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Prelude: The Study of Programming Languages

It seems to me that there have been two really clean, consistent models of programming so far: the C model and the Lisp model. These two seem points of high ground, with swampy lowlands between them.

Paul Graham

As this is a “programming languages” course, you might be wondering: are we going to study new programming languages, much in the way that we studied Python in CSC108 or even Java in CSC207? Yes and no.

You will be introduced to new programming languages in this course; most notably, Racket and Haskell. However, the focus of our teaching is different than in CSC108 and CSC207. In CSC324, we leave the syntax and standard library up to you. How do if-expressions work? What is the function that finds the leftmost occurrence of an element in a list? Why is this a syntax error? These are the types of questions that we expect you to be able to research and solve. (Of course, we’ll provide useful tutorials and links to standard library references to help you along.) Instead, we focus on the ways in which these languages allow us to express ourselves; that is, we focus on particular affordances of these languages compared to the languages you have used to this point.

Programs and programming languages

We start with a simple question: what is a program? Most people are used to thinking about programs in one of two ways: as an active entity on our computer that does something when run; or as the source code itself, which tells the computer what to do. As we develop more and more sophisticated programs for more targeted domains, we often lose sight of one crucial fact: that code is not itself the goal, but instead a means of communication to the computer describing what we want to achieve.

A programming language, then, isn’t just the means of writing code, but a true language in the common sense of the word; it is a means of communication, between humans and their machines. Unlike what linguists call natural languages,
which often carry ambiguity, nuance, and errors, programming languages target machines, and so must be precise, unambiguous, and perfectly understandable by mechanical algorithms alone. This makes the rules governing programming languages quite inflexible, which is often a source of trouble from beginners; yet once mastered, the clarity afforded by these languages enables humans to harness the awesome computational power of modern technology.

But even this lens of programming languages as communication is incomplete. Unlike natural languages, which have evolved over millennia, often organically without much deliberate thought, programming languages are not even a century old, and were explicitly designed by humans. As programmers, we tend to lose sight of this, taking our programming language for granted—quirks and oddities and all. But programming languages exhibit the same fascinating design questions, trade-offs, and limitations that are inherent in all software design. Indeed, the various software used to implement programming languages—that is, to take human-readable source code and enable a computer to understand it—are some of the most complex and sophisticated programs in existence today.

The goal of this course, then, is to stop taking programming languages for granted; to go deeper, from users of programming languages to understanding the design and implementation of these languages.

**Syntax and grammars**

The **syntax** of a programming language is the set of rules governing what the allowed expressions of a programming language can look like; these are the rules governing allowed program structure. The most common way of specifying the syntax of a language is through a **grammar**, which is a formal description of how to generate expressions by substitution. For example, the following is a simple grammar to generate arithmetic expressions:

```plaintext
<expr> = NUMBER
   | '(' <expr> <op> <expr> ')'
<op> = '+' | '-' | '*' | '/'
```

We say that the left-hand side names `<expr>` and `<op>` are **non-terminal symbols**, meaning that we generate valid expressions by substituting for them using these grammar rules. By convention, we’ll put angle brackets around all non-terminal symbols.

The all-caps **NUMBER** is a terminal symbol, representing any numeric literal (e.g., 3 or -1.5). The vertical bar | indicates alternatives (read it as “or”): for example, `<op>` can be replaced by any one of the strings ‘+’, ‘-’, ‘*’, or ‘/’.

It is important to note that this grammar is recursive, as a `<expr>` can be replaced by more occurrences of `<expr>`. This should match your intuition about

---

1 This is not to minimize the work of language deliberative bodies like the Oxford English Dictionary, but to point out that language evolves far beyond what may be prescribed.

2 We’ll typically use INTEGER when we need to instead specify any integral literal.
both arithmetic expressions and programs themselves, which can both contain subparts of arbitrary length and nesting!

With these grammars in hand, it is easy to specify the syntax of a programming language: an expression is **syntactically valid** if and only if it can be generated by the language’s grammar. For the arithmetic expression language above, the expression \((3 + 5)\) is syntactically valid, but \((3 +)\) and \((3 - + * 5)\) and even \(3 + 5\) are not.

**Abstract syntax trees**

The source code is merely the starting point of a program; as text, it is useful for communicating ideas among humans, but not at all useful for a machine. To “run” a program requires the computer to operate on the program’s source code—but as you have surely experienced before, operating on strings alone is often tricky and cumbersome, as that datatype is far less structured than what its contents suggest.

Therefore, one of the key steps for any operation on a program is to **parse** it, which here means to convert the source code—a string—into a more structured representation of the underlying program. Because the source code can only represent a valid program if it is syntactically valid, parsing is always informed by the programming language’s grammar, and is the stage at which syntax errors are detected. The problem of parsing text is a rich and deep problem in its own right, but due to time constraints we won’t spend time discussing parsing techniques in these notes, instead relying on either the simplicity of the language syntax (Racket) or built-in code parsing tools (Python) to resolve this task for us.

What we will focus on is the output of parsing: this “more structured representation of the underlying program.” The most common representation, used by interpreters and compilers for virtually every programming language, is the **abstract syntax tree (AST)**, as trees are a natural way to represent the inherently hierarchical organization of programs into smaller and smaller components. While different programming languages and even different compilers or interpreters for the same language will differ in their exact AST representation, generally speaking all abstract syntax trees share the following properties:

1. A **leaf** of the tree represents an expression that has no subexpressions, e.g. a literal value (5, “hello”) or identifier (person). We call such expressions **atomic**.

2. Each **internal node** of the tree represents a compound expression, i.e. one that is built out of smaller subexpressions. These encompass most of the programming forms you are familiar with, including control flow structures, definitions of functions/types/classes, function calls, etc.

3. Nodes can be categorized by the kind of expression they represent, e.g. through an explicit “tag” string, using an object-oriented approach (a Node class hierarchy), or using algebraic data types. This last one is likely new to you. We’ll study algebraic data types later on in this course!
4. AST node types are often in rough correspondence to the grammar rules of a language. So for example, the language’s syntax could contain a grammar rule for what a function definition looks like:

```
<function-def> = 'def' <id> '(' [<id> ','] ... ')' ':' '

And when parsed, a program’s AST might include a “function definition” node with children for the name, parameters, and body of the function.

As we’ll start to see in the next chapter, abstract syntax trees enable us to avoid the idiosyncracies of program syntax and instead get to interesting operations on programs themselves.

Semantics and evaluation

You may have noticed in the previous section that we used the term expression to describe the inputs of our parsing. This may strike you as a little strange, since the term program usually connotes much more than this. Most modern programming languages, including Python, Java, and C, are fundamentally imperative in nature: inspired by the Turing machine model of computation, programs in these languages are organized around statements corresponding to instructions that the computer should execute. In this model, we think of “running” a program as telling the computer to execute the instructions found in our program; the result of running the program is whatever happens when these instructions are executed.

While it is certainly a familiar model of computation, and tracks closely to what computer hardware actually requires, one of the downsides of this model is its inherent complexity. In order to understand what a program means, we need to understand what each kind of statement does, and how it impacts control flow and underlying memory. The semantics of a programming language are the rules governing the meaning of programs written in that language; for imperative style programs, we need to describe the meaning not just of individual expressions like 3 + 5, but also the meaning of return (interrupts control flow), for (iterates through a specified range), and other keywords.

To simplify matters, we’ll stick with the easier task of understanding expression-based programs, in which a program is just an expression. In this model, running a program means telling the computer to evaluate the expression; the result of running the program is simply the value of the expression after it has been evaluated. (Note that all imperative programs have a notion of “evaluating expressions”; it’s just that those languages include a bunch of other stuff as well.)

So the semantics of an expression-based language govern what the value of such programs are. This might seem simple, but it’s worth spelling out explicitly, because there are actually multiple ways of studying such semantics.

The denotational semantics of a programming language specify the abstract value of a program, drawing on formal definitions (e.g., from mathematics). We

\[ \text{Note that this is not necessarily exact; we are skipping over details of parsing and basic syntax analysis that many compilers often do before producing an AST.} \]
won’t go into the details here, but instead rely on your intuitions from mathematics and basic programming. In the space below, we’ve listed several programs (each consisting of just a single Python expression) that have the same denotational value 10:

```
10
3 + 7
1 + 3 ** 2
ord('\n')
(lambda x: x + 3)(7)
list(range(50000000))[10]
```

That is, each of the above expressions, while written and parsed differently, produces the same result when evaluated; we would say that they have the same mathematical meaning, or the same value.

However, your gut probably tells you that this isn’t the full story. After all, even though these expressions might evaluate to the same value, how they each get to that value is quite different. The operational semantics of a programming language specify the evaluation steps used to determine the value of a program. In imperative-style languages, it is the operational semantics that are hardest to specify, as they deal with complexities of control flow, mutation, and function calls. As we’ll see in the next chapter, specifying the operational semantics of expression evaluation alone—especially in a functional context—is generally straightforward.

While we will focus on denotational and operational semantics in this course, it is worth mentioning one other kind of semantics that comes up in programming languages. This is axiomatic semantics, where rather than focus on evaluation, we focus on what is true about each piece of a code segment. For example, we might argue that “this loop maintains the invariant that sum is the sum of the first i integers in list L”. Sound familiar? You used some of the axiomatic tools—invariants, variants, pre/postconditions—already in CSC236!

Models of computation

It was in the 1930s, years before the invention of the first electronic computing devices, that a young mathematician named Alan Turing created modern computer science as we know it. Incredibly, this came about almost by accident; he had been trying to solve a problem from mathematical logic: the Entscheidungsproblem (“decision problem”) asked whether an algorithm could decide if a logical statement is provable from a given set of axioms. Turing showed no such
algorithm exists. To answer this question, Turing developed an abstract model of mechanical, procedural computation: a machine that could read in a string of 0’s and 1’s, a finite state control that could make decisions and write 0’s and 1’s to its internal memory, and an output space where the computation’s result would be displayed. Though its original incarnation was an abstract mathematical object, the fundamental mechanism of the Turing machine—reading data, executing a sequence of instructions to modify internal memory, and producing output—would soon become the von Neumann architecture lying at the heart of modern computers, and seed the paradigm of imperative programming.

The story of Alan Turing and his machine is one of great genius, great triumph, and great sadness. It is no exaggeration to say that the fundamentals of computer science owe their genesis to this man. Their legacy is felt by every computer scientist, software engineer, and computer engineer alive today. But there is another story, too.

**Alonzo Church**

Shortly before Turing published his paper introducing the Turing machine, the logician Alonzo Church had published a paper resolving the same fundamental problem using entirely different means. At the same time that Turing was developing his model of the Turing machine, Church was drawing inspiration from the mathematical notion of functions to model computation. Church would later act as Turing’s PhD advisor at Princeton, where they showed that their two radically different notions of computation were in fact equivalent: any problem that could be solved in one could be solved in the other. They went a step further and articulated the **Church-Turing Thesis**, which says that any reasonable computational model would be just as powerful as their two models. And incredibly, this bold claim still holds true today. With all of our modern technology, we are still limited by the mathematical barriers erected eighty years ago.

And yet to most computer scientists, Turing is much more familiar than Church; the von Neumann architecture is what drives modern hardware design; the most commonly used programming languages today revolve around state and time, instructions and memory, the cornerstones of the Turing machine. What were Church’s ideas, and why don’t we know more about them?

**The lambda calculus**

The imperative programming paradigm derived from Turing’s model of computation has as its fundamental unit the **statement**, a portion of code representing some instruction or command to the computer. Though such statements are composed of subexpressions, these expressions typically do not appear on their own; consider the following odd-looking, but valid, Python program:

---

4 Finite state controls are analogous to the deterministic finite automata that you learned about in CSC236.

7 To make this amazing idea a little more concrete: no existing programming language and accompanying hardware will ever solve the Halting Problem. None.
```python
def f(a):
    12 * a - 1
    a
    'hello' + 'goodbye'
```

Even though all three expressions in the body of \( f \) are evaluated each time the function is called, they are unable to influence the output of this function. We require sequences of statements (including keywords like return) to do anything useful at all! Even function calls, which might look like standalone expressions, are only useful if the bodies of those functions contain statements for the computer to execute.

In contrast to this instruction-based approach, Alonzo Church created a model called the **lambda calculus** in which expressions themselves are the fundamental, and in fact only, unit of computation. Rather than a program being a sequence of statements, in the lambda calculus a program is a single expression (possibly containing many subexpressions). And when we say that a computer runs a program, we do not mean that it performs operations corresponding to statements, but rather that it evaluates the single expression.

Two questions arise from this notion of computation: what do we really mean by the words “expression” and “evaluate”? Or in other words, what are the syntax and semantics of the lambda calculus? This is where Church borrowed functions from mathematics, and why the programming paradigm that this model spawned is called functional programming. In the lambda calculus, an expression is one of three things:

1. An **identifier**: \( a, x, yolo \), etc.
2. A **function expression**: \( \lambda x \rightarrow x \), for example.\(^9\) This expression represents a function that takes one parameter \( x \), and returns it. In other words, this is the identity function.
3. A **function application**: \( f \ expr \). This expression applies the function \( f \) to the expression \( expr \).

Now that we have defined our allowable expressions, what do we mean by evaluating them? To evaluate an expression means performing simplifications to it until it cannot be further simplified; we’ll call the resulting fully-simplified expression the **value** of the expression.

This definition meshes well with our intuitive notion of evaluation, but we’ve really just shifted the question: what do we mean by “simplifications?” In fact, in the lambda calculus, variables and functions have no simplification rules: in other words, they are themselves values, and are fully simplified. On the other hand, function application expression \( \text{can be simplified, using the idea of substitution from mathematics. For example, suppose we apply the identity function to the variable } \text{hi:} \)

\[
(\lambda x \rightarrow x)\text{hi}
\]

\(^9\) We use non-standard notation here. This function would normally be expressed as \( \lambda x.x \).
We evaluate this by substituting \( hi \) for \( x \) in the body of the function, obtaining \( hi \) as a result.

Pretty simple, eh? As surprising as this may be, function-evaluation-as-substitution is the only simplification rule for the lambda calculus! So if you can answer questions like “If \( f(x) = x^2 \), then what is \( f(5) \)?” then you’ll have no trouble understanding the lambda calculus.

The main takeaway from this model is that function evaluation (via substitution) is the only mechanism we have to induce computation; functions can be created using \( \lambda \) and applied to values and even other functions, and through combining functions we create complex computations. A point we’ll return to again and again in this course is that functions in the lambda calculus are far more restrictive than the functions we’re used to from previous programming experience. The only thing we can do in the lambda calculus when evaluating a function application is substitute the arguments into the function body, and then evaluate that body, producing a single value. These functions have no concept of time to require a certain sequence of steps, nor is there any external or global state that can influence their behaviour.

At this point the lambda calculus may seem at best like a mathematical curiosity. What does it mean for everything to be a function? Certainly there are things we care about that aren’t functions, like numbers, strings, classes and every data structure you’ve studied up to this point—right? But because the Turing machine and the lambda calculus are equivalent models of computation, anything you can do in one, you can also do in the other! So yes, we can use functions to represent numbers, strings, and data structures; we’ll see this only a little in this course, but rest assured that it can be done. And though the Turing machine is more widespread, the beating heart of the lambda calculus is still alive and well, and learning about it will make you a better computer scientist.

\[ A \text{ paradigm shift in you} \]

The influence of Church’s lambda calculus is most obvious today in the \textit{functional programming paradigm}, a function-centric approach to programming that has heavily influenced languages such as Lisp (and its dialects), ML, Haskell, and F#. You may look at this list and think “I’m never going to use these in the real world,” but support for functional programming styles are being adopted in more “mainstream” languages, such as LINQ in C# and lambdas in Java 8. Other languages like Python and JavaScript have supported the functional programming paradigm since their inception.

The goal of this course is not to convert you into the \textit{Cult of FP}, but to open your mind to different ways of solving problems. After all, the more tools you have at your disposal in “the real world,” the better you’ll be at picking the best one for the job.

Along the way, you will gain a greater understanding of different programming language properties, which will be useful to you whether you are exploring new
languages or studying how programming languages interact with compilers and interpreters, an incredibly interesting field in its own right.\textsuperscript{10}

\textit{Course overview}

\textit{Chapter 1}. We will begin our study of functional programming with two new languages: Racket, a dialect of Lisp commonly used for both teaching and language research, and Haskell, a pure functional programming language with an elaborate and powerful static type system. It might seem like overkill to use two different languages, and we are certainly very conscious of this! Our pedagogical goal here is to reinforce the idea that we are not learning specialized idiosyncracies of a particular language. Rather, we want to focus on the guiding high-level principles that exist in many different languages, and we believe that the best way to do this is to explore how these principles are expressed in multiple languages at once, to gain a deeper understanding of these ideas. In the next chapter, we will explore language design features like scope, function call strategies, and tail recursion, comparing Racket with more familiar languages like Python and Java. We will also use this as an opportunity to gain lots of experience with functional programming idioms: (structural) recursion; the list functions map, filter, and fold; and higher-order functions and closures.

\textit{Chapter 2}. In the next section of the course, we’ll do a deep dive into two substantial programming language features: object-oriented programming and (simulating) non-deterministic choices. Rather than study just the features themselves, we’ll take the approach of a programming language designer, and ask the question “How do we implement such a feature?” This lens will give us more than just a greater understanding of these language features. Our quest for user-friendly implementations will lead us to the defining feature of Racket: a powerful macro system that allows us to extend the very syntax and semantics of a programming language.\textsuperscript{11}

\textit{Chapter 3}. If the preceding chapter is all about macros being used to express new concepts and paradigms in a language, our last section of the course is dual to this: expressing constraints in a language, through the creation of types. We are all familiar with types; but as with other aspects of programming languages, in this course we’ll study types with more attention to detail and creativity than you likely have in the past. In particular, we’ll explore Haskell’s powerful static type system, and see how to use it to express not just simple notions like “don’t add a number to a string”, but more abstract concepts like failing computations, mutable state, and external I/O, all with strong guarantees from our Haskell compiler.\textsuperscript{12}

\textsuperscript{10} Those of you who are particularly interested in compilers should take CSC488.

\textsuperscript{11} Put another way, to allow us to introduce new kinds of nodes into an abstract syntax tree.

\textsuperscript{12} One particularly nifty feature we’ll talk about is type inference, a Haskell compiler feature that means we get all the benefits of static typing without the verbosity of Java’s Kingdom of Nouns.
1 Functional Programming: Theory and Practice

Any sufficiently complicated C or Fortran program contains an ad hoc, informally-specified, bug-ridden, slow implementation of half of Common Lisp.

Greenspun's tenth rule of programming

In 1958, John McCarthy created Lisp, a bare-bones programming language based on Church’s lambda calculus. Since then, Lisp has spawned many dialects (languages based on Lisp with some deviations from its original specifications), among which are Common Lisp, Clojure (which compiles to the Java Virtual Machine), and Racket (a language used actively in educational and programming language research contexts).

In 1987, it was decided at the conference Functional Programming Languages and Computer Architecture to form a committee to consolidate and standardize existing non-strict functional languages, and so Haskell was born (we’ll study what the term “non-strict” means in this chapter). Though mainly still used in the academic community for research, Haskell has become more widespread as functional programming has become, well, more mainstream. Like Lisp, Haskell is a functional programming language: its main mode of computation involves defining pure functions and combining them to produce complex computation. However, Haskell has many differences from the Lisp family, both immediately noticeable and profound.

Our goal in this chapter is to expose you to some of the central concepts in programming language theory, and functional programming in particular, without being constrained to one particular language. So in this chapter, we’ll draw on examples from three languages: Racket, Haskell, and Python. Our hope here is that by studying the similarities and differences between these languages, you’ll gain more insights into the deep concepts in this chapter than by studying any one of these languages alone.

*Lisp itself is still used to this day; in fact, it has the honour of being the second-oldest programming language still in use. The oldest? Fortran.*

*Now part of this one: http://www.icfpconference.org/*
The baseline: “universal” built-ins

While one of the major strengths of the lambda calculus as a model of computation is its simplicity, in practice we want built-in data types and functions to scaffold our programs. Of course, programming languages vary in how they build in their data types and the operations that they support, and so we’ll restrict most of our attention in this course to a fairly conservative set of built-ins, common to all three languages:

- Primitive data types and literals: integers, floats, booleans, strings
- Compound data types: lists and maps
- Built-in functions on these built-in types
- Boolean operations (and, or, not) and if expressions (sometimes called “ternary ifs”)

Function expressions

With the built-ins out of the way, we’ll now turn to one of the first “new” aspects, which is one of the central ideas of functional programming: functions are first-class values, meaning that they can be treated and manipulated in the exact same way as other kinds of values in a program.

This is actually a big idea! Many languages do not support functions as first-class values, meaning there are some things you can do with other kinds of values that you can’t do with functions. One of these is simple: what is the equivalent of a “function value” in a language?

Suppose we want to represent the integer value three in a program. We take for granted how easy it is to do so: simply write the numeric literal 3! But suppose we want to represent a function that “takes a number and adds 3 to it”; in Python, you would probably write:

