CSC373 Summer '22

Practice Problems on Linear Programming

Q1 [20 Points] Travel Planning is Hard

Anu is making holiday travel plans. There are 6 cities that she is interested in visiting. For each $k \in \{1, \ldots, 6\}$, it would cost c_k to incorporate city k into the itinerary but she would gain h_k units of happiness from visiting city k (both amounts are regardless of which other cities are on the itinerary). Unfortunately, she has a limited budget of B, so she may not be able to visit all the cities.

There are also other considerations. Cities 3 and 5 are too similar to each other, so Anu prefers to visit at most one of them. The same is true for cities 1, 5, and 6. Some cities can only be enjoyed in the fullest if visited together with some other cities. Anu doesn't want to visit cities 5 or 6, unless at least one of cities 3 and 4 is also on the itinerary. Finally, she wants to visit at least one and at most three among the cities $1, 2, 4, 5, 6$.

Anu obviously wants to find an itinerary (a subset of cities to visit) that will maximize her happiness, given the constraints listed above.

(a) [10 Points] Formulate this problem as a binary integer linear program. Here, "binary" means that every variable x_k in your program should be in $\{0, 1\}$.

(b) [10 Points] Imagine a "relaxation" of the program from part (a), where you allow each variable x_k to take any non-negative real value (i.e., $x_k \geq 0$). This relaxation becomes a linear program. Write the dual of this linear program.

Solution to Q1

(a) In the following binary integer linear program (BILP), for $k \in \{1, \ldots, 6\}$, binary variable x_k indicates whether Anu will visit city k.

$$
\begin{aligned}
\text{Maximize } & \textstyle\sum_{k=1}^{6} h_k \cdot x_k & \\
\text{s.t. } & \textstyle\sum_{k=1}^{6} c_k \cdot x_k \leq B & \text{(Budget)} \\
& x_3 + x_5 \leq 1 & \text{(Similar cities)} \\
& x_1 + x_5 + x_6 \leq 1 & \text{(Similar cities)} \\
& x_5 \leq x_3 + x_4 & \text{(City 5 only if at least one of cities 3 and 4)} \\
& x_6 \leq x_3 + x_4 & \text{(City 6 only if at least one of cities 3 and 4)} \\
& x_1 + x_2 + x_4 + x_5 + x_6 \geq 1 & \text{(At least one of cities 1,2,4,5,6)} \\
& x_1 + x_2 + x_4 + x_5 + x_6 \leq 3 & \text{(At most three of cities 1,2,4,5,6)} \\
& x_k \in \{0, 1\}, \forall k \in \{1, \ldots, 6\} &
\end{aligned}
$$

(b) Let us introduce dual variables $y_1, \ldots, y_7$ corresponding to the seven primal constraints in the order in which they are listed above. Then, the dual is given as follows.

$$
\begin{aligned}
\text{Minimize } & By_1 + y_2 + y_3 - y_6 + 3y_7 \\
\text{s.t. } & c_1y_1 + y_3 - y_6 + y_7 \geq h_1 && (x_1) \\
& c_2y_1 - y_6 + y_7 \geq h_2 && (x_2) \\
& c_3y_1 + y_2 - y_4 - y_5 \geq h_3 && (x_3) \\
& c_4y_1 - y_4 - y_5 - y_6 + y_7 \geq h_4 && (x_4) \\
& c_5y_1 + y_2 + y_3 - y_4 - y_6 + y_7 \geq h_5 && (x_5) \\
& c_6y_1 + y_3 - y_5 - y_6 + y_7 \geq h_6 && (x_6) \\
& y_k \in \{0,1\}, \forall k \in \{1,\ldots,7\}
\end{aligned}
$$

Q2 [20 Points] Tricky Transportation

Anu has begun her much awaited journey. While she planned a bit before leaving, she didn't work out all the details in advance. She now faces the problem of planning her transportation.

There are n buses running along a road (think of it as the real axis). Each bus i starts at location s_i and goes to location t_i (where $t_i > s_i$). If she decides to use bus i, she can board at any point $x \in [s_i, t_i]$, drop off at any later point $y \in [x, t_i]$, and must pay a fixed cost c_i regardless of where she boards and drops off. Anu wants to get from point a to point b on the road (where $b > a$).

