  - For each such median, there are 3 elements $\leq p^*$
  - So there can be at most $^{7n}/_{10}$ elements that can be $> p^*$



Analysis

- Thus, $|A_{more}| \leq {}^{7n}/_{10}$
  - Similarly, $|A_{less}| \leq {}^{7n}/_{10}$
  - (These are rough calculations…)

- How does this factor into overall algorithm analysis?

Analysis

- Divide n elements into ${}^{n}/_{5}$ groups of 5 each
- Find the median of each group
- Find p^* = median of ${}^{n}/_{5}$ medians
- Create A_{less} and A_{more} according to p^*
- Run selection on *one of* A_{less} or A_{more}

$O(n)$

$T(n/5)$

$O(n)$

$T(7n/10)$

- $T(n) \leq T({}^{n}/_{5}) + T({}^{7n}/_{10}) + O(n)$
- Note: ${}^{n}/_{5} + {}^{7n}/_{10} = {}^{9n}/_{10}$
  - Only a fraction of n, so using a similar analysis to the one in the Master theorem, $T(n) = O(n)$

Residual Notes

- Lower bounds on the worst-case running time

  - Note that we only derived *upper bounds* on the worst-case running time of the form $T(n) = O(n^2)$ or $T(n) = O(n)$

  - If we want to claim that our algorithm does not run *faster* than what is claimed in this upper bound, we have to produce a matching *lower bound*, e.g., $T(n) = \Omega(n^2)$ or $T(n) = \Omega(n)$

  - This is typically done by producing a *family of examples*, one for each value of n, such that the algorithm's running time on these examples grows like n^2 or n as the value of n grows

Residual Notes

- Best algorithm for a problem?
  - Typically hard to determine
  - We still don't know best algorithms for multiplying two n-digit integers or two $n \times n$ matrices
    - Integer multiplication
      - Breakthrough in March 2019: first $O(n \log n)$ time algorithm
      - It is conjectured that this is asymptotically optimal
    - Matrix multiplication
      - 1969 (Strassen): $O(n^{2.807})$
      - 1990: $O(n^{2.376})$
      - 2013: $O(n^{2.3729})$
      - 2014: $O(n^{2.3728639})$

Residual Notes

- Best algorithm for a problem?
  - Usually, we design an algorithm and then analyze its running time

  - Sometimes we can do the reverse:
    - E.g., if you know you want an $O(n^2 \log n)$ algorithm
    - Master theorem suggests that you can get it by
$$T(n) = 4\,T\left({}^{n}/_{2}\right) + O(n^2)$$
    - So maybe you want to break your problem into 4 problems of size $n/2$ each, and then do $O(n^2)$ computation to combine

Residual Notes

- Access to input
  - For much of this analysis, we are assuming random access to elements of input
  - So we're ignoring underlying data structures (e.g. doubly linked list, binary tree, etc.)

- Machine operations
  - We're only counting the number of comparison or arithmetic operations
  - So we're ignoring issues like how real numbers are stored in the closest pair problem
  - When we get to P vs NP, representation will matter

Residual Notes

- Size of the problem
  - Can be any reasonable parameter of the problem
  - E.g., for matrix multiplication, we used n as the size
  - But an input consists of two matrices with n^2 entries
  - It doesn't matter whether we call n or n^2 the size of the problem
  - The actual running time of the algorithm won't change