```python
def f(x):
    return x + 3
```

But there’s one big difference here: in the former case, we had the value 3 by itself, whereas in the latter case, there is both the function, and a name f associated with the function. In some programming languages, it is only possible to define function values simultaneously with a given identifier to refer to the value. However, if we want to treat functions as first-class values, then we had better be able to write them standalone, independent of a name binding!

A function value that is expressed independently of an identifier is called an anonymous function.

In the lambda calculus, all function expressions are anonymous: λx → x is a function expression, but doesn’t have an associated name. All three of our chosen languages support anonymous functions as well, using the following
syntax, each inspired in its own way by the lambda calculus.

In Racket:

\[
\text{(lambda (param \ldots) body)}
\]

In Haskell:

\[
\text{\langle param \ldots \rangle \rightarrow body}
\]

In Python:

\[
\text{lambda param \ldots : body}
\]

In each of the above, each \langle param \rangle is called a parameter of the function, and must be an identifier. The \langle body \rangle is an expression called the body of the function.

For programmers that have never seen anonymous functions before, such functions might seem strange: what’s the point of writing a function if you don’t have a name to refer to it? While we’ll see some motivating examples later in this chapter, for now we’ll leave you with a different question: does every expression you write have a name?

**Function application**

Calling functions in each of the three languages is straightforward, but note that both Racket and Haskell use an unfamiliar syntax.

First, in Python a function call looks like most languages you’ve probably worked with before:

\[
\text{function(arg, \ldots)}
\]

In Racket, the function goes inside the parentheses:

\[
\text{(function arg \ldots)}
\]

One thing that trips students up is that in Racket, every parenthesized expression is treated as a function call, except for the ones starting with keywords like `lambda`. This is in stark contrast with most programming languages, in which expressions are often enclosed in (redundant) parentheses to communicate grouping explicitly. Racket doesn’t have the concept of “redundant” parentheses!

\[4\] We’ll use background colours to distinguish languages in code blocks, with yellow for Racket, blue for Haskell, and grey for all others.

\[5\] This syntax is known as Polish prefix notation.
In Haskell, parentheses are not required at all; instead, any two expressions separated by a space are considered a function call:

\[
<\text{function}> \ <\text{arg}> \ ... \n\]

Operators are functions

Consider common binary arithmetic operations like addition and multiplication, which we normally think of as being written \textit{infix}, i.e., between two numeric expressions: \(3 + 4\) or \(1.5 * 10\).

Again in the theme of the centrality of functions to our programming, it’s important to realize that in fact, these operators are just functions, at least from a mathematical point of view.\(^6\) In fact, all three of Racket, Haskell, and Python treat them as functions!

In Racket, operators are just identifiers that refer to built-in functions, and are called the same way any other function would be called:

```racket
> (+ 10 20)
30
> (* 3 5)
15
```

In Python, operators are implemented under the hood by delegating to various “dunder methods” for built-in classes:

```python
>>> 10 + 20
30
>>> int.__add__(10, 20)
30
```

Haskell uses a similar approach to Python. Every infix operator (e.g., \(+\)) is a function whose name is the same as the operator, but enclosed in parentheses (e.g., \((+)):

```haskell
> 10 + 20
30
> (+) 10 20
30
```

\(^6\) Formally, we might write something like \(+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) to represent the + operation as taking two real numbers and outputting a real number.

Again, pretty weird! If you’ve never thought about it before, this might seem overly complex. But it turns out that Racket has the cleanest model (no infix operators, uniform prefix syntax for all functions); Python and Haskell both...
include a level of indirection to support the more familiar infix syntax we learn in mathematics.

*Function purity*

One thing you might notice about our grammar rules for defining functions is that a function body is just a single expression. This is again a consequence of our model of an expression-based language, rather than the imperative-style “sequence of statements” that you normally see in other languages.

The best way to think about functions in this style of programming is by analogy the mathematical functions, e.g. \( f(x) = x + 1 \), which are purely determined by input-output pairs. In this definition, we say that the body of \( f \) is the expression \( x + 1 \), and that we evaluate calls to \( f \) by substituting values for \( x \) into the body, evaluating the body expression, and then returning the result.

In the function definition rules we gave above, all the function values produced behave exactly the same as mathematical functions: their behaviour is entirely determined by what their body expression evaluates to for a set of given arguments. For example, consider the Racket function \((\text{lambda} (x) (+ x 1))\). If we call this function on the number 10,

\[
((\text{lambda} (x) (+ x 1)) 10)
\]

we evaluate this function call by taking the 10 and substituting it for \( x \) in the expression \( (+ x 1) \), producing 11 as the returned value. Note that a \textbf{return} keyword isn’t necessary: given the \texttt{lambda} expression, we know precisely that the value of \( (+ x 1) \) (with something substituted for \( x \)) will be returned.

In programming, we say that a \textbf{(mathematically) pure function} is a function that satisfies the following properties:

1. The function’s behaviour is exactly determined by the \textit{value} of its inputs. For example, the function cannot access data from “outside” its inputs, including standard input, the filesystem, the Internet, etc. This also rules out randomized functions; we say that pure functions must be \textit{deterministic}.
2. The function only \textit{return}s a value, and does nothing else. In parallel to (1), this means that pure functions cannot print to standard output, write to the filesystem, or send data across the Internet. We call such actions \textit{side effects}, and so we say that “pure functions have no side effects.”

This definition might seem overly restrictive: we often want functions to communicate with the “outside world”, either for input or output! As is hopefully becoming a common theme in this course, we start with this type of function because such functions are easiest to reason about: if you understand substitution, then you’re good to go. In the final chapter of these notes, we’ll see how to incorporate side effects like mutation and external I/O into a pure functional framework.
Name bindings

While Alonzo Church and Alan Turing showed that anonymous functions and function application are sufficient to perform all computations that modern programming languages can, even pure functional programming languages like Haskell offer additional conveniences for the programmer. We’ve previously discussed some of these—primitive values and built-in functions—and the following example illustrates another:

```
(((lambda (x)
    ((lambda (f)
        (lambda (n)
            (if (equal? n 0) 1 (* n (f (- n 1))))))
        (lambda (v) ((x x) v)))))
    (lambda (x) ((lambda (f)
        (lambda (n)
            (if (equal? n 0) 1 (* n (f (- n 1))))))
        (lambda (v) ((x x) v))))))
```

This monstrosity of a program evaluates to 3628800, which is 10!, illustrating the power of anonymous functions to successfully implement something of a recursive nature. However, we certainly don’t want to get stuck writing code like this! Instead, every programming language gives us the ability to bind identifiers to values, so that evaluating an identifier results in the value that the identifier is bound to.

You have seen one kind of identifier already: the formal parameters used in function definitions. These are a special type of identifier, and are bound to values when the function is called. In this subsection we’ll look at using identifiers more generally. We’ll use identifiers for two purposes:

1. To “save” the value of subexpressions so that we can refer to them later.
2. To refer to a function name within the body of the function, enabling recursive definitions.

The former is clearly just a convenience to the programmer; the latter does pose a problem to us, but it turns out that writing recursive functions in the lambda calculus is possible, as we illustrated in the above example.\(^7\)

It is important to keep in mind that all other uses of identifiers are as a convenience, to make our programs easier to understand, but not to truly extend the power of the lambda calculus. Unlike the imperative programming languages you’ve used so far, identifier bindings in pure functional programming are immutable: once bound to a particular value, that identifier cannot be re-bound, and so literally is an alias for a value. This leads us to an extremely powerful concept known as referential transparency. We say that an identifier is referentially transparent if it can be substituted with its value in the source code without...
changing the meaning of the program.\textsuperscript{8}

This approach to identifiers in functional programming is hugely different than what we are used to in imperative programming, in which re-binding names is not just allowed, but required for some common constructs,\textsuperscript{9} or subverted by mutable data structures. Given that re-binding and mutation feels so natural to us, why would we want to give it up? Or put another way, why is referential transparency (which is violated by mutation) so valuable?

Mutation is a powerful tool, but also makes our code harder to reason about: we need to constantly keep track of the “current value” of every identifier throughout the execution of a program. Referential transparency means we can use names and values interchangeably when we reason about our code regardless of where these names appear in the program; a name, once defined, has the same meaning for the rest of the time.\textsuperscript{10}

For about 95\% of this course we will not use any mutation at all, and even when we do use it, it will be in a very limited way. Remember that the point is to get you thinking about programming in a different way, and we hope in fact that this ban will simplify your programming!

\textit{Global bindings}

In Racket, the syntax for a global, or top-level, binding uses the keyword \texttt{define}:

\begin{verbatim}
(define <id> <expr>)
\end{verbatim}

In Haskell (and Python), top-level bindings are written using the familiar $=$ symbol:

\begin{verbatim}
<id> = <expr>
\end{verbatim}

These definitions bind the value of $\texttt{expr}$ to the identifier $\texttt{id}$. Here are some examples of these bindings, including a few that bind a name to a function.

\begin{verbatim}
# Racket
(define a 3)
(define add-three
  (lambda (x) (+ x 3)))

# Haskell
a = 3
addThree = \x -> x + 3

# Python
a = 3
add_three = lambda x : x + 3
\end{verbatim}

\textsuperscript{8} This again parallels mathematics. When we write “let $x = 5$” in a statement or proof, any subsequent statement we make about $x$ should make just as much sense if we replace the $x$ with $5$.

\textsuperscript{9} e.g., loops

\textsuperscript{10} In particular, the whole issue of whether an identifier represents a \textit{value} or a \textit{reference} is rendered completely moot.
Because function definitions are so common, each language provides a concise way of binding function values to names; you are already familiar with `def` in Python. We’re going to use this more convenient notation for the rest of the course, but keep in mind that in Racket and Haskell, these are merely “syntactic sugar” for the lambda expression.\footnote{Syntactic sugar is a part of a programming language’s syntax that doesn’t introduce new functionality, but is just another way of writing existing functionality in a simpler, more readable way.}

```
; Racket
(define (add-three x) (+ x 3))
(define (almost-equal x y) (<= (abs (- x y)) 0.001))
```

```
-- Haskell
addThree x = x + 3
almostEqual x y = abs (x - y) <= 0.001
```

Local bindings

Most programming languages support local scopes as well as global scope; among the most common of these is local scope within functions. For example, function parameters are local to the body of the function.

```
(define (f x)
  ; can refer to x here in the body of the function
  (+ x 10))

; can't refer to x out here (try it!)
(+ x 10)
```

In Racket, we can also explicitly create a local scope using the keyword `let*`:

```
(let* ([<id> <expr>] ...) <body>)
```

A `let*` expression takes pairs `[<id> <expr>]`, binds each expression to the corresponding identifier, and evaluates the `<body>` expression using these bindings. The value of the `let*` expression is the value of the `<body>`.

In Haskell, the same local scoping is achieved using the following:\footnote{There’s one subtle difference between Racket’s `let*` and Haskell’s `let`: the latter allows for recursive bindings, while the former does not. If you ever need to do this in Racket, you can use `letrec` instead.}

```
let <id> = <expr>
  ...
in <body>
```
What about Python? Python certainly has the concept of local scope, e.g., function and class definitions, but it does not have special syntactic support for let expressions, i.e., an expression that involves local bindings and that evaluates to produce a value. This is yet another reminder of the way in which Python programs are oriented around statements—assignment statements being the most common of these—while Racket and Haskell are oriented around expressions.

Name bindings and code structure

In the previous chapter, we said that we would focus on programs in expression-based contexts, in which a program is just a single expression to be evaluated. In practice, this is unwieldy, and this section described the use of identifiers to simplify complex expressions. Therefore the actual program form we’ll stick with for this course is:

```plaintext
<prog> = [<binding> ...] <expr>
```

where `<binding>` is a top-level binding expression. Moreover, the `<expr>` can now use let expressions, which are structured in the same style: an arbitrary number of (distinct) name bindings, followed by a single expression to evaluate. For example, here is a Racket program that computes the distance between three distinct points and evaluates to the minimum distance:

```racket
1 (define p1 (list 2 3))
2 (define p2 (list 10 8))
3 (define p3 (list -1 6))
4
5 (define (distance p q)
6   (let* ([dx (abs (- (first p) (first q)))]
7           [dy (abs (- (second p) (second q)))]
8           (sqrt (+ (* dx dx) (* dy dy))))
9
10 (define distance-p1-p2 (distance p1 p2))
11 (define distance-p1-p3 (distance p1 p3))
12 (define distance-p2-p3 (distance p2 p3))
13
14 (min distance-p1-p2 distance-p1-p3 distance-p2-p3)
```

Written in Python, such a program is simply one where every global or local block of code consists of an arbitrary number of assignments to distinct variables, followed by a single expression to evaluate (or return, if inside a function). The important thing to keep in mind in that context is that such assignments are non-mutating: we never bind to the same variable more than once!
Lists and structural recursion

The list data type is one of the most fundamental in programming languages; it is an arbitrary-length compound data type, used to store any number of values. For many of us, lists are the first time we are able to write code that operates on an arbitrary amount of data, without knowing ahead of time how much data there is.

Imperative languages naturally process lists using loops, explicitly relying on notions of time and state to keep track of the “current” list element. This follows a metaphor of a list as a linear sequence of items, in which each item is “processed” one at a time. However, because pure functional programming does not permit the mutation required to re-bind an identifier to each element of the list in turn, we must take another approach to writing programs involving lists.

The key insight here is to identify a recursive structure for the list data type, and use this definition to inform both our representation of and operations on lists. A list is defined recursively as:

- The empty list is a list. (Represented in Racket as ’(), in Haskell and Python as []).
- If x is a value and lst is a list, then we can create a new list whose first element is x and whose other items are the ones from lst. We call this combination the cons operation, calling the produced list “x cons lst.” In Racket, this is represented by the cons function: (cons x lst), and in Haskell by the infix operator x:lst. For example, we could construct the list [1, 2, 3] using Python doesn’t have a built-in function that does this, perhaps evidence that this recursive definition is not central to Python lists (which it isn’t).

Finally, we generally are able to deconstruct non-empty lists into their first and other elements, in Racket using the functions first and rest, and in Haskell by head and tail. We have the identity lst = (cons (first lst) (rest lst)) for all non-empty lists lst.

Structural recursion

This recursive structure informs not just how we represent lists, but how we operate on lists as well. For example, consider the problem of computing the sum of the elements of a list. Whereas an iterative approach would process each element in turn by adding its value to an accumulator, a recursive approach mimics the recursive structure of the list itself:

- The sum of an empty list is equal to 0.
- The sum of “x cons lst” is equal to x plus the sum of lst.

\[\text{In fact, the name “Lisp” comes from “list processing.”}\]

\[15\text{ short for “construct”}\]

\[16\text{ Python doesn’t have a built-in function that does this, perhaps evidence that this recursive representation is not central to Python lists (which it isn’t).}\]
The beauty of this English description is that it translates immediately into a pure function:

```racket
(define (sum lst)
  (if (empty? lst)
      0
      (+ (first lst)
          (sum (rest lst)))))
```

And here is a function that takes a list, and returns a new list containing just the multiples of three in that list. Note that this is a *filtering* operation, something we’ll return to later in this chapter.

```racket
(define (multiples-of-3 lst)
  (cond [(empty? lst) '()]
        [(equal? 0 (remainder (first lst) 3))
         (cons (first lst)
               (multiples-of-3 (rest lst)))]
        [else (multiples-of-3 (rest lst))]))
```

We call this form of recursion *structural recursion* because the code follows the structure of the data type on which it operates. What’s remarkable about this technique is that because it is based solely on the form of the input data, we can apply it to create a “code template” that works for many list functions:

```racket
(define (f lst)
  (if (empty? lst)
      ...
      (f (rest lst))))
```

Moreover, this technique can be used for any data type with a recursive definition, not just lists. The other fundamental example we’ll return to again and again in this course is the recursive definition of trees, and specifically the abstract syntax trees that we use to represent programs.
Exercise Break!

1.1 Write a function to determine the length of a list.
1.2 Write a function to determine if a given item appears in a list.
1.3 Write a function to determine the number of duplicates in a list.
1.4 Write a function to remove all duplicates from a list.
1.5 Given two lists, output the items that appear in both lists (intersection). Then, output the items that appear in at least one of the two lists (union).
1.6 Write a function that takes a list of lists, and returns the list that contains the largest item (e.g., given (list (list 1 2 3) (list 45 10) (list) (list 15)), return (list 45 10)).
1.7 Write a function that takes an item and a list of lists, and inserts the item at the front of every list.
1.8 Write a function that takes a list with no duplicates representing a set (order doesn’t matter). It returns a list of lists containing all of the subsets of that list.
1.9 Write a function that takes a list with no duplicates, and a number \( k \), and returns all subsets of size \( k \) of that list.
1.10 Modify your function to the previous question so that the parameter \( k \) is optional, and if not specified, the function returns all subsets.
1.11 Write a function that takes a list, and returns all permutations of that list (recall that in a permutation, order matters, so (list 1 2 3) is distinct from (list 3 2 1)).
1.12 A sublist of a list is a series of consecutive items of the list. Given a list of numbers, find the maximum sum of any sublist of that list. (Note: there is a \( O(n) \) algorithm that does this, although you should try to get any algorithm that is correct first, as the \( O(n) \) algorithm is a little more complex.)

Pattern-matching

We’ll now take a short detour away from the pure lambda calculus to introduce one of the most underutilized programming language features: pattern-matching, which allows the programmer to specify conditional behaviour depending on the structure of a given value in a concise and understandable syntax. As an example, consider the following value-based branching function definition:

```haskell
f x =
    if x == 0 then 10
    else if x == 1 then
```
While this is fine, we can shorten this substantially in Haskell by using value pattern-matching in the function definition.\textsuperscript{17}

Rather than giving one generic function signature \( f \ x = \ldots \), we give three pattern-based definitions:

\begin{verbatim}
    f 0 = 10
    f 1 = 20
    f x = x + 30
\end{verbatim}

Essentially, Haskell’s function definition syntax allows us to eliminate the explicit use of if-else tests; instead, we provide the patterns (in this case, 0, 1, or \( x \)—an identifier matches anything\textsuperscript{18}) and the Haskell compiler generates the equivalent code that does the testing for us whenever this function is called.

Here is the equivalent code in Racket. The syntax isn’t quite as terse (note the use of the define/match keyword), but is certainly more concise than using ifs or even cond:

\begin{verbatim}
    (define/match (f x)
        [(0) 10]
        [(1) 20]
        [(_) (+ x 30)] ; The underscore matches anything
    )
\end{verbatim}

While the above example is cute, you might be skeptical that it generalizes; after all, how often do we write explicit special cases when defining functions? But it turns out that pattern-matching goes far beyond simple value-equality checks.

Consider now the recursive list function from the previous section, which we generalized into a structural recursion template.

\begin{verbatim}
    (define (sum lst)
        (if (empty? lst)
            0
            (+ (first lst)
                (sum (rest lst))))
    )
\end{verbatim}

Both Racket and Haskell support structural pattern matching, a concise syntax for decomposing a value into subparts and binding each part to a new identifier that can be referred to independently. In Racket patterns, we use the expression (list) to pattern-match on an empty list, and we use cons to perform a structural decomposition pattern match of a list into its “first and rest”.\textsuperscript{19}

\textsuperscript{17} Here, we’ll focus on pattern-matching for function definitions only, although both Racket and Haskell support pattern-matching as an arbitrary expression form as well.

\textsuperscript{18} Order matters! The patterns are checked top-down, with the first match being selected.

\textsuperscript{19} This kind of decomposition is sometimes referred to as destructuring in other languages.
(define/match (sum lst)
  [([list]) 0]
  [([cons x xs]) (+ x (f xs))])

In Haskell, the syntax is even terser: [] represents an empty list, while the infix colon : plays the role of the cons.

```
sum [] = 0
sum (x:xs) = x + sum xs
```

---

**Exercise Break!**

1.13 Redo your solutions to the previous set of list exercises using pattern-matching in both Racket and Haskell.

---

**Tail call elimination**

As you have studied in previous courses, function calls are stored on the call stack, a part of memory that stores information about the currently active functions as the program runs. In non-recursive programs, the size of the call stack is generally not an issue, but with recursion the call stack quickly fills up with recursive calls. Here is a quick Python example, in which the number of function calls is \( \Theta(n) \):

```
def f(n):
  if n == 0:
    return 0
  else:
    return f(n - 1)

# Produces a RuntimeError because the call stack fills up
f(10000)
```

In fact, the same issue of recursion taking up a large amount of memory occurs in Racket and Haskell as well. However, these languages (and many others) perform **tail call elimination**, which can significantly reduce the space requirements for recursive functions. A tail call is a function call that happens as the last instruction of a function before the return; the \( f(n-1) \) call in the previous example has this property. When a tail call occurs, there is no need to remember where it was called from, because the only thing that’s going to happen afterwards is that the value will be returned to the original caller.\(^{20}\) This property of tail calls is common to all languages; however, some languages take advantage of it to optimize recursive computations by performing **tail call optimization**.\(^{21}\)

\(^{20}\) Simply put: if \( f \) calls \( g \) and \( g \) just calls \( h \) and returns its value, then when \( h \) is called there is no need to keep any information about \( g \); just return the value to \( f \) directly!
of this, and others do not. Racket is one that does: when it calls a function that it detects is in tail call position, it first removes the calling function’s stack frame from the call stack, leading to a very small ($O(1)$) space usage for the equivalent Racket function:

```scheme
(define (f n)
  (if (equal? n 0)
      0
      (f (- n 1))))
```

Our `sum` implementation from the previous section is not tail-recursive, because the result of the recursive call must first be added to `(first lst)` before being returned. However, we can use a tail-recursive helper function instead.