(a) [15 Points] Write a binary integer linear program that helps Anu achieve this goal while minimizing her total cost. Your program only needs to find the minimum total cost. It is OK if it does not find the locations at which Anu should board and drop off buses. Your program must have a finite (and ideally, a polynomial) number of constraints. Prove that your program is correct.

(b) [5 Points] Suppose each bus is operated by one of two companies. Let T_1 and T_2 be the indices of buses operated by companies 1 and 2, respectively ($T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 = \{1,\ldots,n\}$). For some reason, Anu wants to make sure that she don't spend more than 70% of the total cost on any single company. Note that this constraint might raise the minimum total cost she needs to pay in order to get from point a to point b. How would you add this constraint to the program in part (a)?

Solution to Q2

(a) In the following binary integer linear program (BILP), variable x_i indicates whether bus i is taken.

$$
\begin{aligned}
\text{Minimize } & \textstyle\sum_{i=1}^{n} c_i \cdot x_i \\
\text{s.t. } & \sum_{i:s_i \leq a < t_i} x_i \geq 1 \\
& \sum_{i:s_i \leq t_j < t_i} x_i \geq 1, \forall j : a \leq t_j < b \\
& x_i \in \{0,1\}, \forall i \in \{1,\ldots,n\}
\end{aligned}
$$

Let OPT be the minimum cost of any valid itinerary and OBJ be the optimal objective value of the BILP. We want to show $OPT = OBJ$. We show $OBJ \leq OPT$ and $OPT \leq OBJ$ separately.

OBJ ≤ OPT: Consider an optimal itinerary with cost OPT. Construct the natural solution x where $x_i = 1$ if Anu takes bus i in this itinerary, and 0 otherwise. Clearly, $\sum_i c_i x_i = OPT$. We show that x is a feasible solution of the BILP. This would establish that the minimum objective value of the BILP must be $OBJ \leq OPT$.

- First constraint: Anu must be boarding some bus i at point a that she leaves at a later point; the corresponding $x_i = 1$ will satisfy the first constraint.
- Second constraint: Fix any j such that $a \leq t_j < b$. Anu must be on some bus i at point t_j that she leaves at a point after t_j; the corresponding $x_i = 1$ will satisfy the second constraint for this j.

OPT ≤ OBJ: Let x be an optimal feasible solution to the BILP with objective value OBJ. We show that there is a valid itinerary with cost at most OBJ; then, it follows that the optimal itinerary has cost $OPT \leq OBJ$. We construct the itinerary as follows.

- At point a, we board a bus i for which $s_i \leq a < t_i$ and $x_i = 1$; such a bus exists due to x satisfying the first constraint. We leave this bus at $\min(b, t_i)$. If we reach point b, we are done. Otherwise, we leave the bus at t_i.
- Whenever we leave a bus at some point t_j, we board another bus i such that $s_i \leq t_j < t_i$ and $x_i = 1$; such a bus exists due to x satisfying the second constraint. Once again, we leave this bus at $\min(b, t_j)$.

Based on the above argument, we see that the only point where Anu can leave a bus where she would not be boarding another bus is b. Hence, the itinerary must be able to get her to point b. We also notice that Anu takes buses in strictly increasing order of their end points (whenever she drops bus j at t_j, she takes another bus i with $t_i > t_j$); hence, she never takes a bus twice. Finally, she only takes buses i for which $x_i = 1$. Hence, this valid itinerary has cost at most $\sum_{i=1}^{n} c_i x_i = OBJ$, as needed.

Note: There is another correct solution in which the second constraint is replaced by $\sum_{i: s_i \leq t_j < t_i} x_i \geq x_j$. Although this is less imposing than the constraint mentioned above, it is sufficient to guarantee that the feasible solutions correspond to valid itineraries because we only need the second constraint enforced at points t_j where $x_j = 1$. In other words, both programs would have the same optimal objective value.

(b) We would add the following two constraints to the program from part (a). The first constraint says that the total cost spent on buses from T_1 is at most 0.7 times the total cost, and the second constraint says the same for the total cost spent on buses from T_2.

$$\sum_{i \in T_1} c_i \cdot x_i \leq 0.7 \cdot \sum_{i=1}^{n} c_i \cdot x_i$$
$$\sum_{i \in T_2} c_i \cdot x_i \leq 0.7 \cdot \sum_{i=1}^{n} c_i \cdot x_i.$$