```scheme
(define (sum lst)
  (sum-helper lst 0))

(define (sum-helper lst acc)
  (if (empty? lst)
      acc
      (sum-helper (rest lst) (+ acc (first lst)))))
```

Because recursion is extremely common in functional programming, converting recursive functions to tail-recursive functions is an important technique that you want to practice. The key strategy is to take your function and add an extra parameter to accumulate results from earlier recursive calls, eliminating the need to do extra computation with the result of the recursive call.

In the above example, the parameter `acc` plays this role, accumulating the sum of the items “processed so far” in the list. We use the same idea to accumulate the “multiples of three seen so far” for our `multiples-of-3` function:

```scheme
(define (multiples-of-3 lst)
  (multiples-of-3-helper lst '()))

(define (multiples-of-3-helper lst acc)
  (if (empty? lst)
      acc
      (multiples-of-3-helper (rest lst) (+ acc (if (equal? 0 (remainder (first lst) 3)) (append acc (list (first lst))) (list (first lst)))))
```

Exercise Break!
1.14 Rewrite your solutions to the previous exercises using tail recursion.

Higher-order functions

So far, we have kept a strict division between our types representing data values—numbers, booleans, strings, and lists—and the functions that operate on them. However, as we said earlier, functions are values, so it is natural to ask: can functions operate on other functions?

The answer in our course is a most emphatic yes, and in fact this is the heart of functional programming: the ability for functions to take in other functions and use them, combine them, and even output new ones. Let’s see some simple programming examples.

The differential operator, which takes as input a function \( f(x) \) and returns its derivative \( f'(x) \), is another example of a “higher-order function,” although most calculus courses won’t use this terminology. By the way, so is the indefinite integral.

<table>
<thead>
<tr>
<th>Code</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>(define (apply-to-1 f) (f 1))</code></td>
<td></td>
</tr>
<tr>
<td><code>(apply-to-1 even?)</code></td>
<td>#f</td>
</tr>
<tr>
<td><code>(apply-to-1 list)</code></td>
<td>'(1)</td>
</tr>
<tr>
<td><code>(apply-to-1 (lambda (x) (+ 15 x)))</code></td>
<td>16</td>
</tr>
<tr>
<td><code>(define (apply-two f1 f2 x))</code></td>
<td></td>
</tr>
<tr>
<td><code>(list (f1 x) (f2 x))</code></td>
<td></td>
</tr>
<tr>
<td><code>(apply-two even? odd? 16)</code></td>
<td>'#t #f'</td>
</tr>
<tr>
<td><code>(define (apply-twice f x))</code></td>
<td></td>
</tr>
<tr>
<td><code>(f (f x))</code></td>
<td></td>
</tr>
<tr>
<td><code>(apply-twice sqr 3)</code></td>
<td>81</td>
</tr>
</tbody>
</table>

Higher-order list functions

With the discussion in the previous section, you might get the impression that people who use functional programming spend all of their time using recursion. But in fact this is not the case! Instead of using recursion explicitly, code often uses three critical higher-order functions to compute with lists. The first two, map and filter are extremely straightforward:

<table>
<thead>
<tr>
<th>Code</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>(map f lst)</code></td>
<td></td>
</tr>
<tr>
<td><code>(filter pred lst)</code></td>
<td></td>
</tr>
</tbody>
</table>

22 Indeed, in the pure lambda calculus all values are functions

23 The differential operator, which takes as input a function \( f(x) \) and returns its derivative \( f'(x) \), is another example of a “higher-order function,” although most calculus courses won’t use this terminology. By the way, so is the indefinite integral.

24 Of course, these higher-order functions themselves are implemented recursively.
To illustrate the third common higher-order function, let’s first demonstrate how we might write \texttt{map} and \texttt{filter} using loops and mutation:

\begin{verbatim}
def map(f, lst):
    acc = []
    for x in lst:
        acc.append(f(x))
    return acc

def filter(pred, lst):
    acc = []
    for x in lst:
        if pred(x):
            acc.append(x)
    return acc
\end{verbatim}

Both of these functions use the same accumulator pattern, using an \textit{accumulator variable} to store a value that gets updated at each loop iteration, and is finally returned at the function’s end. We can generalize this pattern into a higher-order function accepting two additional arguments: an initial value and a function to update the accumulator inside the loop.

\begin{verbatim}
def accumulate(combine, init, lst):
    acc = init
    for x in lst:
        acc = combine(acc, x)
    return acc
\end{verbatim}

This more general loop pattern is codified in a recursive fashion in both Racket and Haskell in a function called \texttt{foldl}:

\begin{verbatim}
(define (foldl combine init lst)
  (if (empty? lst)
      init
      (foldl combine
              (combine (first lst) init)
              (rest lst))))
\end{verbatim}

\texttt{foldl} combine init lst =

\texttt{if} null lst
\texttt{then}

\texttt{Note that the order of the arguments of the “combine” functions are different in the two languages.}
Though all three of \texttt{map}, \texttt{filter}, and \texttt{foldl} are extremely useful in performing most computations on lists, both \texttt{map} and \texttt{filter} are constrained in having to return lists, while \texttt{foldl} can return any data type. In fact, given the flexibility of this function illustrated in its corresponding loop version, it should be clear at least conceptually that it is possible to implement \texttt{map} and \texttt{filter} in terms of \texttt{foldl}—doing so is a great exercise at this point in your learning.

\section*{Exercise Break!}

1.15 You might notice that the Racket implementation of \texttt{foldl} is very similar to the implementation of \texttt{sum-helper} from the previous section. Use \texttt{foldl} to implement \texttt{sum} in both Racket and Haskell. Note that you should be able to pass in the “plus” function directly (i.e., not using a lambda), but this may require a bit of research when doing so in Haskell.

1.16 Implement a function that takes a predicate (boolean function) and a list, and returns the number of items in the list that satisfy the predicate.

1.17 Is \texttt{foldl} tail-recursive? If so, explain why. If not, rewrite it to be tail-recursive.

1.18 Reimplement all of the previous exercises using \texttt{map}, \texttt{filter}, and/or \texttt{foldl}, \textit{without} using explicit recursion.

1.19 Write a function that takes a list of unary functions, and a value \texttt{arg}, and returns a list of the results of applying each function to \texttt{arg}.

1.20 Implement \texttt{map} and \texttt{filter} using \texttt{foldl}.

1.21 The “\texttt{l}” in \texttt{foldl} stands for “left”, because items in the list are combined with the accumulator in order from left to right.

\begin{verbatim}
(foldl f 0 (list 1 2 3 4))
; equivalent to...
(f 4 (f 3 (f 2 (f 1 0))))
\end{verbatim}

Write another version of \texttt{fold} called \texttt{foldr}, which combines the items in the list from right to left:

\begin{verbatim}
(foldr f 0 '(1 2 3 4))
; equivalent to...
(f 1 (f 2 (f 3 (f 4 0))))
\end{verbatim}

Hint: this can be done using basic structural recursion—start by mentally dividing the input list into first and rest.
Currying

Currying is a powerful feature of functional programming languages that allows a function to be applied to only some of its arguments. We'll talk more about Currying when we discuss Haskell’s static type system, but for now our interest is in how Currying simplifies functions whose bodies call higher-order functions.

Suppose we want to write a function `big` that takes a list `lst` and returns a list containing only the elements from `lst` that are larger than 5. We can accomplish this using the `filter` higher-order function:

```scheme
big lst = filter (\ x -> 5 < x) lst
```

But notice that the anonymous function there is simply calling the `<` function with its first argument set to 5. That is, it’s the `<` function “partially applied” to one of its two arguments. Currying allows us to dispense with the anonymous function, writing simply:

```scheme
big lst = filter ((<) 5) lst
```

This kind of flexibility would be very strange indeed for imperative languages like C and Java. But it is this flexibility that gives functional programming its power: it lets us adapt the arity of a function to contexts for which the function may not have been explicitly designed!

The higher-order function `apply`

Next, we will look at one more fundamental higher-order function. As a warm-up, consider the following mysteriously-named function:

```scheme
define ($ f x) (f x))
```

This function takes two arguments, a function and a value, and then applies the function to that value. This is fine for when `f` is unary, but what happens when it’s not? For example, what if we wanted to give `$` a binary function and two more arguments, and apply the function to those two arguments? Of course, we could write another function for this purpose, but then what about a function that takes three arguments, or one that takes ten? What we would like, of course, is a higher-order function that takes a function, then any number of additional arguments, and applies that function to those extra arguments. In Racket, we have a built-in function called `apply` that does almost this:

```scheme
; (apply f lst)
; Call f with arguments taken from the elements of lst
```
(apply + (list 1 2 3 4))
; equivalent to...
(+ 1 2 3 4)

; More generally,
(apply f (list x1 x2 x3 ... xn))
; equivalent to...
(f x1 x2 x3 ... xn)

Note that apply differs from map, even though the types of their arguments are very similar (both take a function and a list). Remember that map calls its function argument once for each value in the list separately, while apply calls its function argument just once, on all of the items in the list at once. The return value of map is always a list; the return value of apply is whatever its function argument returns.

In Haskell, the story is similar but not identical. We have a binary infix operator $, which acts as unary function application:

f $ x
-- equivalent to
f x

However, due to constraints imposed by Haskell’s type system, it does not provide an equivalent to Racket’s apply, which works on functions of arbitrary arity. We’ll discuss Haskell’s type system in detail later in this course.

Exercise Break!

1.22 Look up rest arguments in Racket, which allow you to define functions that take in an arbitrary number of arguments. Then, implement a function ($$ f x1 ... xn), which is equivalent to (f x1 ... xn).

Functions returning functions

We have now seen functions that take primitive values and other functions, but so far they have all had primitive values as their output. Now, we’ll turn our attention to another type of higher-order function: a function that returns a function. This is extremely powerful: in its most general form, it allows us to define new functions dynamically, that is, while the program is running. Here is a simple example of this:

(define (make-adder x)
  ; Notice that the body of this function (which is what's returned)
(lambda (y) (+ x y)))
(make-adder 10) ; #<procedure>
(define add-10 (make-adder 10))
add-10 ; #<procedure>
(add-10 3); 13

The function add-10 certainly seems to be adding ten to its argument; using the substitution model of evaluation for (make-adder 10), we see how this happens:

(make-adder 10)
; (substitute 10 for x in the body of make-adder)
(lambda (y) (+ 10 y))

---

Exercise Break!

1.23 Write a function that takes a single argument \( x \), and returns a new function that takes a list and checks whether \( x \) is in that list or not.

1.24 Write a function that takes a unary function and a positive integer \( n \), and returns a new unary function that applies the function to its argument \( n \) times.

1.25 Write a function \( \text{flip} \) that takes a binary function \( f \), and returns a new binary function \( g \) such that \((g x y) = (f y x)\) for all valid arguments \( x \) and \( y \).

1.26 Write a function that takes two unary functions \( f \) and \( g \), and returns a new unary function that always returns the max of \( f \) and \( g \) applied to its argument.

1.27 Write the following Racket function:

```racket
#| (fix f n x)
  f: a function taking m arguments
  n: a natural number, 1 <= n <= m
  x: an argument

  Return a new function g that takes m-1 arguments, which acts as follows:
  (g a_1 ... a_{n-1} a_{n+1} ... a_m)
  = (f a_1 ... a_{n-1} x a_{n+1} ... a_m)

  That is, \( x \) is inserted as the \( n \)th argument in a call to \( f \).
#
> (define f (lambda (x y z) (+ x (* y (+ z 1)))))
> (define g (fix f 2)())
> (g 2 4); equivalent to (f 2 100 4)
```
(Hint: recall rest arguments from an earlier exercise.)

1.28 Write a function curry, which does the following:

```scheme
#|
(curry f)
  | f: a binary function
  |
Return a new higher-order unary function g that takes an
argument x, and returns a new unary function h that takes
an argument y, and returns (f x y).
>
> (define f (lambda (x y) (- x y)))
> (define g (curry f))
> ((g 10) 14) ; equivalent to (f 10 14)
-4
```

1.29 Generalize your previous function to work on a function with $m$ arguments,
where $m$ is given as a parameter.

---

### Programming with abstract syntax trees

A natural generalization of lists are trees, in which the “recursive” part contains
an arbitrary number of subcomponents rather than just one. In this section,
we’ll start looking at how abstract syntax trees (ASTs) are represented in real
code, and how our discussion of structural recursion allows us to operate on
such trees very easily.

#### Racket: Quoted expressions

First, let’s see how we can represent ASTs in Racket. This is actually one of
the fundamental strengths of all Lisp languages: the parenthesization of the
source code immediately creates a nested list structure, which is simply another
representation of a tree. To make this even more explicit in the language, any
Racket expression (no matter how complex or deeply nested) can be turned into
a static nested list simply by prefixing it with an apostrophe ‘. We call this
**quoting** an expression.

```scheme
> (+ 1 2) ; A regular Racket expression
3
> '(+ 1 2) ; Quoting the expression: a list of three elements.
'(+ 1 2)
```
> (first '(+ 1 2))
> '+'
> (second '(+ 1 2))
> 1
> (third '(+ 1 2))
> 2
> (+ (* 2 3) (* 4 5)); This is a nested list
> '(+ (* 2 3) (* 4 5))

Even though in Racket a quoted expression really is just a list, we will call it a *Racket datum* to distinguish it from a regular list. Formally, here are the types of values that comprise the “tree” in a Racket datum:

1. Quoted literals (5, #t, “hello”) represent those same values as in the original code. In fact, quoting a literal technically has no effect: (equal? '5 5) returns true.
2. Quoted identifiers and keywords become *symbols*; a symbol is a primitive datatype that essentially functions as a unique identifier. For example, the datum '(+ 1 2) has as its first element the symbol '+, which can be compared using equal? to other values, and used in pattern-matching. Note that keywords like define also get quoted into symbols, and so the only way to determine which symbols correspond to identifiers and which correspond to keywords is to start with a fixed set of keywords to check.
3. Quoted compound expressions become lists, in which each element is the quoted version of the corresponding subexpression. '(+ 2 3) is equivalent to (list '+ '2 '3), which is in turn equivalent to (list '+ 2 3).

Given this structure, we can use structural pattern-matching to take in a datum and perform recursive computations on it. For example, here is a function that takes a Racket datum consisting of just numeric literals, identifiers, and arithmetic function calls, and returns the number of times + is used:

```scheme
(define/match (num-plus datum)
  ;; For a function call (which in this case is *any* parenthesized
  ;; expression), recurse on each subexpression and add up the total
  ;; number of occurrences. Note the use of (list expr ...) to bind
  ;; lists of arbitrary length to expr.
  [[[list expr ...)]
    (apply + (map num-plus expr))]
  ;; Value pattern-match on '+
  [('+) 1]
  ;; For any other atomic value (identifier or literal), return 0
  [(...) 0])
)

> (num-plus '+)
1
> (num-plus 3)
```

<sup>plural: “datums”, not “data” to avoid confusion</sup>
Haskell: type constructors

As a consequence of Haskell’s type system, we cannot store values of different types in the same list, making it cumbersome to directly represent Racket datums in the same way (note that those Racket lists freely mix nested lists, symbols, and literals of all different types). Instead, Haskell’s type system permits the concise declaration of, and pattern-match on, a type with multiple different constructors. We will study this more formally later in the course, but for now we think you’ll find the following representation of an “expression” type, which would serve as the node of an abstract syntax tree, to be quite intuitive:

```haskell
data Expr = NumLiteral Int   -- An integer literal
          | Identifier String -- An identifier
          | Call Expr [Expr]   -- A function call, where the first
                               -- Expr is the function being called,
                               -- and the list [Expr] contains the
                               -- argument expressions.
```

Again, the syntax is not too important here; the main point is that NumLiteral, Identifier, and Call are three constructors that take in, respectively, an integer, a string, and another expression and list of expressions, and return a value of type Expr. Note that NumLiteral and Identifier are atomic expressions (they are leaves in the AST), while Call is recursive. As an example, we would represent the Racket expression \((+ 2 3)\) by the value:

```
Call (Identifier "+") [NumLiteral 2, NumLiteral 3]
```

This might seem cumbersome at first, but because of structural pattern-matching, it is just as easy to implement the num-plus function in Haskell as it is in Racket.\footnote{We explicitly differentiate between the Identifier and NumLiteral case for illustrative purposes only. As long as we kept the first two lines the same, we could have used a single \_ in the third pattern, in the same way we did for Racket.}
Undefined programs and evaluation order

So far in this chapter, we have focused on the denotational semantics of pure functional programs: what they mean, or what they evaluate to.\textsuperscript{28} We have seen how the tools of recursion and higher-order functions can be used to express interesting computations in this setting, but even then have focused on the simple rule of function application as substitution.

While the denotational semantics of a language determine what expressions must evaluate to, the operational semantics determine how expressions are evaluated. For simple expressions like 5, (+ 20 30), and ((lambda (x) x) 1), this is not very interesting: literals are evaluated simply by using their literal value, while the other two expressions reach a numeric value of 50 and 1, respectively, in just a single simplification step each.

However, operational semantics must cover not just these simple cases, but more complex expressions as well. Consider the following:

\begin{verbatim}
1 ((lambda (x) (+ x 6)) (+ 4 4))
\end{verbatim}

In this expression, we have a choice about how to evaluate it. We could evaluate the (+ 4 4) first, and then call the lambda expression on it:

\begin{verbatim}
1 ((lambda (x) (+ x 6)) (+ 4 4))
2 ; evaluate (+ 4 4)
3 ((lambda (x) (+ x 6)) 8)
4 ; substitute 8 for x
5 (+ 8 6)
6 14
\end{verbatim}

Or, we could substitute the entire (+ 4 4) expression into the body of the lambda first, and then evaluate the result:

\begin{verbatim}
1 ((lambda (x) (+ x 6)) (+ 4 4))
2 ; substitute (+ 4 4) for x
3 (+ (+ 4 4) 6)
4 ; evaluate (+ 4 4)
5 (+ 8 6)
6 ; evaluate (+ 8 6)
7 14
\end{verbatim}

Perhaps unsurprisingly, it turns out that these two are equivalent. In fact, the Church-Rosser Theorem says (informally) that for any program of this form in the lambda calculus, every possible order of function application must result in the same final value. Informally, every road leads to the same destination.

However, even this informal version of the theorem only applies to valid function
applications, and not ones that result in an error or non-terminating computation. What happens in this case?

It is the answer to this question that leads to our discussion of the different evaluation orders used to deal with function applications, which has significant implications for a programming language.\textsuperscript{29}

Consider, for example, a binary function \( f \), and the function call \((f \ 10 \ (/ \ 1 \ 0))\). Is this always guaranteed to raise an error?

\textit{Strict evaluation semantics}

Your intuition for the previous question is probably a resounding \textit{yes}, as this is the behaviour used in almost every modern programming language. First, we say that an expression has an \textit{undefined value} if it represents a computation that does not complete, e.g., due to its corresponding abstract mathematical meaning (e.g., division by 0), or because it represents a non-terminating computation.\textsuperscript{30} For every expression type, we say that the type has \textit{strict evaluation denotational semantics} if and only if:

- Whenever an expression of that type contains a subexpression whose value is undefined, the value of the expression itself is also undefined.

Most modern programming languages use strict denotational semantics for function calls. In this context, the function call has two kinds of subexpressions: the function being called, and the arguments to the function; if any of these contains an error, the function call itself generates this error. This is most commonly implemented by (i.e., induces an operational semantics of) a \textit{left-to-right eager evaluation} of function call expressions:

- When evaluating a function call, first evaluate the subexpression representing the function being called (often, but not always, an identifier).
- Then evaluate each argument subexpression, in left-to-right order.
- Finally, “call” the function by substituting the \textit{value} of each subexpression into the body of the function.

Because arguments are evaluated before being substituted into the body of a function, this guarantees that if one of the argument subexpressions (or even the function subexpression itself) is undefined, this is discovered before the function is actually called.

Similarly, most languages use strict semantics for name bindings; for example, in Racket the top-level binding \((\text{define} \ <\text{id}> \ <\text{expr}>)\) first evaluates \(<\text{expr}>\), and then binds the \textit{value} of that expression to the \(<\text{id}>\).\textsuperscript{31}

\textit{Non-eager syntactic forms}

In your programming experience so far, you have probably taken for granted eager evaluation order in function calls. However, you also learned that there

\textsuperscript{29} While we’ll stick to the pure functional world here, evaluation order is even more important in imperative-style languages that support functions with \textit{side effects}.

\textsuperscript{30} This “undefined” value is commonly referred to as \textit{bottom} in the programming languages literature.

\textsuperscript{31} In introductory classes, we normally learn this as “first evaluate the right-hand side of the =, then assign the value to the left-hand side.”
are certain types of expressions that do not eagerly evaluate all of their subexpressions: boolean operators and the ternary if. In Racket, these are and, or, if, and cond (a generalization of if); even though these may look like ordinary functions, they aren’t! To illustrate the point, suppose we tried to write our own “and” function, simply as a wrapper for the built-in Racket and:

```scheme
(define (my-and x y) (and x y))
```

Even though my-and looks identical to and, it’s not, all because of evaluation order. Whereas the built-in and can stop evaluating its arguments when it reaches a “false” value, the same is not true of my-and—the latter is a function, and so eagerly evaluates all of its arguments before passing their values to and:

```scheme
(and #f (/ 1 0)) ; evaluates to #f
(my-and #f (/ 1 0)) ; raises an error
```

Because of this, we say that and, or, if, and cond are syntactic forms, rather than identifiers that refer to built-in functions. This point is actually rather subtle, and in fact is not specific to Racket at all! In any programming language that uses eager evaluation for functions, short-circuiting boolean operations and conditional are all implemented as syntactic constructs, not simple function calls.

**Function bodies and delaying evaluation**

Now that we’ve given a name to eager evaluation, you might look for other places where languages eagerly evaluate expressions. Consider the following Racket function:

```scheme
(define (f x)
  (length (range 3000000)))
```

Note that the body of this function doesn’t depend on the parameter x; when the Racket interpreter evaluates this function, does it “eagerly” evaluate the body of f? How about in other languages that you know?

No, it doesn’t: in general, a function’s body is not evaluated until the function is called, even if the body doesn’t depend on the function’s parameters.32 This means that we can delay the evaluation of an expression simply by putting it inside a function body. We give the name thunk to any nullary function whose purpose is to delay the evaluation of its body expression. For example, `(lambda () (/ 1 0))` is a thunk, wrapping around the error-raising division expression. When we evaluate the lambda, this doesn’t evaluate its body:

```scheme
> (lambda () (/ 1 0)) ; A thunk as an anonymous function
#<procedure>
```

32 This isn’t entirely true for modern compilers, which may evaluate constant expressions during compilation to speed up program execution time. This optimization, known as constant folding, is outside the scope of this course.
Using this idea, we can simulate non-strict semantics for function calls by passing in thunks that wrap the “actual” arguments:

```
; x and y are now thunks
(define (my-and x y)
  (and (x) (y)))

> (my-and (lambda () #f) (lambda () (/ 1 0)))
#f ; The (/ 1 0) is never evaluated!
```

---

**Exercise Break!**

1.30 Given the following nested function call expression, write the order in which the functions are evaluated:

```
(f (a b (c d) (d)) e (e) (f (f g)))
```

1.31 Draw the expression tree associated with the previous expression, where each internal node represents a function call, whose children are the arguments to that function.

1.32 Racket has `lambda` and `define` expressions. Neither of these are functions, but are instead also syntactic forms (like `and`); how do you know?

---

**Non-strict semantics in Haskell**

One of the design features that sets Haskell apart from other languages is that it uses non-strict semantics for both function calls and name bindings. During a function call, the argument subexpressions are not evaluated right away. Instead, they are only evaluated if and when they are used, and so if an argument is never used, it is never evaluated. This evaluation strategy is known as lazy evaluation. Consider the following example:

```
> onlyFirst x y = x
> onlyFirst 15 (head [])
15
> onlyFirst (head []) 15
*** Exception: Prelude.head: empty list
```
The function onlyFirst only requires its first argument to evaluate its body, so in the first call, the argument (head []) is never evaluated. This means that we can define a short-circuiting “and” function just like any other Haskell function:

```haskell
> myAnd x y = if x then y else False
```

In fact, the (&&) and (||) boolean operators really are just built-in functions for Haskell, unlike most other programming languages!

Things get even more surprising when it comes to name bindings. In Haskell, the binding <id> = <expr> does not evaluate <expr>, but instead binds the expression to the identifier. It’s only when that identifier is required to be evaluated does <expr> get evaluated:

```haskell
> x = error "This is an error"  -- No error when x is bound.
> x                            -- Error when x is evaluated.
```

Indeed, you may have wondered what the syntax is for defining nullary functions in Haskell, given that the general function definition syntax (<id> <param> ... = <body>) seems to coincide with a “simple” name binding (<id> = <expr>) when there are no parameters. Now we are able to answer this question: every name binding in Haskell defines a function! The binding <id> = <expr> defines a thunk, analogous to (define (<id>) <expr>) in Racket, making it clear that the expression is not evaluated at the time of definition.

**The trouble with Haskell’s foldl**

To wrap up this section, we’ll revisit one particularly famous wart of Haskell’s implementation. Even though we have not discussed efficiency (either time or space) too much in this course, there is one place where this has already come up: the notion of tail call optimization, which is central to mitigating call stack growth when using recursion. To start, let’s look at how our old friend foldl is implemented in Haskell (this mimics the definition in Racket):

```haskell
foldl _ acc [] = acc
foldl f acc (x:xs) =
    let acc' = f acc x
       in
    foldl f acc' xs
```

We use a let here to split off an intermediate definition of acc’ from the recursive call, to make it more obvious that the recursive call to foldl is in tail call
position. And indeed, Haskell does perform tail call optimization, as you would expect from a pure functional language. However, when we try running a large computation like \texttt{foldl \ max \ 0 \ [1..100000000]}, not only does the computation take a long time, but its memory usage increases linearly over time. What gives?

The key to understanding this is in understanding Haskell’s lazy evaluation order. Even though laziness might seem like a neat feature, it does have its drawbacks. The evaluation of functions becomes more unpredictable, making it harder to correctly reason about both the time and space efficiency of a program. In the case of our \texttt{foldl} implementation, laziness means that the \texttt{let} bindings don’t actually get evaluated until absolutely necessary—and when is that? Certainly not when we make a recursive call:

\begin{verbatim}
let acc' = f acc x
in
foldl f acc' xs
\end{verbatim}

reduces to

\begin{verbatim}
foldl f (f acc x) xs
\end{verbatim}

That is, the second argument passed to the recursive call is a \textit{thunk} whose body is \((f \ acc \ x)\), not the value produced by actually calling \(f\). Let’s make that a bit more concrete. Consider calling \texttt{foldl max 0 \ [1, 2, 3]}.

In the initial call, the body of \texttt{foldl} becomes

\begin{verbatim}
let acc' = max 0 1
in
foldl max acc' [2, 3]
\end{verbatim}

In a language with strict binding semantics, we would expect \texttt{max 0 1} to evaluate to \(1\) before being bound to \texttt{acc’}, leading to the recursive call \texttt{foldl max 1 [2, 3]}. This is the case in, say, Racket, and combined with tail call optimization this leads to space usage that is constant with respect to the size of the input list, since the space used to store the accumulator is fixed (a single integer). In the case of Haskell, however, \texttt{acc’} is bound to a thunk containing \((\texttt{max 0 1})\). So then even when the recursive call undergoes tail call optimization, the resulting argument data takes a bit more space than before: a thunk instead of the initial integer \(0\). This space usage \textit{accumulates} through the recursive calls, with each new binding of \texttt{acc’} resulting in a new thunk being stored, leading to the linearly increasing space usage we observe when we call this function. Indeed, these thunks are only called when we reach the end of the list, and \texttt{acc} is returned and printed (assuming we call this directly in the Haskell REPL).

Because laziness presents its own unique problems, the Haskell language provides ways of explicitly \textit{forcing} the evaluation of an expression, even when that

\footnote{pun definitely intended}
expression’s value is not “necessary” for evaluating its containing outer expression. One approach is with the compiler extension Bang patterns, whose use can be illustrated as follows.

```haskell
{-# LANGUAGE BangPatterns #-}
foldl _ !acc [] = acc
foldl f !acc (x:xs) =
  let acc' = f acc x
  in
  foldl f acc' xs
```

The only difference, of course, is the first line (telling the compiler to use the Bang patterns extension), and the exclamation mark preceding the acc parameter in the function definition. This has the effect of making acc a strict parameter: whenever foldl is called, the argument expression corresponding to acc must be evaluated before being passed to foldl, and of course this includes recursive calls. With this implementation of foldl we get the constant space evaluation of foldl max 0 [1..100000000], taking advantage of both tail call optimization and eager evaluation of the calls to f to ensure that only an integer accumulator is passed through the recursive calls.

A final note: why did we call foldl one of Haskell’s “famous warts”? It turns out that even though this is a well-known issue in the Haskell community, for historical reasons the more naturally-named function, foldl, behaves in the lazy way and so is generally discouraged. Instead, users wanting to use the more space-efficient version can import foldl' from Data.List, paying the price of both an extra import and a clumsier name. Go figure.

**Lexical closures**

To end this chapter, we’ll study in more detail one of the most interesting implementation details of a functional programming language: supporting the dynamic creation of functions in a space-efficient manner. Recall the make-adder function, which takes a number x and returns a new “add x” function:

```racket
(define (make-adder x)
  (lambda (y) (+ x y)))
```

Suppose we call make-adder multiple times: (make-adder 10), (make-adder -1), (make-adder 9000), etc. A naive (but correct!) implementation of Racket would create and store in memory brand-new functions for each call:

```racket
; (make-adder 10)
(lambda (y) (+ x 10))
; (make-adder -1)
```
But it seems rather wasteful for Racket to create and store in memory brand-new functions each time, when really all of the function values have essentially the same body, differing only in their value of \( x \).

A better implementation factors out the literal \( \lambda \) expression \((\lambda (y) (+ x y))\), which is common to all calls to \texttt{make-adder}, and instead stores simply the \textit{binding for} \( x \), which can vary for each function call. And in fact, this is exactly what Racket does! An abstract, but more accurate, representation of the previous example calls is:

\[
\begin{align*}
; & (\texttt{make-adder 10}) \quad (x: 10) \quad ((\lambda (y) (+ x y)) \\
; & (\texttt{make-adder -1}) \quad (x: -1) \quad ((\lambda (y) (+ x y)) \\
; & (\texttt{make-adder 9000}) \quad (x: 9000) \quad ((\lambda (y) (+ x y))
\end{align*}
\]

The data structure storing the reference to the function code, and the binding for \( x \), is called a \textbf{closure}. All along, when we’ve referred to \texttt{lambda}s as “function values”, each one actually been implemented as a closure! Why has this never come up before? Closures are conceptually necessary only when the function body has a \textbf{free identifier}, which is an identifier that is not local to the function. This should make intuitive sense: suppose we have a function where every identifier in its body is either a parameter, or bound in a local \texttt{let} expression. Then every time the function is called, all of the data necessary to evaluate that function call is contained in the arguments and the function body—no additional lookup necessary. In the definition of \texttt{make-adder}, the inner \texttt{lambda} expression has a free identifier \( x \), and this is the name whose value will need to be looked up in the closure. When this closure is itself called, say in the expression \(((\texttt{make-adder 10}) 3)\), we take the function \((\lambda (y) (+ x y))\), and look up the value of \( x \) in the closure to retrieve the 10.

In summary, a closure is a data structure containing a pointer to a function, as well as a collection of name-value bindings for all free identifiers for that function. If the body doesn’t have any free identifiers, then the closure can be identified with the function body itself, since its name-value binding is empty. Note that in order for the closure to be evaluated, all of the identifiers inside the function body must be in scope \textit{when the function is defined}, just not necessarily local to the function. This was true for the previous example: while \( x \) was a free identifier for the inner \texttt{lambda}, it was still in scope because it was a parameter of the enclosing \texttt{make-adder}. In contrast, the following variation would raise a runtime error because of an unbound identifier:

\[
; (\texttt{make-adder 10}) \quad (x: 10) \quad ((\lambda (y) (+ y y))
\]
Lexical and dynamic scope

We have just finished saying that closures—and in particular, name bindings—are created when a `lambda` expression is evaluated. But this is actually not the full story; there is yet another consideration to take into account when reasoning about how closures work. Consider the variation of `make-adder`:

```scheme
(define (make-adder x)
  (lambda (y) (+ z y)))
```

```scheme
(define z 10)
(define (make-z-adder) (lambda (x) (+ x z)))
(define add-z (make-z-adder))
> (add-z 5)
15
```

So far, so good. The body of `make-z-adder` is a function with a free identifier `z`. Calling `make-z-adder` evaluates the `lambda`, returning a closure in which the free `z` refers to the global one, and so binds to the value 10. But what happens when we shadow this `z`, and then call `make-z-adder`?

```scheme
(let* ([z 100])
  (let* ([add-z-2 (make-z-adder)])
    (add-z-2 5)))
```

Now the `lambda` expression is evaluated when we call the function in the local scope, but what value of `z` gets bound in the closure? Put more concretely, does this expression output 15 (because `z` is bound to 10) or 105 (because `z` is bound to 100)? It turns out that this expression evaluates to 15, just like the previous example. Our goal is to understand why.

The question of how to resolve free identifiers when creating closures is very important, even though we usually take it for granted. In Racket, the value used is the one bound to the name that is in scope where the `lambda` expression appears in the source code. That is, though closures are created at runtime, how the closures are created (i.e., which bindings are used) is based only on where they are written in the source code. In the previous example, the `z` in the closure returned by `make-z-adder` is bound to the value of the global `z`, which is the one that is in scope where the `lambda` expression is written.

More generally, the binding for any identifier in Racket can be determined by taking its location in the source code, and proceeding outwards through enclosing expressions until you find where this identifier is first introduced (either through a binding or a parameter declaration). We say that identifiers in Racket
obey **lexical scope** or **static scope**, referring to the fact that this resolution can be done by analysing the source code itself, prior to actually running any code.

In contrast to this is **dynamic scope**, in which identifiers are resolved based on when the identifier is used during program execution, rather than where it appears in the code. If Racket used dynamic scope, the above example would output 105, because the closure created by `make-z-adder` would now bind the enclosing local `z`.

Here is an example of how dynamic scoping works in the bash shell scripting language:

```bash
X=hey

function printX {
  echo $X
}

function localX {
  local X=bye
  printX
}

printX  # will print hey
localX  # will print bye
```

In other words, lexical scoping resolves names based on context from the source code, whereas dynamic scoping resolves names based on context from the program state at runtime. Early programming languages used dynamic scope because it was easier to implement, as names could be resolved by looking for the nearest binding on the function call stack, or by recursively accumulating name bindings in an interpreter. However, dynamic scope makes programs very difficult to reason about, as the values of non-parameter names of a function depended on the myriad places the function is used, rather than the single place where it is defined. Lexical scope was a revolution in how it simplified programming tasks, and is used by every modern programming language today.\(^{37}\) And it is ideas like that which motivate research in programming languages!

\(^{37}\) ALGOL was the first language to use lexical scope, in 1958.

### A Python puzzle

Closures are often used in the web programming language JavaScript to dynamically create and bind *callbacks*, which are functions used to respond to events like a user clicking a button or entering some text. A common beginner mistake when creating these functions exposes a very subtle misconception about closures when they are combined with mutation. We’ll take a look at an analogous example in Python.
The fact that Python also uses lexical scope means that the closure of each of the three `print_i` functions refers to the same `i` variable (in the enclosing scope). That is, the closures here store a *reference*, and not a *value.* After the loop exits, `i` has value 3, and so each of the functions prints the value 3. Note that the closures of the functions store this reference even after `make_functions` exits, and the local variable `i` goes out of scope!

By the way, if you wanted to fix this behaviour, one way would be to not use the `i` variable directly in the created functions, but instead pass its value to another higher-order function. In the code below, each `print_i` function has a closure looking up `x`, which is bound to different values and not changed as the loop iterates.

---

```python
def make_functions():
    flist = []
    for i in [1, 2, 3]:
        def print_i():
            print(i)
            print_i()
        flist.append(print_i)
    print('End of flist')
    return flist

def main():
    flist = make_functions()
    for f in flist:
        f()

>>> main()
1
2
3
End of flist
3
3
3
```

---

Since we have been avoiding mutation up to this point, there hasn’t yet been a distinction between the two.
def main():
    flist = make_functions()
    for f in flist:
        f()

Secret sharing

Here's one more cute example of using closures to allow “secret” communication between two functions in Python.\(^{39}\)

```python
def make_secret():
    secret = ''
    
    def alice(s):
        nonlocal secret
        secret = s

    def bob():
        nonlocal secret
        print(secret)
        # Reset secret
        secret = ''

    return alice, bob

>>> alice, bob = make_secret()
>>> alice('Hi bob!')
Hi bob!
>>> bob()
>>> secret
Error ...
```

Exercise Break!

1.33 In the following lambda expressions, what are the free identifiers (if any)? (Remember that this is important to understand what a closure actually stores.)

```python
(lambda x y (+ x (* y z))) ; (a)
(lambda x y (+ x (w y z))) ; (b)
```
Write a snippet of Racket code that contains a function call expression that will evaluate to different values depending on whether Racket uses lexical scope or dynamic scope.

Summary

In this chapter, we looked at the basics of functional programming in two new languages, Racket and Haskell. Discarding mutation and the notion of a “sequence of statements,” we framed computation as the evaluation of functions using higher-order functions to build more and more complex programs. However, we did not escape notion of time entirely; in our study of evaluation order, we learned precisely how Racket (and most other languages) eagerly evaluate function call arguments, and how special syntactic forms distinguish themselves from functions precisely because of how their arguments are evaluated. We contrasted this with Haskell, which lazily evaluates its arguments only if and when they are needed.

Our discussion of higher-order functions culminated in our discussion of closures, allowing us to create functions that return new functions, and so achieve an even higher level of abstraction in our program design. Along the way, we discovered the important difference between lexical and dynamic scope, an illustration of one of the big wins that static analysis yields to the programmer. Finally, we saw how closures could be used to share internal state between functions without exposing that data. In fact, this encapsulation of data to internal use in functions should sound familiar from your previous programming experience, and will be explored in the next chapter.
2 Macros, Objects, and Backtracking

In most programming languages, syntax is complex. Macros have to take apart program syntax, analyze it, and reassemble it… A Lisp macro is not handed a string, but a preparsed piece of source code in the form of a list, because the source of a Lisp program is not a string; it is a list. And Lisp programs are really good at taking apart lists and putting them back together. They do this reliably, every day.

Mark Jason Dominus

Now that we have some experience with functional programming, we will briefly study two other programming language paradigms. The first, object-oriented programming, will likely be very familiar to you, while the second, logic programming, will likely not. Because of our limited time in this course, we will not treat either topic with as much detail as it deserves. In particular, we will stay in Racket, rather than switching programming languages. Instead, we will build support for these paradigms into the Racket language itself, and so kill two birds with one stone: you will learn how to use an advanced macro system to fundamentally extend a language, and gain a deeper understanding of these paradigms by actually implementing simplified versions of them.

Object-oriented programming: a re-introduction

OOP to me means only messaging, local retention and protection and hiding of state-process, and extreme late-binding of all things.

Alan Kay

Because we have often highlighted the stark differences between imperative and functional programming, you may be surprised to learn that our study of func-
tions has given us all the tools we need to implement a simple object-oriented system.

Recall the definition of a closure: a function together with name bindings that are saved when the function is created. It is this latter part—“stored values”—that is reminiscent of objects. In traditional object-oriented languages, an object is a value that stores data in attributes, with associated functions called methods that can operate on this data.

This paradigm was developed as a way to organize and encapsulate data, while exposing a public interface to define how others may operate on this data. Unlike the pure functions we have studied so far, a method always takes a special argument, an associated object that we say is calling the method. Though internal attributes of an object are generally not accessible from outside the object, they are accessible from within the body of methods that the object calls.

Historically, the centrality of the object itself to call methods and access (public) attributes led to the natural metaphor of entities sending and responding to messages to model computation. We have already seen how closures provide encapsulation when they are returned from function calls. If we put the notion of an object as something that receives and responds to a message into the context of functional programming, well, an object is just a particular type of function. Here is a simple example of that idea:

```
(define (point msg)
  (cond [(equal? msg 'x) 10]
        [(equal? msg 'y) -5]
        [else "Unrecognized message"]))

> (point 'x)
10
> (point 'y)
-5
```

Of course, this “point object” is not very compelling: it only has attributes but no methods, making it more like a C struct, and its attribute values are hard-coded.

One solution to the latter problem is to create a point class, a template that specifies both the attributes and methods for a type of object. In class-based object-oriented programming, every object is an instance of a class, obtaining their attributes and methods from the class definition.4 Objects are created by calling a class constructor, a function whose purpose is to return a new instance of that class, often initializing all of the new instance’s attributes.

To translate this into our language of functions, a constructor for Point is a function that takes two numbers, and returns a function analogous to the one above, except with the 10 and 5 replaced by the constructor’s arguments.5

---

4 This “calling object” is often an implicit argument with a fixed keyword name like this or self.

5 We’ll use the convention in these notes of treating messages to objects as symbols naming the attribute or method to access.

---

4 Even though class-based OOP is the most common approach, it is not the only one. JavaScript uses prototypal inheritance to enable behaviour reuse; objects are not instances of classes, but instead inherit attributes and methods directly from other objects.

5 We follow the convention of giving the constructor the same (capitalized) name as the class itself.
(define (Point x y)
  (lambda (msg)
    (cond [(equal? msg 'x) x]
          [(equal? msg 'y) y]
          [else "Unrecognized message"])))

> (define p (Point 2 -100))
> (p 'x)
  2
> (p 'y)
  -100

Of course, this is where we require the use of closures: in the returned function, \(x\) and \(y\) are free, and so must have values bound in a closure when the \(\text{Point}\) constructor returns. Moreover, because the identifiers \(x\) and \(y\) are local to the \(\text{Point}\) constructor, even though their values are stored in the closure, once the constructor has returned they can’t be accessed without passing a message to the object.\(^6\) This is the property of closures that enables encapsulation: even though both \(x\) and \(y\) values are accessible by passing the right messages to the object, it is certainly possible to implement “private” attributes in this model—and a nice thing to think about even now!

One other point: \textit{lexical closures} is absolutely required to maintain proper encapsulation of the attributes. Imagine what would happen if the following code were executed in a dynamically-scoped language, and what implications this would have when we create multiple instances of the same class.

(define p (Point 2 3))
(let ([x 10])
 (p 'x))

\textit{Basic methods}

Next, let’s add two simple methods to our \textit{Point} class. Because Racket functions are first-class values, we can treat attributes and methods in the same way: a method is just an attribute that happens to be a function.\(^7\) Of course, one of the characteristics that distinguishes methods from arbitrary functions is that in the body of a method, we expect to be able to access all instance attributes of the calling object. It turns out that this is not an issue; study the example below, and see if you can determine why not!

(define (Point x y)
  (lambda (msg)
    (cond [(equal? msg 'x) x]
          [(equal? msg 'y) y]
          [(equal? msg 'to-string) ...]})

\(^6\)This is analogous to the \textit{secret} variable in the example at the end of the last chapter.

\(^7\)This is a starkly different view than how, say, Java treats methods, which are completely separate from data attributes.
More on re-usability

Cool! We have seen just the tip of the iceberg of implementing class-based objects with pure functions. As intellectually stimulating as this is, however, the current technique is not very practical. Imagine creating a series of new classes—and all of the boilerplate code\(^8\) you would have to write each time.

What we’ll study next is a way to augment the very syntax of Racket to achieve the exact same behaviour in a much more concise, natural way:

\(^8\) e.g., the message handling with cond and equal?, “Unrecognized message”, etc.
Exercise Break!

2.1 First, carefully review the final implementation of the Point class we gave above. This first question is meant to reinforce your understanding about function syntax in Racket. Predict the output of each of the following expressions (many of them are erroneous—make sure you understand why).

```
1 Point
2 (Point 3 4)
3 (Point 3)
4 (Point 3 4 'x)
5 ((Point 3 4))
6 ((Point 3 4) 'x)
7 ((Point 3 4) 'distance)
8 ((Point 3 4) 'distance (Point 3 10))
9 (((Point 3 4) 'distance) (Point 3 10))
```

2.2 Take a look at the Person example given above. Even though it is currently invalid Racket code, its intended semantics should be quite clear. Write a Person class in the same style as the Point class above. This will ensure that you understand our approach for creating classes, so that you are well prepared for the next section.
Pattern-based macros

As we have previously touched on, Racket’s extremely simple syntax—programs are nested lists—not only makes code easy to parse, but also makes it easy to manipulate. One neat consequence of this is that Racket has a very powerful macro system, with which developers can quite easily extend the language by adding new keywords, and even entire domain-specific languages.

In programming, we are used to thinking about functions as operating on program values: a function is an entity that takes values as inputs, and outputs a new value. As the building blocks of programs, functions are extremely versatile and powerful for programmers, and so it is easy to forget that they have limitations: their syntax and semantics are defined by the programming language itself, and not the programmer who writes them. A macro is an abstract function that operates on program source code: a macro is an entity that takes code as input, and outputs new code. Many of the features of Integrated Development Environments (IDEs) that we take for granted are macros: code formatting, identifier renaming, automatic refactoring, getter and setter generation, etc.

But IDEs are themselves software that operate on text files containing source code, so it is perhaps not too surprising that they implement these features. What is far more interesting, and the topic of this section, is when programming languages themselves allow the programmer to define their own macros, just as they can define their own runtime functions. In such languages, there is an additional step between the reading of source code and generation of machine code or runtime evaluation, called macro expansion, in which macros (both built-in and user-defined) are applied to the abstract syntax tree to generate a new abstract syntax tree. That is, the AST that is actually used to generate machine code or to be interpreted is not necessarily the one originally produced by the parser, and might actually look very different, depending on the macros applied to it!

This may seem kind of abstract: why might we want to use macros? The main use of macros we’ll see is to introduce new syntax into a programming language. In this section, we’ll build up a nifty syntactic construct: the list comprehension. First, let’s see how list comprehensions work in Haskell:

```
>>> [x + 2 | x <- [0, 10, -2]]
[2,12,0]
```

The list comprehension consists of three important parts: an output expression, a variable name, and an initial list. Expressed as a grammar rule, we have:

```
<list-comp> = "[" <out-expr> "|" <id> "<-" <list-expr> "]"
```

The other characters in the expression are just syntax necessary to indicate that this is indeed a list comprehension expression instead of, say, a list expression. What we would like to do is mimic this concise syntax in Racket:

```
[<x + 2 | x <- [0, 10, -2]]]
```

For a technical reason, we have to use : instead of | in Racket.
> (list-comp (+ x 2) : x <- (list 0 10 -2))
'(2 12 0)

Let's first talk about how we might implement the high-level functionality in Racket, ignoring the syntactic requirements. Recalling your work in the previous chapter, you might notice that a list comprehension is essentially a map:

> (map (lambda (x) (+ x 2)) (list 0 10 -2))
'(2 12 0)

Now, we do some pattern-matching to generalize:

; Putting our examples side by side...
(list-comp (+ x 2) : x <- (list 0 10 -2))
(map (lambda (x) (+ x 2)) '(0 10 -2))

; leads to the following generalization.
(list-comp <out-expr> : <id> <- <list-expr>)
(map (lambda (<id>) <out-expr>) <list-expr>)

This step is actually the most important one, because it tells us (the programmers) what syntactic transformation the interpreter will need to perform: every time it sees a list comprehension, it should transform it into a map. It remains to actually tell the Racket interpreter to do this transformation; we do this by writing a pattern-based macro:

(define-syntax list-comp
  (syntax-rules (: <-)
    [(list-comp <out-expr> : <id> <- <list-expr>)
     (map (lambda (<id>) <out-expr>) <list-expr>)]))

Let's break that down. The top-level define-syntax is a syntactic form that defines a name binding for a new macro (analogous to define for values). Its first argument is the name of the macro—in our case, list-comp—and the second argument is the macro expression. In this course, we'll stick to pattern-based macros, which are declared using syntax-rules. syntax-rules itself takes two or more arguments: the first is a list of the literal keywords in the syntax—in our case, : and <-—followed by one or more syntax pattern rules, which are pairs of expressions, similar to define/match. In each rule, the first expression is a pattern to match (starting with list-comp), while the second expression is called a template that specifies how the new syntax should be generated from parts of the pattern. We only have one pattern rule right now, but that will change shortly.
With this macro defined, we can now evaluate our Racket list-comp expression:

```racket
> (list-comp (+ x 2) : x <- (list 0 10 -2))
'(2 12 0)
```

While this might look just like a plain function call, it isn’t! There are two phases involved to produce the result ’(2 12 0):

1. First, the list-comp expression is transformed into a map expression, by applying the syntax pattern rule we defined. This is a code transformation; you could achieve the same effect by literally typing in `(map (lambda (x) (+ x 2)) (list 0 10 -2))` instead.
2. Second, the map expression is evaluated, using plain old function call semantics.

One way to see the difference between list-comp and a regular function is to use the syntax in incorrect ways:

```racket
> (list-comp 1 2 3)
list-comp: bad syntax in: (list-comp 1 2 3)
> (list-comp (+ x 2) : x <- 10)
map: contract violation
  expected: list?
  given: 10
  argument position: 2nd
  other arguments...:
    #<procedure>
```

The first error is a syntax error: Racket is saying that it doesn’t have a syntax pattern rule that matches the given expression. The second error really demonstrates that a syntax transformation occurs: `(list-comp (+ x 2) : x <- 10)` might be syntactically valid, but it expands into `(map (lambda (x) (+ x 2)) 10)`, which raises the runtime error you see above.

**The purpose of literal keywords**

Our syntax pattern rule makes use of both pattern variables and literal keywords. A pattern variable is an identifier that can be bound to an arbitrary expression when the pattern matches; by convention, we’ll name these with enclosing angle brackets, but this is not required by Racket. During macro expansion, when Racket finds a pattern-match, it binds the pattern variable to the corresponding expression, and then substitutes this expression where ever the variable appears in the rule’s template. This is similar to function calls, in which argument values are bound to parameter names and substituted into the body of a function, but there is a crucial difference: in macro expansion, expressions
aren’t evaluated before they are bound! This should make intuitive sense as long as you remember that macros are fundamentally about transforming code, not about evaluation; when we say that an expression is bound to a pattern variable, we literally mean that entire expression is bound to the variable, not the value of the expression.

On the other hand, literal keywords are parts of the syntax rule that must appear literally in the syntax expression. These play the same role as keywords in other programming languages (e.g., else, : in Python), namely to give structure to the program syntax. If we try to use list-comp without the two keywords, we get a syntax error—the Racket interpreter does not recognize the expression, because it no longer pattern-matches our rule:

```racket
> (list-comp (+ x 2) 3 x "hi" (list 0 10 -2))
list-comp: bad syntax ...
```

Using multiple pattern rules

In Haskell, list comprehensions support optional filtering of the input list:

```haskell
>>> [x + 2 | x <- [0, 10, -2], x >= 0]
[2, 12]
```

To achieve this form of list comprehension in Racket, we simply add an extra syntax rule to our macro definition, much like we can add an extra pattern rule to a define/match!

```racket
(define-syntax list-comp-if
  (syntax-rules (: <- if)
    [(list-comp-if <out-expr> : <id> <- <list-expr>)
      (map (lambda (<id>) <out-expr>) <list-expr>)]
    [(list-comp-if <out-expr> : <id> <- <list-expr> if <condition>)
      (map (lambda (<id>) <out-expr>)
        (filter (lambda (<id>) <condition>) <list-expr>))])))
```

```racket
> (list-comp-if (+ x 2) : x <- (list 0 10 -2))
'(2 12 0)
> (list-comp-if (+ x 2) : x <- (list 0 10 -2) if (> x 0))
'(2 12)
```

Note that as with functions defined using pattern-matching in both Racket and Haskell, with these macros the pattern rules are checked top-to-bottom, with
the first match being the one that is chosen. It doesn’t matter in this example because the rules are mutually exclusive, but in general it’s certainly possible to define two syntax pattern rules that overlap, so be careful about this when writing your own macros!

Text-based vs. AST-based macros

If you’ve heard of macros before learning a Lisp-family language, it was probably from C or C++. The C macro system operates on the source text itself, rather than a parsed AST. In the macro example above we noted that we couldn’t use | or , as literal keywords; this is because they are special characters that affect the parsing of Racket code into an AST, and so have an effect even before the macro expansion phase. This limitation aside, it is far easier and safer to define AST-based macros. In this section, we look at one small example of how text-based C macros can lead to a common pitfall.

Identifiers in C macros are treated just as strings, and in particular their scope is not checked when the substitution happens. This means that inside a macro, it is possible to refer to identifiers defined outside of the macro, a phenomenon known as name capture. This is illustrated in the following example.

```c
#include <stdio.h>
#define INCI(i) {a = 0; ++i;}

int main(void) {
    int a = 0, b = 0;
    INCi(a);
    INCi(b);
    printf("a is now %d, b is now %d\n", a, b);
    return 0;
}
```

The #define line is a macro: it searches the source for text of the form INCI(...), and when it finds it, it replaces it with the corresponding text in the body of the macro. After macro expansion, the code becomes the following:

```c
#include <stdio.h>

int main(void) {
    int a = 0, b = 0;
    {a = 0; ++a;};
    {a = 0; ++b;};
    printf("a is now %d, b is now %d\n", a, b);
    return 0;
}
```
This expansion of INCI illustrates name capture: both of the substituted assignments \( a = 0; \) reset the value of the \( a \) declared in \texttt{main}, and so this program prints \texttt{a is now 0, b is now 1}. In essence, what \( a \) in the INCI macro refers to depends on which \( a \) is in scope \textit{where the macro is used}. If this sounds familiar, it should! Identifiers in C macros obey \textit{dynamic scope}.

Let’s look at a similar example in Racket.

```racket
(define-syntax make-adder
  (syntax-rules ()
    [(add-a <x>)
      (+ <x> a)])
)
```

It may look like \( a \) is undefined in the body of \texttt{make-adder}, much like \( a \) was undeclared in the body of \texttt{INCI} in our C example. The question is, what happens with:

```racket
(let ([a 10])
  (add-a 100))
```

Using a literal textual substitution, after macro expansion this expression would become

```racket
(let ([a 10])
  (+ 100 a))
```

This is certainly a semantically-valid Racket expression, and would evaluate to 110. However, this is not what happens!

```racket
> (let ([a 10])
    (add-a 100))
ERROR a: undefined;
cannot reference an identifier before its definition
```

That is, the \( a \) in the \texttt{add-a} macro template obeys \textit{lexical scope}, and cannot be bound dynamically where the macro is used. This is true for all Racket macros: free identifiers in the macro obey lexical scope, meaning they are bound to whatever identifier is in scope where the macro is originally defined, and cannot accidentally names when it is used. Because this was seen as a massive improvement over the previous dynamically-scoped macros, early developers of Lisp termed this form of macro \textit{hygienic macros}.

As a consequence of macro hygiene, identifiers defined within a macro body are \textit{not} accessible outside of the macro body—that is, they’re local to the macro—even when a straight textual substitution would suggest otherwise. For example:
(define-syntax defs
  (syntax-rules ()
    [(defs)
      ; We use begin to enclose multiple expressions
      (begin
        (define x 1)
        (define y (+ x 10))
        ; x and y are both in scope here.
        (+ x y))]]
  (defs) ; This evaluates to 12
  x ; This is an error! x and y are not in scope.
)

Contrast this behaviour against simply using the begin instead of our defs macro:

; This still evaluates to 12
(begin
  (define x 1)
  (define y (+ x 10))
  ; x and y are both in scope here.
  (+ x y))

; But now x is defined as well!
; The following evaluates to 1.
  x

Macro ellipses

It is often the case that we want a macro that can be applied to an arbitrary number of expressions.\(^\text{16}\) This poses a challenge when writing macro pattern rules, as we can’t write an explicit pattern variable for each expression we expect to match. Instead, we can use the ellipsis \(\ldots\) in a pattern: \(<\text{pat}>\ldots\) matches zero or more instances of the pattern \(<\text{pat}>\), which could be a pattern variable or a complex pattern itself.\(^\text{17}\)

Here is one example of using the ellipsis in a recursive macro that implements cond in terms of if. To trigger your memory, recall that branching of “else if” expressions can be rewritten in terms of nested if expressions:

(\begin{verbatim}
(cond [c1 x1]
      [c2 x2]
      [c3 x3]
      [else y])
end{verbatim})
Here is our macro that achieves this behaviour:

```
(define-syntax my-cond
  (syntax-rules (else)
    [(my-cond) (void)]
    [(my-cond [else <val>]) <val>]
    [(my-cond [<test> <val>] [next-pair] ...) (if <test> <val> (my-cond [next-pair] ...)))
  ))
```

This example actually illustrates two important concepts with Racket’s pattern-based macros. The first is how this macro defines not just a syntax pattern, but a nested syntax pattern. The first pattern rule will match the expression (my-cond [else 5]), but not (my-cond else 5)—bracketing matters. The second is the <next-pair> ... part of the second pattern, which binds all arguments after the first one. For example, here’s a use of the macro, and one step of macro expansion, which should give you a sense of how this recursive macro works:

```
(my-cond [c1 x1]
  [c2 x2]
  [c3 x3]
  [else y]))
```

```
; <test> is bound to c1, <val> is bound to x1,
; and <next-pair> ... is bound to ALL of
; [c2 x2] [c3 x3] [else y]
(if c1
  x1
  (my-cond [c2 x2] [c3 x3] [else y]))
```

**Warning:** don’t think of the ellipsis as a separate entity, but instead as a modifier
for <next-pair>. We say that the ellipsis is “bound” to <next-pair>, and must appear together with <next-pair> in the rule’s template. A beginner mistake is to try to treat the ellipsis as an identifier in its own right, bound to the “rest” of the arguments:

```scheme
(define-syntax my-cond-bad
  (syntax-rules (else)
    [(my-cond-bad [else <val>]) <val>]
    [(my-cond-bad [<test> <val>] ...) ; This would make sense if ... was a variable representing the
       ; remaining arguments, but it isn't.
      ; Instead, the rule below is treated as a syntax error; the ...
      ; cannot be used independently of the pattern it's bound to.
      (if <test> <val> (my-cond-bad ...))))
```

This ellipsis binding goes both ways: <next-pair> can’t appear without the ellipsis, either.

To summarize, a pattern variable that is modified with an ellipsis is no longer bound to an individual expression, but instead to the entire sequence of expressions. We’ll see how to make powerful use of this sequence in the next section.

---

**Exercise Break!**

2.3 Explain how a macro is different from a function. Explain how it is similar to a function.

2.4 Below, we define a macro, and then use it in a few expressions. Write the resulting expressions after the macros are expanded. Note: do not evaluate any of the resulting expressions! This question is a good check to make sure you understand the difference between the macro expansion and evaluation phases of the Racket interpreter.

```scheme
(define-syntax my-mac
  (syntax-rules ()
    [(my-mac x) (list x x)]))

(my-mac 3)
(my-mac (+ 4 2))
(my-mac (my-mac 1000))
```

2.5 Write macros to implement and and or in terms of if. Note that both syntactic forms take an arbitrary number of arguments.

2.6 Why could we not accomplish the task in the previous question with functions?

2.7 Consult the official Haskell documentation on list comprehensions. One additional feature we did not cover is list comprehensions on multiple variables.
Modify our existing list-comp macro to handle this case. Hint: first convert the above expression into a list comprehension within a list comprehension, and use apply append.

2.8 Add support for if statements in list comprehensions with multiple variables. Are there any syntactic issues you need to think through?

Objects revisited

Now that we have seen how to define Racket macros, let’s return to our basic implementation of objects and classes. To start, here is an abbreviated Point class with only the two attributes, and no methods.

```racket
(define (Point x y)
  (lambda (msg)
    (cond [(equal? msg 'x) x]
          [(equal? msg 'y) y]
          [else "Unrecognized message!"]))))
```

The goal now is to abstract away the details of defining a constructor function and the message handling to enable easy declaration of new classes. Because we are interested in defining new identifiers for class constructors, our previous tool for abstraction, functions, is insufficient for this task. Instead, we turn to macros to introduce a new syntactic form into the language, my-class:

```racket
; This expression should expand into the definition above.
(my-class Point (x y))
```

Performing the same pattern-matching procedure as before, we can see how we might write a skeleton of the macro (accepting not just two but an arbitrary number of attribute names).

```racket
(define-syntax my-class
  (syntax-rules ()
    [(my-class <class-name> (<attr> ...))
     (define (<class-name> <attr> ...) (lambda (msg)
      (cond ???
             [else "Unrecognized message!"]))))]))
```

Unfortunately, this macro is incomplete; without the ???, it would generate the following code:
What’s missing, of course, are the other expressions in the cond that match messages corresponding to the attribute names. Let’s consider the first attribute \( x \). For this attribute, we want to generate the code

\[
\text{[\text{equal? msg 'x) x]}
\]

Now, to do this we need some way of converting an identifier into a symbol, which we can do using the syntactic form \( \text{quote} \):

\[
\text{[\text{equal? msg (quote x) x]}
\]

For (\text{class} \ Point \ (x \ y)), we actually want this expression to appear for both \( x \) and \( y \):

\[
\text{[\text{equal? msg (quote x) x]} \ \text{[\text{equal? msg (quote y) y]}]
\]

So the question is how to generate one of these expressions for each of the attributes bound to \(<\text{attr}> \ldots \); that is, repeat the above pattern an arbitrary number of times, depending on the number of attributes. It turns out that macro ellipses support precisely this behaviour:\(^{20}\)

\[
\text{(define-synatx my-class}
\]

\[
\text{(syntax-rules ()
\]

\[
\text{[(my-class <class-name> (<attr> \ldots))
\]

\[
\text{(define <class-name> <attr> \ldots)
\]

\[
\text{(lambda (msg)
\]

\[
\text{(cond [(\text{equal? msg (quote <attr>)}) <attr>]
\]

\[
\text{; Repeat the previous expression once per expression
\]

\[
\text{; in <attr> \ldots, replacing just occurrences of <attr>
\]

\[
\text{...}
\]

\[
\text{[else "Unrecognized message!"])])])})
\]

Adding methods

Now, let us augment our macro to allow method definitions. Even though we previously made a stink about methods just being any other type of attribute,
there is one important difference: it isn’t enough to provide just the name of the
method in the class definition, we need to provide the method body as well. We
will use the following syntax, mimicking Racket’s own macro pattern-matching
syntax:

```
(my-class <class-name> (<attr> ...)
  (method (<method-name> <param> ...) <body>)
...)
```

Once again, the easiest way to write the pattern-based macro is to create an
instance of both the syntactic form we want to write, and the expression we
want to expand it into. To use our ongoing point example:

```
; What we want to write:
(my-class Point (x y)
  (method (distance other-point)
    ; This let expression is the body of the method.
    (let ([dx (- x (other-point 'x))]
          [dy (- y (other-point 'y))])
      (sqrt (+ (* dx dx) (* dy dy)))))

; What we want it to expand into:
(define (Point x y)
  (lambda (msg)
    (cond [(equal? msg 'x) x]
          [(equal? msg 'y) y]
          [(equal? msg 'distance)
             (lambda (other-point)
               (let ([dx (- x (other-point 'x))]
                     [dy (- y (other-point 'y))])
                 (sqrt (+ (* dx dx) (* dy dy))))]))
```

After carefully examining these two expressions, the pieces just fall into place:21

```
(define-syntax my-class
  (syntax-rules (method)
    [(my-class <class-name> 
      ; This ellipsis is paired with <attr>
      <attr> ...)
     (define (<class-name> <attr> ...)
       (lambda (msg)
         (cond [(equal? msg (quote <attr>)) <attr>]
          ...)
     (define (<class-name> <param> 
      ; This ellipsis is paired with <param>
      (method (<method-name> <param> ...) <body>)
      ; This ellipsis is paired with the whole previous pattern
      ...)
     (define (<class-name> <attr> ...)
       (lambda (msg)
         (cond [(equal? msg (quote <attr>)) <attr>]
```

21 This showcases the use of nested ellipses in a pattern expression. Holy cow, that rocks! Racket’s insistence in always pairing an ellipsis with a named identifier and strict parenthesization really help distinguish the ellipsis pairings.
The object __dict__

You might notice that the repeated (equal? msg <symbol>) clauses implement a (slow!) dictionary lookup, where the name of an attribute is keyed to its value for that object. Indeed, in Python every object has a special attribute __dict__ that refers precisely to its dictionary of attributes (but not methods—more on this later):

```python
class A:
    def __init__(self, x):
        self.x = x
        self.y = 10
        self.z = 'hello!'

>>> a = A(True)
>>> a.__dict__
{'x': True, 'y': 10, 'z': 'hello!'}
```

We can improve our my-class macro by using a built-in hashing data structure in Racket to store our attributes and methods.

```racket
(define-syntax my-class
  (syntax-rules (method)
    [(my-class <Class> (<attr> ...) (method (<method-name> <param> ...) <body>) ...)  ; Use the same name as Python
     (define <Class> <attr> ...)
     (let* ([__dict__ ; Look up the given attribute in the object's dictionary.
             (make-immutable-hash
              (list (quote <attr>) <attr>)
              ...  
              (quote <method-name>)
              (lambda (<param> ...) <body>))
              ...)])
     (lambda (msg)
          ; Call this if the attribute isn't found.
          (lambda () "Unrecognized message!"))))))
```
Exercise Break!

2.9 Modify the \_\_dict\_\_\_\_-based class implementation to enable accessing the \_\_dict\_\_\_ value from outside the object, as we can in Python. For example:

```
> (define p (Point 2 3))
> (p '__dict__)
'#:hash((x . 2) (y . 3))
```

Note that in Python, the \_\_dict\_\_\_\_ attribute itself doesn’t appear in the object’s dictionary. Can you achieve the same effect in Racket?

The problem of self

So far, our implementation of classes may seem to work, but has one major deficiency: we cannot explicitly reference the calling object in a method. Consider the following example in Python:

```
class Point:
    def __init__(self, x, y):
        self.x = x
        self.y = y

    def size(self):
        return math.sqrt(self.x ** 2 + self.y ** 2)

    def same_size(self, other):
        return self.size() == other.size()
```

Our current implementation has no trouble implementing the initializer method \_\_init\_\_\_\_ (it does this automatically), and it can handle simple attribute access of \(x\) and \(y\) because those identifiers would be bound in the closure returned by our class constructor. However, \texttt{self.size()} poses a problem:

```
(my-class Point (x y)
  (method (size) (sqrt (+ (* x x) (* y y))))
  (method (same-radius other))
  (equal? ??? ((other 'size))))
```

Unlike \(x\) and \(y\), there is no \texttt{size identifier} in scope in the closure: instead, \texttt{'size} is a key in the dictionary, but we have no way to automatically convert it! Perhaps the most direct way of solving this problem is to modify our macro to introduce
identifiers for each method as well, making them exactly parallel to the “data” attributes that are the constructor parameters:\textsuperscript{22}

\begin{verbatim}
(method (same-radius other)
  (equal? (size) (other 'size)))
\end{verbatim}

However, this has the drawback that we still wouldn’t have a way of referencing self, e.g., to pass the calling object to a completely different function. So instead, we’ll look at an approach to make self an accessible identifier, referencing the calling object:

\begin{verbatim}
(method (same-radius other)
  (equal? ((self 'size)) ((other 'size)))
\end{verbatim}

\textit{Pythonic inspiration}

To guide our design and implementation, let’s first review how self works in Python. You are already familiar with the most visible part: self is an explicit parameter written in every instance method of the class!\textsuperscript{23} In fact, if we adopt this convention then we can write the following Racket code without making any changes to our macro at all:

\begin{verbatim}
(my-class Point (x y)
  (method (size self) (sqrt (+ (* (self 'x) (self 'x))
                   (* (self 'y) (self 'y))))))
  (method (same-radius self other)
          (equal? ((self 'size)) ((other 'size))))
\end{verbatim}

Look at that last line carefully—it’s uglier than you think. What’s going on there? Before, (self ‘size) returned a function that takes no arguments, and so we could just call it (by enclosing it in parentheses). Now, (self ‘size) returns a function that takes a single argument, self, and so we need to pass that to it, and the same is true when we call (other ‘size) too. Of course, this is very redundant: since we’re calling (self ‘size) already, we know what the calling object is, and shouldn’t need to pass it in again!

This is exactly what Python does: an instance method call self.size() automatically binds the value to the left of the period, self, to the first parameter of the size method; any arguments passed in the parentheses get bound to the remaining method parameters. So in order to achieve this behaviour, we do need to modify this macro to achieve this “automatic self binding.” For this, we’ll again turn to Python.

Recall that in our initial look at Python \_\_dict\_\_ values we noted that an object’s \_\_dict\_\_ only stored its data attributes, and not its methods. It turns out that this is because methods are stored in the \_\_dict\_\_ of the class itself:

\textsuperscript{22} This models the notion of an implicit this in, e.g., Java.

\textsuperscript{23} In fact, the name self is just a convention: technically, Python interprets the first parameter as the one that is bound to the calling object, whatever the parameter name actually is.
This gives us a path towards an implementation of methods in our Racket macro:

1. Store methods in a separate class-level dictionary that’s shared among all instances.
2. When methods are looked up in this dictionary, don’t just return the method as-is, instead bind the first parameter of the method to the calling object, and return a new method instead!\(^{24}\)

Here’s an implementation of (1), which is pretty straightforward:

```racket
(define-syntax my-class
  (syntax-rules (method)
    ([my-class <Class>                   
      (non-function) attributes     
        (<attr> ...)                 
      ; methods                       
      (method (<method-name> <params> ...) <body>) ...) 
      ; The template
      (begin
        ; This is a dictionary of the methods associated with a class.
        ; (Very much analogous to Point.__dict__ in Python.)
        (define class__dict__
          (make-immutable-hash (list
                                    (cons (quote <method-name>)
                                      (lambda (<params> ...) <body>))
                                    ...)))
        (define (<Class> <attr> ...) 
          (let* ([self__dict__        
                     ; The object’s __dict__ now only stores non-function
                     ; attributes. Methods are looked up in class__dict__.
                     (make-immutable-hash
                       (list (cons (quote <attr>) <attr>) ...))]
              (lambda (attr)
                (cond
                  ; Note the lookup order (first object, then class).
                  ; This is the same as Python.
                  [(hash-has-key? self__dict__ attr)])))
```
This is a little longer, but conceptually the only thing we’ve done is take our original object dictionary and split it into two, putting the data attributes in one and the methods in the other. However, this alone doesn’t solve (2). First, we introduce the function \texttt{fix-first},\textsuperscript{25} which is a higher-order function that takes a function \texttt{f} that takes \(n+1\) arguments, and a value \(x\), and returns a function \texttt{g} that takes \(n\) arguments, such that \(g(x_1, x_2, \ldots, x_n)\) is equivalent to \(f(x, x_1, x_2, \ldots, x_n)\). So then in the second branch of the \texttt{cond}, we can do \(\texttt{fix-first ??? (hash-ref class\_dict\_ attr)}\). But what goes in the ???? We need a way to refer to the object itself—and remember that the “object” is the entire \texttt{lambda} expression itself. In other words, this expression needs to be \textit{recursive}; we can do this by using \texttt{letrec}:

\begin{verbatim}
(define-syntax my-class
  (syntax-rules (method)
    [[(my-class <Class>
      ; (non-function) attributes
      (<attr> ...)
      
      ; methods
      (method (<method-name> <params> ...) <body>) ...)
      
      ; The template
      (begin
        ; This is a dictionary of the methods associated with a class.
        ; (Very much analogous to Point.__dict__ in Python.)
        (define class\_dict__
          (make-immutable-hash (list
            (cons (quote <method-name>)
              (lambda (<params> ...) <body>))
            ...)))
        (define (<Class> <attr> ...)
          ; We use letrec to give a name to the object we're returning.
          ; This is so that we can bind the object to the 'self'
          ; parameter in the call to fix-first.
          (letrec
            ([self\_dict__
              ; The object's __dict__ now only stores non-function
              ; attributes. Methods are looked up in class\_dict__.
              (make-immutable-hash
                (list (cons (quote <attr>) <attr>) ...))]
             [me (lambda (attr)
                (cond
                  [(hash-has-key? self\_dict__ attr)
                    [(hash-ref self\_dict__ attr)]
                    [(hash-has-key? class\_dict__ attr)
                      (hash-ref class\_dict__ attr)]
                    [else “Unrecognized message!”]))]))))))
\end{verbatim}

\textsuperscript{25}We leave the implementation of this function as an exercise.
The power of chained lookups

In the previous section, we used the idea of a lookup chain to separate instance attributes and methods. The idea of using multiple dictionaries to store different kinds of “attributes” is a versatile technique, and is commonly used in interpreters to implement different forms of inheritance. In Python, for example, since each class has its own \texttt{\_\_dict\_\_}, it is straightforward to implement method inheritance simply by inspecting the sequence of superclasses until the method is found.\footnote{This works even though Python supports multiple inheritance; the built-in \texttt{mro} function, standing for method resolution order, returns a sequence of the superclasses in the order they are checked for a given method.}

JavaScript dispenses with the notion of separate “classes”, and instead supports a chain of objects directly. In this language, almost every object keeps a reference to another object, called its prototype. When an attribute is accessed on an object, the object’s own “dictionary” is checked first; if the attribute isn’t found, the object’s prototype is checked, and then the object’s prototype’s prototype, and so on until the root “base object” (with no prototype) is reached. this form of inheritance is known as prototypal inheritance, distinguishing it from the class-based inheritance of Python and other languages. But fundamentally, both class-based and prototypal inheritance can be thought of a variations of the same idea of chained lookups!

Exercise Break!

Each of these exercises involves extending our basic class macro in interesting ways. Be warned that these are more challenging than the usual exercises.

2.10 Modify the \texttt{my-class} macro to add support for private attributes.

2.11 Modify the \texttt{my-class} macro to add support for some form of inheritance, as described in the previous section.

Manipulating control flow I: streams

In the first half of this chapter, we saw how to use macros to introduce new syntactic forms to reduce boilerplate code and create name bindings for classes. Hopefully this gave you a taste at the flexibility and power of macros: by changing the very syntax of our core language, we enable new paradigms, and hence
new idioms and techniques for crafting software. Now, we’ll start to look more explicitly at another common usage of macros: manipulating control flow to circumvent function eager evaluation order in Racket.

Our first foray into this area will be to implement streams, a “lazy” analogue of the familiar list data type. First, recall the recursive definition of a list:

- A list is either empty, or
- A value “cons’d” with another list.

In Racket, \texttt{cons} is a function, and so obeys eager evaluation. This means that all lists built up using \texttt{cons} are eagerly evaluated: all list elements are evaluated when the list is constructed.\textsuperscript{27} However, this is often not necessary: when we traverse a list, we often only care about accessing one element at a time, and do not need the other elements to have been evaluated at this point.

This motivates our implementation of a stream in Racket, which follows the following recursive definition:

- A stream is either empty, or
- A \texttt{thunk} wrapping a value “cons’d” with a \texttt{thunk} wrapping another stream.

As we discussed in the previous chapter, the \texttt{thunks} here are used to delay the evaluation of the elements of the stream until it is called. While we could implement this stream version of \texttt{cons} by explicitly passing \texttt{thunks} into the built-in \texttt{cons}, we again use macros to reduce some boilerplate:

\begin{verbatim}
; Empty stream value, and check for empty stream.
(define s-null 's-null)
(define (s-null? stream) (equal? stream 's-null))

#| (s-cons <first> <rest>)
<first>: A Racket expression.
<rest>: A stream (e.g., s-null or another s-cons expression).

Returns a new stream whose first value is <first>, and whose other items are the ones in <rest>. Unlike a regular list, both <first> and <rest> are wrapped in a thunk, so aren't evaluated until called.

|#
(define-syntax s-cons
 (syntax-rules ()
  [(s-cons <first> <rest>)
   (cons (thunk <first>) (thunk <rest>))])))

; These two define the stream-equivalents of "first" and "rest".
; We need to use `car` and `cdr` (similar to `first` and `rest`)
; for a technical reason that isn't important for this course.
; Note that s-rest both accesses the "rest thunk" and calls it,
\end{verbatim}
(define (s-first stream) ((car stream)))
(define (s-rest stream) ((cdr stream)))

(define-syntax make-stream
  (syntax-rules ()
    [(make-stream) s-null]
    [(make-stream <first> <rest> ...)
      (s-cons <first> (make-stream <rest> ...))])))

The beauty of this macro-based stream implementation is that its public interface is identical to built-in lists:

> (define s1 (s-cons 3 (s-cons 4 (s-cons 5 s-null))))
> (s-first s1)
3
> (s-first (s-rest s1))
4
> (s-first (s-rest (s-rest s1)))
5
> (s-null? (s-rest (s-rest (s-rest s1))))
#t

The difference, of course, is in the creation of these streams:

> (define items1 (list 1 2 (/ 1 0)))
ERROR /: division by zero
> (define items2 (make-stream 1 2 (/ 1 0)))
> (s-first items2)
1

Our use of thunks to wrap the arguments to s-cons may have looked familiar: basically what we did is create a “lazy” version of cons. But in Haskell, cons (being a function) is already lazy! So in fact, the list data type we’ve been using in Haskell is actually a stream!
In this section, we’ll take a brief look at one of the flashiest consequences of using streams to delay evaluation of the elements in a sequence: constructing and manipulating infinite sequences. While we could do this in Racket, we’ll use Haskell’s built-in list data types to underscore the fact that Haskell’s lists really are streams.

Here are some simple examples.

```haskell
myRepeat x = x : myRepeat x

> take 3 (myRepeat 6)
[6,6,6]

plus1 x = x + 1
nats = 0 : map plus1 nats

> take nats 5
[0,1,2,3,4]
```

What’s going on with that wild `nats` definition? To understand what’s going on, let’s look at a simpler example: `head nats`. Let’s carefully trace it using the central mechanic of pure functional programming: substitution.

1. In Haskell, `head` is defined through pattern-matching as `head (x:_)=x`.
2. `nats` is defined as `0 : map plus1 nats`. When we unify `nats` against this pattern, we get `x = 0`, and 0 is returned. Note that `head` completely ignores the rest of the list, and so we don’t need to look at the recursive part at all!

Now something a bit more complicated: `head (tail nats)`. Again, let’s substitute, keeping in mind the definition for `tail`, `tail (_:xs) = xs`.

```haskell
-- We can't call 'head' until we can pattern-match on (x:__).
-- To do this, we'll need to expand '(tail nats)' a bit.
head (tail (0 : map plus1 nats))

-- We can evaluate the inner part using pattern-matching on the
-- implementation of tail:
-- tail (_:xs) = xs
head (map plus1 nats)

-- Now we need the following definition of map:
-- map _ [ ] = []
-- map f (x:xs) = (f x) : (map f xs)
-- In order to pattern-match, we need to expand nats again.
head (map plus1 (0 : map plus1 nats))

-- This allows us to pattern-match: x = 0, and xs = map plus1 nats.
-- We do the substitution into the second rule, but remember:
```
-- we aren't evaluating the inner expressions!
head ((plus1 0) : (map plus1 (map plus1 nats)))

-- At this point, we can *finally* pattern-match against 'head'.
-- As before, we get to discard the complex "tail" of the argument.
plus1 0

-- Finally, we evaluate (plus1 0) directly to print the result.
1

So the first two elements are 0 and 1. The expansion we did suggests what would happen if we access further elements. The part that we discarded, (map plus1 (map plus1 nats)), adds one, twice. In subsequent recursive expansions of nats, the map plus1 function calls accumulate, leading to larger and larger numbers.

Since all of the common higher-order list functions work recursively, they apply equally well to infinite lists:

`squares = map (\x -> x * x) nats`
`evens = filter (\x -> x `mod` 2 == 0) nats`

> take 4 squares
[0,1,4,9]

We'll wrap up with a cute example of the Fibonacci numbers. Exercise: how does this work?

```
-- an example of zipWith, which combines two lists element-wise
zipWith (+) [1,2,3] [4,5,6] -- 4 10 18
fibs = 1 : 1 : zipWith (+) fibs (tail fibs)
```

**Exercise Break!**

2.12 Define an infinite list containing all negative numbers. Then, define an infinite list `ints` containing all integers such that `elem x ints` halts whenever `x` is an integer.

2.13 Define an infinite list containing all rational numbers.

2.14 (Joke) Define an infinite list containing all real numbers. Bonus: why is this funny?

2.15 Look up Pascal's Triangle. Represent this structure in Haskell.

2.16 Repeat your work in this section using the stream interface we developed in Racket (i.e., `s-cons, make-stream`, etc.).
Manipulating control flow II: the ambiguous operator -<

In the last section, we looked at a rather simple application of macros to create an implementation of streams, a lazy data structure used to decouple the creation of data from the consumption of that data. For the remainder of this chapter, we’ll explore this idea in a different context, one in which the data we create is specified by expressions using non-deterministic choice, and the consumption of that data is exposed explicitly through an impure function next. Our implementation of such expressions will introduce one new technical concept in programming language theory: the continuation, a representation of the control flow at a given point in the execution of a program.

Defining -< and next

Because you will likely be less familiar with this language feature, we will describe the functionality of the main two expressions before discussing their implementation. The main one is a macro called -< (pronounced “amb”, short for “ambiguous”), and it behaves as follows:

- -< takes an arbitrary number of argument expressions, representing the possible choices for the expression.
- If there are no arguments, it returns a special constant DONE.
- If there is at least one argument, it evaluates and returns the value of the first argument.
- In addition, if there is more than one argument, -< stores a “choice point” that (somehow) contains the remaining arguments so that they can be accessed, one at a time, by calling a separate function next.
- Once all of the arguments have been evaluated, subsequent calls to next return DONE.

Note that here we’re describing the denotational semantics of -< and next, but not their operational semantics! That is, before we get to any implementation at all, we want you to understand the expected behaviour of these two forms. Here is an example of how we would like to use them in tandem:

```
> (-< 1 2 (+ 3 4)) ; The expression first evaluates to 1.
1
> (next) ; Calls to (next) evaluate and return the other "choices".
2
> (next)
7
> (next) ; Here our constant DONE is represented as a symbol 'done.
'done
```
<- as a self-modifying stream

To get started, let’s write a skeleton definition for both <- and next. Our inspiration—which might be yours too, based on the previous example—is a self-modifying stream: a sequence of values that is accessed one at a time using next, but that actually uses mutation to keep track of which item is next.

The high-level description of our implementation is the following: we’ll use a private “stream-like” variable next-choice that is initialized by calling <- and is accessed and mutated by calling next:

```
#| A (private) global variable storing the result of choice expressions.
#| This should either be (void) or a *thunk* that both returns a value
#| and mutates `choices`.

You can think about this value as a self-mutating stream (loosely, when the "rest thunk" is called, rather than returning a lazy list, it stores it in 'choices').
|#

(define choices (void))

; A constant representing the state of having *no more choices*.
(define DONE 'done)

(define-syntax <-
  (syntax-rules ()
    [(-<)
      (begin
        (set! choices (void))
        DONE)]
    [(-< <expr1> <expr2> ...)
      (begin
        ; 1. Update `choices` to store a *thunk* that wraps the
        ;    remaining expressions. None of them are evaluated.
        (set! choices (thunk (-< <expr2> ...))))
        ; 2. Evaluate and return the first expression.
        (<expr1>)]))

(define (next)
  (if (void? choices)
      DONE
      (choices)))
```
Because we are (for the first time!) using mutation in our approach, our code looks a little different than all of the code we’ve written so far. In particular, we use the set! syntactic form to bind a new value to an existing name, and begin to enclose a sequence of expressions, evaluating each in turn and returning the value of the final one.31

We can verify that our macro works as intended on the example above:

```
> (-< 1 2 (+ 3 4))
1
> (next)
2
> (next)
7
> (next)
'done
```

Essentially what’s happening is the initial call to -< is returning the first argument (1) and storing the other expressions in a “stream” in choices. Calling next is like calling s-first on the “stream” to return the first value, but because of the -< macro, this also has the effect of replacing choices with something akin to (s-rest choices); i.e., permanently consuming the first element so that only the rest of the stream remains.

One particularly nice feature of this implementation is that because the set! expression is evaluated before <expr1> in the macro template, choices will update even if the next expression raises an error:

```
> (-< 1 2 (/ 1 0) 4)
1
> (next)
2
> (next)
ERROR /: division by zero
> (next)
4
```

This implementation is more complicated than it needs to be, if this were the limit of what we wanted to achieve. But we aren’t done yet! While this implementation is able to store the other choices, that’s all it saves.
When the first expression is evaluated, \((-\times 1 \, 2)\) returns 1 to the outer expression, which is why the output is 4. However, our implementation sets next to \((\text{thunk} \; (-\times 2 \; (+ \, 3 \, 4)))\), and so when it is called, it simply returns 2.

In other words, even though the remaining choices in the choice expression are saved, the computational context in which the expression occurs, is not. To fix this problem, we’ll need some way of saving this computational context—and this brings us to the realm of continuations.

Continuations

In our studies to date, we have been largely been passive participants in the operational semantics of our languages, subject to the evaluation orders of function calls and special syntactic forms. Even with macros, which do allow us to manipulate evaluation order, we have done so only by rewriting expressions into a fixed set of built-in macros and function calls. However, Racket has a data structure to directly represent (and hence manipulate) control flow from within a program: the continuation. Consider the following simple expression:

\[(+ \; (+ \; (* \; 3 \; 4) \; \text{first} \; (\text{list} \; 1 \; 2 \; 3)))\]

By now, you should have a very clear sense of how this expression is evaluated, and in particular the order in which the subexpressions are evaluated.\(^33\) For each subexpression \(s\), we say that its continuation is a representation of what remains to be evaluated after \(s\) itself has been evaluated. Put another way, the continuation of \(s\) is a representation of the control flow from immediately after \(s\) has been evaluated, up to the final evaluation of the enclosing top-level expression as.\(^34\)

For example, the continuation of the subexpression \((-\times 3 \, 4)\) is, in English, “evaluate \((\text{first} \; (\text{list} \; 1 \; 2 \; 3))\) and then add it to the result.” We represent this symbolically using an underscore to represent the “hole” in the remaining expression: \((+ \; _\; (\text{first} \; (\text{list} \; 1 \; 2 \; 3)))\).\(^35\) While it might seem like we can determine continuations simply by literally replacing a subexpression with an underscore, this isn’t exactly true. The continuation of the subexpression \((\text{first} \; (\text{list} \; 1 \; 2 \; 3))\) is \((+ \; 12 \; _\;)\); because of left-to-right evaluation order in function calls, the \((-\times 3 \, 4)\) is evaluated before the call to \(\text{first}\), and so the continuation of the latter call is simply “add 12 to the result.”

Since both identifiers and literal values are themselves expressions, they too have continuations. The continuation of \(4\) in the above expression is \((+ \; (* \; 3 \; _) \; (\text{first} \; (\text{list} \; 1 \; 2 \; 3)))\), and the continuation of \(\text{first}\) is \((+ \; 12 \; (_\; (\text{list} \; 1 \; 2 \; 3)))\). Finally, the continuation of the whole expression \((+ \; (* \; 3 \; 4) \; (\text{first} \; (\text{list} \; 1 \; 2 \; 3)))\) is simply “return the value,” which we represent simply as \(_\;\;\).
let/cc: reifying continuations

So far, continuations sound like a rather abstract representation of control flow. What’s quite spectacular about Racket is that it has continuations as a first-class data type; we say that Racket reifies continuations, meaning exposes continuations as values that a program and access and manipulate as easily as numbers and lists.

To reify continuations, we need to be able to access the current continuation at a point during execution. We as humans can read a whole expression and determine the continuations for each of its subexpressions (like we did in the previous section). How can we write a program that does this kind of meta-analysis for us?

Though it turns out to be quite possible to implement reified continuations purely in terms of functions, this is beyond the scope of the course. Instead, we will use Racket’s syntactic form let/cc to capture continuations for us. Here is the relevant syntax pattern:

1 (let/cc <id> <body>)

A let/cc expression has the following denotational semantics: the value of a let/cc expression is equal to the value of the body expression, when the latter is evaluated in the current environment, with one additional binding: <id> is bound to the continuation of the let/cc expression.

For example, here’s a slight modification to the previous example:

1 > (+ (* 3 (let/cc cont 4))
2   (first (list 1 2 3)))
3 13

As before, the continuation of the expression (let/cc cont 4) is (+ (* 3 _)
(first (list 1 2 3))), and this continuation is bound to the name cont. Note that the body of the let/cc, which is simply 4, gets evaluated and returned, producing the same result as before. Before moving on, make sure you understand the following output:

1 > (+ (* 3 (let/cc cont (* 5 20)))
2   (first (list 1 2 3)))
3 301

Of course, just binding the continuation to the name cont it is not very useful. Moreover, cont is local to the let/cc expression, so we cannot refer to it after the expression has been evaluated. We’ll use mutation to save the stored continuation. You might have some fun thinking about how to approach this and the rest of the chapter without using any mutation.

36 Those interested should look up “continuation-passing style.”  
37 You might have some fun thinking about how to approach this and the rest of the chapter without using any mutation.
expressions within the `let/cc`; as with `begin`, `let/cc` returns the value of its last body expression.

```scheme
(define global-cont (void))
(+ (* 3
   (let/cc cont
     (set! global-cont cont)
     4))
   (first (list 1 2 3)))

> global-cont
#<continuation>
```

Now that we have saved the continuation, we can call it, using the same syntax as function calls: `(cont <arg>)`, where `<arg>` is first evaluated then passed to the continuation.38

```scheme
> ; global-cont is (+ (* 3 _) (first (list 1 2 3)))
> (global-cont 4)
13
> (global-cont 100)
300
```

**Calling continuations vs. calling functions**

On the syntax level, calling a continuation cannot be distinguished from calling a function, and yet these two operations are very different. Here is a simple example to distinguish the difference:

```scheme
(define global-cont (void))
(+ (* 3
   (let/cc cont
     (set! global-cont cont)
     4))
   (first (list 1 2 3)))

(define f
  (lambda (x) (+ (* 3 x) (first (list 1 2 3)))))
```

It might seem like `global-cont` and `f` do the same thing:

```scheme
> (f 4)
13
```
But they don’t!

> (+ 1 (f 4)) ; (f 4) returns 13 to the continuation (+ 1 _)
14

> (+ 1 (global-cont 4)) ; global-cont *discards* the (+ 1 _)
13

The function \( f \) exists within the larger control flow of evaluating the expression
\( (+ 1 (f 4)) \): the continuation of \( (f 4) \) is “add 1 to the result”. The continuation
of \( (global-cont 4) \) is exactly the same, but calling a continuation discards the
enclosing continuation, and replaces it with the continuation being called.\(^{39}\) To
drive the point home, we can see that continuations even interrupt Racket’s
normal eager evaluation control flow:

> (+ (global-cont 4) (/ 1 0)) ; The (/ 1 0) is never evaluated!
13

Back to \(<\<\)

Recall that our motivation for learning about continuations was that our current
\(<\<\) macro implementation stored the choices, but not the computational context
around each choice. We now have a name for that computational context: it’s
just the continuation of the \(<\<\) expression! Let’s see how to improve our macro
using let/cc. Here’s a first attempt, replacing the begin with a let/cc:

```scheme
(define-syntax -<
  (syntax-rules ()
    [(-<)
      (begin
        (set! choices (void))
        DONE)]
    [(-< <expr1> <expr2> ...)
      ; cont is bound to the continuation of the (-< ...> expression
      (let/cc cont
        (set! choices (thunk (-< <expr2> ...))))
        <expr1>)]))
```
For example, in the example `(+ 1 (⟨ 1 2 3 ⟩))`, `cont` would be bound to `(+ 1 _)`.

Of course, our macro is incomplete, as it doesn’t actually “save” `cont`. Doing this is straight-forward: inside our saved thunk, rather than simply returning the next choice (with `(⟨ expr2 ⟩ ...)`), we call `cont` on the result!

```scheme
(define-syntax -<
  (syntax-rules ()
    [⟨⟩
      (begin
        (set! choices (void))
        DONE)]
    [⟨ expr1⟩ ⟨ expr2 ⟩ ...
      ; cont is bound to the continuation of the ⟨ ⟨ ⟩ ⟩ expression
      (let/cc cont
        (set! choices (thunk (cont ⟨ ⟨ expr2 ⟩ ⟩))))
        ⟨ expr1⟩)])
```

And now we get our desired behaviour:

```scheme
> (+ 1 (⟨ 10 20 ⟩)) ; cont is (+ 1 _)
11
> (next) ; choices is, abstractly, (thunk ((+ 1 _) ⟨ ⟨ 20 ⟩ ⟩)))
21
> (next)
'done
```

It is rather amazing that such a small change unlocks a huge number of possibilities for our macro; this truly illustrates the power of continuations and macros in Racket.

**Branching choices**

Even with the improvement of the previous section, we aren’t quite done yet. Our overwriting of `choices` means that at any time, our program is only aware of one choice point. So in the example below, evaluating the second choice clobbers the first:

```scheme
> (+ (⟨ 10 20 ⟩) (⟨ 2 3 ⟩))
12
> (next)
13
> (next)
'done
```
Now, as program language designers we might stop here and say, “Well, too bad! You can only use one `<expression per program.” But this is rather unsatisfying for our users, and so let’s work a little harder to support multiple choice points. Obviously, just adding more global variables (choices1, choices2, etc.) isn’t an option. We’ll need to be able to represent an arbitrary number of choice points, and so a collection-type data structure is most appropriate. For simplicity, we’ll use a straightforward one: a stack implemented with a list, with the front of the list representing the top of the stack.

Before we modify our macro, we introduce a new global variable choices to represent the stack of choices, as well as two helper functions to emulate stack behaviour.

<table>
<thead>
<tr>
<th>Line</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>; The stack of choice points, represented as a list.</td>
</tr>
<tr>
<td>2</td>
<td>(define choices null)</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>; Generic push and pop (mutating) functions</td>
</tr>
<tr>
<td>5</td>
<td>(define-syntax push!</td>
</tr>
<tr>
<td>6</td>
<td>(syntax-rules ()</td>
</tr>
<tr>
<td>7</td>
<td>[(push! &lt;id&gt; obj)</td>
</tr>
<tr>
<td>8</td>
<td>(set! &lt;id&gt;</td>
</tr>
<tr>
<td>9</td>
<td>(cons obj &lt;id&gt;))])</td>
</tr>
<tr>
<td>10</td>
<td>)</td>
</tr>
<tr>
<td>11</td>
<td>(define-syntax pop!</td>
</tr>
<tr>
<td>12</td>
<td>(syntax-rules ()</td>
</tr>
<tr>
<td>13</td>
<td>[(pop! &lt;id&gt;)</td>
</tr>
<tr>
<td>14</td>
<td>(let ([obj (first &lt;id&gt;)]</td>
</tr>
<tr>
<td>15</td>
<td>(set! &lt;id&gt; (rest &lt;id&gt;))</td>
</tr>
<tr>
<td>16</td>
<td>obj)])</td>
</tr>
<tr>
<td>17</td>
<td>)</td>
</tr>
<tr>
<td>18</td>
<td>; Aliases for use in `&lt; and next.</td>
</tr>
<tr>
<td>19</td>
<td>(define (add-choice obj)</td>
</tr>
<tr>
<td>20</td>
<td>(push! choices-stack obj))</td>
</tr>
<tr>
<td>21</td>
<td>)</td>
</tr>
<tr>
<td>22</td>
<td>(define (get-choice)</td>
</tr>
<tr>
<td>23</td>
<td>(pop! choices-stack))</td>
</tr>
<tr>
<td>24</td>
<td></td>
</tr>
</tbody>
</table>
| 25   | The idea is pretty straight-forward: every time we encounter a choice expression, we push the corresponding thunk onto the choices stack, and every time we call next, we pop off a thunk off the stack and call it." | 41This approach is the same as a stack-based implementation of depth-first search—this is not a coincidence.
(define-synta< ~<
 (syntax-rules ()
 ; If there are zero or one choice, no need to update `choices`.
 [(~<) DONE]
 [(~< <expr1>) <expr1>]]

 ; If there are two or more values, return the first one and
 ; store the others in a thunk,
 ; but *also stores the continuation of the -< expression*.
 [(~< <expr1> <expr2> ...)
  (let/cc cont
   ; 1. Store the current continuation and the remaining choices.
   (add-choice (thunk (cont (~< <expr2> ...)))))
   ; 2. Evaluate and return the first expression.
   (<expr1>))]]

> (+ (~< 1 2) (~< 3 10))
4
> choices
'(#:procedure #:procedure)

After evaluating the expression containing two choice expressions, we see that
choices now stores two thunks. To access them, we need to make the following
modification to next:

(define next)
(if (null? choices)
  DONE
  ((get-choice))))

And then we have the desired behaviour:

> (+ (~< 1 2) (~< 3 10))
4
> (next)
11
> (next)
5
> (next)
12
> (next)
'done
An important detail

Even though it is gratifying that the implementation yields the correct result, it should be at least somewhat surprising that this works so well. In particular, let us trace through that previous example again, but inspect the contents of choices each time:

```
> choices
'(())
> (+ (-< 1 2) (-< 3 10))
4
> choices
'(#<procedure> #<procedure>)
```

As before, we know that the first element contains the choice (-< 10), while the second contains (-< 2).

```
> (next)
11
> choices
'(#<procedure>)
```

After the first call to (next), the top choice is popped and evaluated, yielding 10. But why is 11 output? The captured continuation, of course: (+ 1 _)—note that by the time the second choice expression has been evaluated, the first was already evaluated to 1.

Only the single choice (-< 2) remains on the stack; because it contains just one argument, the natural thing to expect when calling next again is for this choice to be popped from the stack, leaving no choices. However, this is not what happens:

```
> (next)
5
> choices
'(#<procedure>)
```

Why is a 5 output, and why is there still a function on choices? The answer to both of these questions lies in the continuation of the first expression (-< 1 2): (+ _ (-< 3 10))). That’s right, the continuation of the first choice expression contains the second one! This is why a 5 is output; the choice (-< 2) is evaluated and passed to the continuation, which then evaluates the second choice expression (-< 3 10), which both returns a 3 and pushes a new thunk onto the stack.

Note that the new function on the stack has the same choice as earlier, (-< 10),
but its continuation is different: “add 2” rather than “add 1.” This is what causes the final output:

```lisp
> (next)
12
> choices
'(())
> (next)
done
```

**Towards declarative programming**

Even though we hope you find our work for the choice macro intellectually stimulating in its own right, you should know that this truly is a powerful mechanism that can change the way you write programs. Just as how in our discussion of streams we saw how thunks could be used to decouple the production and consumption of linear data, our choice library allows this decoupling to occur even when how we produce data is non-linear.

So far in this course, we have contrasted functional programming with imperative programming, with one of the main points of contrast being writing code that is more descriptive of the computation you want the computer to perform, rather than describing how to do it—that is, writing programs that are more declarative in nature.\(^4\)

For example, here is a simple expression which generates all possible binary strings of length 5, following the English expression “take five characters, each of which is a 0 or 1”:\(^4\)

```
> (string (-< #\0 #\1)
   (-< #\0 #\1)
   (-< #\0 #\1)
   (-< #\0 #\1)
   (-< #\0 #\1))
"00000"
> (next)
"00001"
> (next)
"00010"
> (next)
"00011"
> (next)
"00100"
```

This easy composability is quite remarkable, and is arguably as close to a human English description as possible.\(^4\) Though we can achieve analogous behaviour

---

\(^4\) In this course, we take the view of declarative as being a spectrum rather than a yes/no binary characteristic. So we won’t say that a language or style is declarative, but rather that it’s more declarative than what we’re used to.

\(^4\) Racket uses the #\ to indicate a character literal.
with plain higher-order list functions (or nested loops), the simplicity of this approach is quite beautiful.

---

**Exercise Break!**

2.17 Suppose we tried to generate all binary strings of length 5 by doing the following:

```scheme
(let* ([char (\#\0 \#\1)])
  (string char char char char char))
```

What does this generate? Explain why. Fix this problem using thunks.

---

**Predicates and backtracking**

Generating “all possible combinations” of choices is fun and impressive, but not necessarily that useful without some kind of mechanism to filter out unwanted combinations. While we could do this by collecting choices into a stream and applying a traditional filter operation, we’ll instead take a different approach to combine the automatic generation that choice expressions give us with automatic backtracking, which is at the core the technique used by logic programming languages like Prolog. The logic programming paradigm is centred on one simple question: “is this true?” Programs are not a sequence of steps to perform, nor the combination of functions; instead, a logic program consists of the definition of a search space (the universe of values known to the program) and the statement of queries for the computer to answer based on these values. Based on this description, it is probably not hard to see that logic programs are generally more declarative than either imperative or functional programs, given that they fundamentally specify what is being queried, but leave it up to the underlying implementation of the language to determine how to answer the query.45

Now, we already have a mechanism for defining a search space, using combinations of choices. What about queries? To start, we will define a query operator `?-`, which takes a unary predicate and an expression, and returns the value of the expression if it satisfies the predicate, and `DONE` if it doesn’t.

```scheme
(define (?- pred expr)
  (if (pred expr)
      expr
      DONE))
```

> (?- even? 10)
10
> (?- even? -3)
'done

45 A more familiar instance of this idea is the database query language SQL, in which programmers write queries specifying what data they want, without worrying (too much) about how this data will be retrieved.
Now, all of the work we did in the previous sections now pays off: we can call this function with a choice expression!

```
> (?- even? (\< 1 2 3 4))
'done
> (next)
2
> (next)
'done
> (next)
'done
```

When the ?- is originally called, it gets passed the first value in the choice expression (i.e., 1); but the continuation of the choice expression is (?- even? \_), and so each time next is called, the next choice gets passed to the query. To use some terminology of artificial intelligence, calling next causes backtracking, in which the program goes back to a previous point in its execution, and makes a different choice.

When making queries with choices, it is often the case that we’ll only care about the choices which succeed (i.e., satisfy the predicate), and ignore the others. Therefore it would be nice not to see the intermediate ‘done outputs; we can make one simple change to our predicate to have this available:

```
(define (?- pred expr)
  (if (pred expr)
      expr
      ; Rather than giving up, just try the next choice!
      (next))))
```

```
> (?- even? (\< 1 2 3 4))
2
> (next)
4
> (next)
'done
```

Now, every time the (pred expr) test fails (next) is called, which calls ?- on the next choice; in other words, we automatically backtrack on failures.

**Search examples**

Probably the easiest way to appreciate the power that this combination of choice and querying yields is to see some examples in action. We have already seen the use of this automated backtracking as a lazy filter operation, producing values in a list of arguments which satisfy a predicate one at a time, rather than
all at once. We can extend this fairly easily to passing in multiple choice points, with the one downside being that our ?- function only takes unary predicates, and so we’ll need to pass all of the choices into a list of arguments.

Our problem will be the following: given a number \(n\), determine all triples of numbers \((a, b, c)\) whose product is \(n\). Here is our definition of the input data; unlike the past examples, we won’t walk through the code with you, so we strongly recommend pausing here to study the code carefully!

```
1 #| 2 (num-between start end) 3 | start, end: integers. Precondition: start <= end 4 | Returns a choice of a number in the range [start .. end], inclusive. 5 |# 6 (define (num-between start end) 7 | (if (equal? start end) 8 | (-< end) 9 | (-< start (num-between (+ 1 start) end))))
10 (define (triples n) 11 | (list (num-between 1 n) 12 | (num-between 1 n) 13 | (num-between 1 n)))
```

And here is our predicate, expressed using a higher-order function:

```
1 (define (product? n) 2 | (lambda (triple) 3 | (equal? n (apply * triple))))
```

Finally, here is a use of the query on the number 12:

```
1 > (?- (product? 12) (triples 12)) 2 '1 1 12 3 > (next) 4 '1 2 6 5 > (next) 6 '1 3 4 7 ... 8 > (next) 9 '6 2 1 10 > (next) 11 '12 1 1 12 > (next) 13 'done
```
See that? In just a few lines of code, we expressed a computation for finding factorizations 12 in a purely declarative way: specifying what we wanted to find, rather than how to find the solutions. This is the essence of pure logic programming: we take a problem and break it down into logical constraints on certain values, give our program these constraints, and let it find the solutions. This should almost feel like cheating—isn’t the program doing all of the hard work?

Remember that we are studying not just different programming paradigms, but also trying to understand what it means to design a language. We spent a fair amount of time implementing the language features `<`, `next`, and `?->`, expressly for the purpose of making it easier for users of our augmented Racket to write code. So here we’re putting on our user hats, and take advantage of the designers’ work!

**Satisfiability**

One of the most famous problems in computer science is a distilled essence of the above approach: the *satisfiability problem*, in which we are given a propositional boolean formula, and asked whether or not we can make that formula true. Here is an example of such a formula, written in the notation of Racket:

\[
\begin{align*}
1 & \text{and} (\text{or} \ x1 (\text{not} \ x2) \ x4) \\
2 & \quad (\text{or} \ x2 \ x3 \ x4) \\
3 & \quad (\text{or} (\text{not} \ x2) (\text{not} \ x3) (\text{not} \ x4)) \\
4 & \quad (\text{or} (\text{not} \ x1) \ x3 (\text{not} \ x4)) \\
5 & \quad (\text{or} \ x1 \ x2 \ x3) \\
6 & \quad (\text{or} \ x1 (\text{not} \ x2) (\text{not} \ x4))
\end{align*}
\]

With our current system, it is easy to find solutions:

\[
\begin{align*}
1 & \text{define (sat lst)} \\
2 & \quad (\text{let ([x1 (first lst)]} \\
3 & \quad \quad [x2 (second lst)] \\
4 & \quad \quad [x3 (third lst)] \\
5 & \quad \quad [x4 (fourth lst)])) \\
6 & \quad (\text{and} (\text{or} \ x1 (\text{not} \ x2) \ x4) \\
7 & \quad \quad (\text{or} \ x2 \ x3 \ x4) \\
8 & \quad \quad (\text{or} (\text{not} \ x2) (\text{not} \ x3) (\text{not} \ x4)) \\
9 & \quad \quad (\text{or} (\text{not} \ x1) \ x3 (\text{not} \ x4)) \\
10 & \quad \quad (\text{or} \ x1 \ x2 \ x3) \\
11 & \quad \quad (\text{or} \ x1 (\text{not} \ x2) (\text{not} \ x4))))
\end{align*}
\]

> (?- sat (list (-< #t #f) \\
13 & \quad (\text{let ([x1 (first lst)]} \\
14 & \quad \quad [x2 (second lst)] \\
15 & \quad \quad [x3 (third lst)] \\
16 & \quad \quad [x4 (fourth lst)])) \\
17 & \quad (\text{and} (\text{or} \ x1 (\text{not} \ x2) \ x4) \\
18 & \quad \quad (\text{or} \ x2 \ x3 \ x4) \\
19 & \quad \quad (\text{or} (\text{not} \ x2) (\text{not} \ x3) (\text{not} \ x4)) \\
20 & \quad \quad (\text{or} (\text{not} \ x1) \ x3 (\text{not} \ x4)) \\
21 & \quad \quad (\text{or} \ x1 \ x2 \ x3) \\
22 & \quad \quad (\text{or} \ x1 (\text{not} \ x2) (\text{not} \ x4))))
'(#t #t #t #f)
> (next)
'(#t #t #f #f)
> (next)
'(#t #f #t #t)
> (next)
'(#t #f #t #f)
> (next)
'(#f #f #t #t)
> (next)
'(#f #f #t #f)
> (next)
'done
3 Type systems

I conclude that there are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies and the other way is to make it so complicated that there are no obvious deficiencies.

Tony Hoare

Here is a problematic Racket function.

```racket
(define (f x)
  (+ x (first x)))
```

Trying to run this function on any input will fail! This is a type error: the parameter `x` cannot simultaneously be a number and a list. Our goal for this chapter is to study the different type systems that are available in different languages. While you certainly have an intuitive notion of types coming from our past programming experience, you may not yet have appreciated the complex design decisions that go into how types are represented and used in a programming language, and how differences in these decisions can have major impacts on the programs written in these languages.

We start with a few definitions to make sure we’re on the same page. A type is a set of values together with a set of behaviours on those values. Note that this is a pretty abstract notion that differs among programming languages: an integer in Racket can take on a different set of values than an int in C, and the (built-in) operations on String values in Haskell are different from the String methods in Java. What is common to all of these languages is the notion of what information a type conveys: if we say that an expression `E` has type “integer”, this constrains both the possible values we expect that expression to have (i.e., what its denotational meaning is), and how we can use this expression in the context of a large program (e.g., `(+ 1 E)` is okay but `E[10]` is not). A type system is the set of rules in a programming language governing the semantics of types in the language. This includes how types are defined, the syntax rules governing where and how types can be written, and how types affect the operational and denotational semantics of the language.

1 Try adding two very large integers in each language!
Describing type systems

Our plan for this chapter is to focus mainly on Haskell’s type system, though we’ll make frequent comparisons to other languages. To begin, we’ll look at two general spectrums on which a programming language’s type system can reside; these are some of the most commonly-cited characteristics of a type system.

Strong and weak typing

One of the more confused notions is the difference between strong/weak typing and static/dynamic typing. Let’s start with strong/weak typing. However, please keep in mind that there is some disagreement about the exact meaning of strong/weak typing, and we present only one such interpretation here.

In a **strongly-typed** language, every value has a fixed type at every point during the execution of a program.\(^2\) Even though this might seem fairly obvious, keep in mind that this property isn’t strictly necessary: all data is stored in memory as 0’s and 1’s, and it is possible that the types are not stored at all!

Most modern languages are strongly-typed, which is why we get type errors when we try to call a function on arguments of the wrong type. On the other hand, a **weakly-typed** language has no such guarantees: values can be implicitly interpreted as having a completely different type at runtime than what was originally intended, in a phenomenon known as **type coercion**. For example, in many languages the expression "5" + 6 is semantically valid, raising no runtime errors. Many languages will convert the integer 6 into the string "6", and concatenate the strings to obtain the string "56".\(^3\) A Java feature called **auto-boxing** allows primitive values to be coerced into their corresponding wrapper classes (e.g., int to Integer) during runtime.

The previous examples of type coercion might be surprising, but aren’t necessarily safety concerns. In C, the situation is quite different:\(^4\)

```
#include <stdio.h>

int main(void) {
    printf("%d", 1);
}
```

Depending on who you ask, different levels of type coercion might warrant the title of strong or weak typing. For example, basically every programming language successfully adds \(3.5 + 4\) without raising a type error. So as with many properties of languages, the takeaway message is that strong/weak typing is not a binary property, but a set of nuanced rules about what types can be coerced and when. It goes without saying that mastering these rules is important for facility in any programming language.
Static and dynamic typing

Even assuming that we are in a very strongly-typed language with little or no type coercion, there is still the fundamental question of “when does the program know about (and check) types?” The answer to this question is one of the key characteristics of a programming language, as it deeply influences both the language’s semantics and the implementation of any interpreter or compiler for that language.

In statically-typed languages, the type of every expression is determined directly from the source code, before any code is actually executed. In languages that permit mutation, static typing requires that even as variables change values, they can only change to values within the same type. Variables typically have a fixed type; even in languages that permit mutation, variables cannot be reassigned to values of a different type than the one they started with. Most of these languages require an explicit type to be specified at the declaration of each variable, although as we’ll discuss later, this is not an absolute requirement for static typing.

In contrast, dynamically-typed languages do not perform any type-checking until the program is run. In such languages, a type error is a runtime error—in the same category as division by zero or array out-of-bounds errors.

As we mentioned earlier, types really form a collection of constraints on programs, so it shouldn’t come as a surprise that static typing is really a form of program verification (like proving program correctness from CSC236). If a compiler can check that nowhere in the code does a variable take on a different type, we are guaranteed that at no point in the program’s runtime does that variable take on a different type. By checking the types of all expressions, variables, and functions, the compiler is able to detect code errors (such as invalid function calls) before any code is run at all. Moreover, static type analysis can be sometimes used as a compiler optimization: if we have compile-time guarantees about the types of all expressions during the program’s run, we can then drop the runtime type checks when functions are called. As we’ll discuss in the next section, statically-typed languages perform type checking before programs are run, essentially rejecting certain programs because they have type errors. This sounds great, and it very often is. However, type systems gain their power by enforcing type constraints; the stricter the constraints, the more erroneous programs can be rejected. This can also have the consequence of rejecting perfectly correct programs\(^5\) that do not satisfy these constraints. However, it’s also worth pointing out that most statically-typed languages have a way of “escaping” the type system with a universal type like “Object” or “Any”, and in this way allow one to write code without any type constraints.

\(^5\) We’ll see a few examples of this when we study Haskell’s type system in the next section.
The basics of Haskell’s type system

In Haskell, types are denoted by capitalized names like `Bool`, `Char`, and `Integer`. In the Haskell interpreter, you can see the type of any expression with `:type`, or `:`t for short. Some examples follow; since Haskell’s numeric types are a bit complex, we’re deferring them until later.

```
1 > :t True
2 True :: Bool
3 > :t 'a'
4 'a' :: Char
5 > :t "Hello"
6 "Hello" :: [Char]
```

Note that the square brackets surrounding the `Char` indicates a list: like C, Haskell interprets strings simply as a list of characters. One important difference between Haskell and Racket is that lists must contain values of the same type, so the expression `[True, 'a']` is rejected by Haskell. This restriction also applies to `if` expressions: both the `then` and `else` subexpressions must have the same type:

```
1 > :t (if 3 > 1 then "hi" else "bye")
2 (if 3 > 1 then "hi" else "bye") :: [Char]
```

The above example illustrates the static nature of Haskell’s type-checking: the type of the entire `if` expression is given, without evaluating it! Haskell can do this precisely because of the aforementioned restriction: because the Haskell compiler knows that both branches evaluate to strings, it is able to determine that the type of the overall expression is a string *without* knowing which branch would actually be evaluated!

Function types and currying

In Haskell, all functions have a type that can be inspected in the same way.

```
1 > :t not
2 not :: Bool -> Bool
```

The type signature `Bool -> Bool` means that `not` is a function that takes as input a `Bool` value, and then outputs a `Bool` value. The very existence of a type signature for a function has huge implications for how functions can be used. For instance, functions in Haskell must have a fixed number of arguments, fixed types for all their arguments, and one fixed return type. Contrast this with Racket’s `member` function, which returns either a list or `#f` (a boolean).
So far, we’ve only considered functions that take in a single parameter. What about functions that take in two parameters? For example, what is the type of the “and” function (&&)?

```
1 > :t (&&)
2 (&&) :: Bool -> Bool -> Bool
```

Huh, that’s weird. One probably would have expected something like (Bool, Bool) -> Bool to denote that this function takes in two arguments. So what does Bool -> Bool -> Bool actually mean? The -> operator in Haskell is right-associative, which means that the proper grouping of that signature is Bool -> (Bool -> Bool). This is quite suggestive: Haskell interprets (&&) as a unary higher-order function that returns a function with type Bool -> Bool! This is the great insight from the lambda calculus known as currying: any multi-parameter function can be broken down into a nested composition of single-valued functions.

Consider the following function definition, which you can think of as a partial application of (&&):

```
1 newF y = (&&) True y
```

The type of newF is certainly Bool -> Bool: if y is True then the function evaluates to True, and if y is False then the expression is False. By fixing the value of the first argument to (&&), we have created a new function. But in fact the y is completely unnecessary to defining newF, and the following definition is completely equivalent.

```
1 newF = (&&) True
```

This makes sense when we think in terms of referential transparency: every time we see an expression like newF True, we can substitute this definition of newF to obtain the completely equivalent expression (&&) True True.

It turns out that Haskell treats all functions as unary functions, and that function application is indicated by simply separating two expressions with a space. An expression like (&&) False True might look like a single function application on two arguments, but the true syntactic form parsed by Haskell is ((&&) False) True, where (&&) False is evaluated to create a new function, which is then applied to True. In other words, the following definitions of (&&) are equivalent:

```
1 -- Usual definition
2 (&&) x y = if x then y else False
3 -- As a higher-order function
4 (&&) x = \y -> if x then y else False
```
These two definitions are *completely equivalent* in Haskell, even though they are not in most other programming languages! Going a step further, the following ternary function can be defined in Haskell in four completely equivalent ways:

```haskell
1 f x y z = x + y * z
2 f x y = \z -> x + y * z
3 f x = \y -> (\z -> x + y * z) -- parentheses shown for clarity
4 f = \x ->
5   (\y -> (\z -> x + y * z))
```

Because binary operators are primarily meant to use infix, Haskell provides a special syntax for currying infix operators called **sectioning**. Let’s return to `(&&)` as an example of this syntax.

```haskell
1 f = (True &&) -- equivalent to f x = True && x
2 g = (&& True) -- equivalent to g x = x && True
```

Note that unlike standard currying, in which partial application must be done with a left-to-right argument order, sectioning allows us to partially apply an operator by fixing either the first or the second argument.

In practice, partial application is a powerful tool for enabling great flexibility in combining and creating functions. Consider the function `addToAll`, which adds a number to every item in a list.

```haskell
addToAll n lst = map (\x -> x + n) lst
```

This function is already quite concise because of our use of `map`, but using what we’ve learned in this section, we can given an even more elegant definition.

First, rather than creating a new anonymous function `(\x -> x + n)`, we can use sectioning to express an “add n” function directly:

```haskell
addToAll2 n lst = map (+ n) lst
```

Taking advantage of automatic currying, we can remove `lst` from both sides, leaving just

```haskell
addToAll3 n = map (+ n)
```

It might seem almost magical to define this function in such a short fashion. As you get more comfortable with functional programming and thinking in terms of combining and creating functions, you can interpret the final definition quite elegantly as “`addToAll3` maps the ‘add n’ function.”
Defining types in Haskell

Now that we have been introduced to Haskell’s type system, we’ll study how to define our own types in this language.

The running example we’ll use in this section is a set of types representing simple geometric objects. There’s quite a bit of new terminology in this section, so make sure you read carefully! First, a point \((x, y)\) in the Cartesian plane can be represented by the following Point type.

```haskell
data Point = Point Float Float
```

You can probably guess what this does: on the left side, we are defining a new type `Point` using the `data` keyword. On the right, we define how to create instances of this type; the `Point` is a `value constructor`, which takes two `Float` arguments and returns a value of type `Point`. Let’s play around with this new type.

```haskell
> let p = Point 3 4
> :t p
p :: Point
> :t Point
Point :: Float -> Float -> Point
```

There is a common source of confusion when it comes to Haskell’s treatment of types that we are going to confront head-on. In Haskell, there is a strict separation between the `expression language` used to define names and evaluate expressions, and the `type language` used to represent the types of these expressions. We have seen some artifacts of this separation already: for example, the 
`-` operator is used in expressions as part of the syntax for a lambda expression, and in types to represent a function type. We have also seen the mingling of these in type annotations, which come in the form `<expression> :: <type>.

A type declaration is a different instance of this mingling. In our above example, the `Point` on the left is the name of a new `type`, while the `Point` on the right is the name of a `function` that returns values of that type. In typical object-oriented languages, the `constructor` of a class is a special method, and its significance is enforced semantically (in part) by having its name be the same as the class name. This is actually not true in Haskell, as we illustrate below:

```haskell
data Point = MyPoint Float Float
```

```haskell
> let p = MyPoint 3 4
> :t p
p :: Point
> :t MyPoint
MyPoint :: Float -> Float -> Point
```
Operating on points

This type definition is much more in the spirit of C structs than classes in the familiar object-oriented setting, in that this allows us to define members of the type (in the case of points, two floats), but not to bundle operations on these types. In Haskell, we provide such operations by defining top-level functions, but naively doing so almost immediately poses a problem:

```
-- Compute the distance between two points
distance :: Point -> Point -> Float
distance p1 p2 = ???
```

The problem is clear: how can we access the Float attributes of the point values? Just as we define functions by pattern matching on primitive values and lists, we can also pattern match on any value constructor.9

```
distance :: Point -> Point -> Float
distance (Point x1 y1) (Point x2 y2) =
  let dx = abs (x1 - x2)
      dy = abs (y1 - y2)
  in
      sqrt (dx*dx + dy*dy)
```

In another testament to the beauty of Haskell’s syntax, the left side of this function definition perfectly matches how we would call the function:

```
> distance (Point 3 4) (Point 1 2)
```

Here’s another example where we use the value constructor to create new Point values.

```
-- Take a list of x values, y values, and return a list of Points
makePoints :: [Float] -> [Float] -> [Point]
makePoints xs ys = zipWith (\x y -> Point x y) xs ys

-- Or better:
makePoints2 = zipWith (\x y -> Point x y)
```

Even though `makePoints2` looks quite simple already, it turns out there’s an even simpler representation using the fact that the `Point` value constructor is just another function! So in fact, the most elegant way of writing this function is simply:

```
makePoints3 = zipWith Point
```
Unions

The previous section introduced struct-based types in Haskell, in which a constructor like Point is used essentially as a mechanism to group pieces of data into one value. In this section, we’ll look at how Haskell supports union types, starting with the most basic form that you may be familiar with already: enumerations.

Consider a type used to represent a day of the week: Monday, Tuesday, etc. While we could use either a string or integer to represent such a type, both of these primitive types have the drawback that they are more expressive than required, and so checks for “invalid” days could not be caught by the type system. Instead, since there are a small and finite number of possible days, we can create an enumeration type by explicitly giving a list of possible constructors:

```haskell
data Day = Monday --
           | Tuesday --
           | Wednesday --
           | Thursday --
           | Friday --
           | Saturday --
           | Sunday --
```

Here, the vertical bar `|` should be read as an “or”, separating the different possible value constructors for Day. Note that Monday, Tuesday, etc. are indeed value constructors, same as Point in the previous section, except that they do not take in any arguments:\(^{10}\)

```haskell
> :t Monday
Monday :: Day
```

This use of alternative constructors in Haskell is pretty standard in other languages as well. What is more unusual is Haskell’s ability to combine constructors that take arguments with alternatives to declare types that have radically different forms:

```haskell
data Shape = Circle Point Float -- centre and radius
           | Rectangle Point Point -- top-left and bottom-right corners
```

Here it is really important to get the terminology correct. This code creates a new type Shape, which has two value constructors: Circle and Rectangle. Here’s how we might use this new type:\(^{11}\)

---

\(^{10}\) Due to Haskell’s laziness, we can view these constructors either as nullary functions, or simply as singleton values.

\(^{11}\) Remember: Circle and Rectangle are functions, while Shape is a type.
Here’s another example of using pattern-matching to write a function operating on Shapes. Note that each constructor gets its own pattern-match rule, and we can even nest patterns.

```
area :: Shape -> Float
area (Circle _ r) = pi * r * r
area (Rectangle (Point x1 y1) (Point x2 y2)) =
  abs ((x2-x1) * (y2-y1))
```

```
> area (Circle (Point 3 4) 1)
3.141592653589793
> area (Rectangle (Point 4 5) (Point 6 2))
6.0
```

### Unions in C

You might be familiar with `struct` and `union` types in C, which together approximate algebraic data types from Haskell. That said, the C syntax is quite a bit more verbose. And for unions, there is additional legwork required of programmers to keep track of what part of the union is active. Here is how our Shape example might look in C. Note that the `circle` and `rectangle` members will overlap in memory: only one of them is available at a time.

```
#include <stdio.h>

struct Point {
  float x, y;
};

union Shape {
  struct Circle {
    struct Point centre;
    float radius; } circle;
  struct Rectangle {
    struct Point corner1, corner2; } rectangle;
};

int main(void) {
  union Shape s;
```
Now, what happens if we want to pass a union `Shape` to a function?

```c
void area(union Shape s) {
    if (... s is a circle)
        ... operate on the circle
    else
        ... operate on the rectangle
}
```

How do we know whether `s` is a circle or a rectangle? We don’t! And, unlike in Haskell, we certainly can’t pattern-match to find out. The C solution is for the programmer to explicitly maintain a *tag field* that indicates what the union is currently storing.\(^\text{12}\) Unfortunately, this requires that we embed the union in a struct that houses both the tag field and the union – compare to Haskell, which keeps track of all of this for us!

```
#include <stdio.h>
#include <math.h>
#define pi 3.141592653589793

struct Point {
    float x, y;
};

struct Shape {
    enum {CIRCLE, RECTANGLE} shape_tag;
    union {
        struct Circle {
            struct Point centre;
            float radius; } circle;
        struct Rectangle {
            struct Point corner1, corner2; } rectangle;
    } shape;
};
```

\(^{12}\) This is reminiscent of what happens when we pass an array to a function; we have to manually keep track of the array’s length.
float area(struct Shape s) {
    if (s.shape_tag == CIRCLE) {
        float r = s.shape.circle.radius;
        return pi * r * r;
    } else { // rectangle
        float x1 = s.shape.rectangle.corner1.x;
        float y1 = s.shape.rectangle.corner1.y;
        float x2 = s.shape.rectangle.corner2.x;
        float y2 = s.shape.rectangle.corner2.y;
        return abs ((x2-x1) * (y2-y1));
    }
}

int main(void) {
    struct Shape s;
    s.shape_tag = CIRCLE;
    s.shape.circle = (struct Circle){.radius=1};
    printf("%f\n", area(s));
    s.shape_tag = RECTANGLE;
    s.shape.rectangle = (struct Rectangle) {{4, 5}, {6, 2}};
    printf("%f\n", area(s));

    return 0;
}

Algebraic data types

You now have the two main building blocks you need to create and understand user-defined types in Haskell. We use value constructors (like Circle) to group individual values together into a compound type with multiple fields, and use unions (indicated by the vertical bar |) to specify the different ways that a value of a particular type can be constructed. We call types that are created using combinations of constructors and unions algebraic data types.\(^{13}\)

Polymorphism I: type variables and generic polymorphism

Let’s now turn our attention to a more advanced feature of Haskell’s type system: its support of polymorphism. The term polymorphism comes from the Greek words poly, meaning “many”, and morphe, meaning “form” or “shape.” In programming language theory, polymorphism refers to the ability of an entity (e.g., function or class) to have valid behaviour in contexts of different types. That’s a pretty abstract definition—let’s look at what this means in Haskell starting from a familiar concept: the list.
The list type, for real

As we have previously discussed, Haskell’s type system imposes a restriction on programmers, that a list value must contain elements of the same type:

1. \( :t [True, False, True] \)
2. \([True, False, True] :: [Bool] \)
3. \( :t ((\&\&) True, not) \)
4. \(((\&\&) True, not) :: [Bool -> Bool] \)

At the same time, lists are quite generic: as long as we adhere to this restriction, we Haskell programmers can create lists of any type. But this raises the question: what is the type of functions that operate on lists, like head? We know that head takes as input a list and returns its first element, but the type of this element depends on the type of the list, and we just said this could be anything!

The type signature of head is \([a] -> a\) and differs from all other type signatures we have seen so far because it contains a type variable \(a\).\(^{14}\) A type variable is an identifier in a type expression that can be instantiated to any type, such as Bool or Char. This type signature tells us that head works on any type of the form \([a]\), and returns a value of type \(a\). Each time we call head, we match the parameter type \([a]\) against the type of the argument, and use this to instantiate the type variable to a concrete type to do the type-checking. For example, here is what happens when we type-check the expression head \([True, False, True]\):

1. Haskell determines that the type of the argument list is \([Bool]\).
2. Haskell matches the argument type \([Bool]\) against the parameter type \([a]\), and instantiates \(a = Bool\) in the function type.
3. Haskell takes the return type \(a\), with \(a\) instantiated to \(Bool\), to recover the final concrete type \(Bool\) for the return type, and therefore the overall type of the function call.

Let’s look at a more complicated example. As we briefly mentioned earlier, Haskell has a list filter function, which can be used as follows:\(^{15}\)

1. \( \text{filter } (\geq 1) [10,0,-5,3] \)
2. \([10,3]\)

What is the type of filter? We can think of filter in the usual sense of taking

\(^{14}\) In type expressions, type variables must start with a lowercase letter, distinguishing them from types themselves.

\(^{15}\) Another instance of sectioning: \((\geq 1)\) is equivalent to \(\lambda x \rightarrow x \geq 1\).
two arguments (putting currying aside). The second argument is a list of a’s, or [a]. The first argument must be a function mapping each a to a Bool, i.e., a -> Bool. filter returns a list of elements that have the same type as the original list (since these elements were elements of the original list). Putting this together we get that the type of filter is

\[
\text{filter :: (a -> Bool) -> [a] -> [a]}
\]

These are some examples of built-in list functions that have type variables—but what about the list type itself? How does Haskell know to allow the programmer to create lists of different types? This is in the very definition of the list type itself. Before we get to the “real” definition, let’s build some intuition by trying to define a list type ourselves. Here’s a type that represents a list of integers. Such a list is either empty, or an integer “cons’d” with another list of integers.

```
data IntList = Empty | Cons Int IntList
```

What if we wanted a list of strings instead?

```
data StrList = Empty | Cons String StrList
```

The only thing that differs in this definition is the type of the first argument to Cons, which should make sense, as this represents the type of the items actually being stored in the list. To generalize this definition, we do exactly the same thing as with functions: introduce a parameter to abstract away the type being “cons’d”!

```
data List a = Empty | Cons a (List a)
```

Here, a is a type variable (same terminology as above), and can be instantiated with different types (like Int or String) to create different types. Note how this exactly mimics the syntax for function definitions! We call the List itself a type constructor, as it is not exactly a type, but rather a function that takes a type a and creates a new type List a. So in this context, a type variable is a parameter of a type constructor.

```
data [] a = [] | (::) a ([] a)
```

Now remember, these value constructors are just functions! So the type of the second constructor (::) shouldn’t be a surprise:

16 Haskell generally avoids the boolean coercions of other languages; it doesn’t have a notion of “truthy” or “falsey” values.

17 So in this context, a type variable is a parameter of a type constructor.

18 The trickiest part of this definition is that the symbol [] plays two different roles: in the type language, [] is a type constructor that takes a type a and returns a new type representing a “list of as”; in the expression language, [] represents an empty list, which is one of two list constructors. Note that [] a is the expanded form of the more familiar [a].
Perhaps more surprising is the type of the first constructor, since it doesn’t take any arguments:

```
> :t []
[] :: [a]
```

Its type contains a type variable, which is what allows the empty list expression [] to be mixed with lists of any type, without any additional overhead on the programmer’s part.

**Generic polymorphism (in Haskell and beyond)**

In Haskell, lists are an example of *generic polymorphism*, a form of polymorphism in which an entity (e.g., function or class) behaves in the *same way* regardless of the type context. For example, Haskell lists are generically polymorphic (or just “generic” for short): they can be used to store elements of any type, and how these lists are constructed ([] and (:)) is the same regardless of what type of element is being stored. Similarly, almost every built-in list function is generic, meaning they operate on their input list regardless of what this input list contains. Put more poetically, generic functions operate on the “outer shape” of their input data (e.g., the fact that the data is a linear sequence) without care to the “inner shape” of the data’s contents.

In terms of types, there is for now a good approximation to tell whether a given function is generic: if its type signature contains a type variable, then it’s generic, and if it contains no type variables, it is not generic—in the latter case, we say that the type is *concrete*. Please note that this heuristic is not quite complete, and we’ll formalize this properly a bit later in this chapter.

As an aside, the concept of generic polymorphism is extremely useful, as it allows for great reductions in code repetition by writing one data structure or function once and having it work on all sorts of different types. If you’ve programmed in Java or C++, generics and type variables shouldn’t be novel. For example, Java’s standard `ArrayList` follows basically the same principle:19

```
19 as does Java’s entire Collections framework
```
In the class definition of `ArrayList`, `T` is a type variable, and is explicitly instantiated as `Integer` when `ints` is created. Of note is the creation of a new empty list; prior to Java 7, the `Integer` type annotation was required to make this code type-check, making it more verbose than the simple `[]` in Haskell, which is itself a generic value. Java 7 updated the language so that programmers would be able to omit the `Integer` from the right-hand side, naming the empty `<>` the diamond operator.

Templates in C++ play the same role, allowing type variables in both functions and classes. Here’s an example of a function template, which also illustrates a more limited form of type inference found in C++.

```cpp
#include <iostream>

template<typename Type>
void f(Type s) {
    std::cout << s << \n;
}

int main() {
    f<double>(1);    // instantiates and calls f<double>(double)
    f<\'a\'>();      // instantiates and calls f<char>(char)
    f(7);           // instantiates and calls f<int>(int)
    void (*ptr)(std::string) = f; // instantiates f<string>(string)
}
```

Polymorphism II: Type classes and ad hoc polymorphism

In the previous section, we looked at generic polymorphism, a useful tool in promoting abstraction by defining types and functions that would operate on not one but many data types. However, this form of polymorphism isn’t the only useful one, and indeed isn’t appropriate in a lot of circumstances. In this section, we’ll look at another form of polymorphism supported by Haskell, starting with a simple example: how can we see `Shape`s in the interpreter?

**Showing shapes**

Recall our `Point` and `Shape` types from earlier in this chapter:

```haskell
data Point = Point Float Float

data Shape = Circle Point Float -- centre and radius
            Rectangle Point Point -- top-left and bottom-right corners
```

Unfortunately, right now we can’t view any instances of our custom types di-

\[20\] This example is taken from http://en.cppreference.com.
rectly in the interpreter, because GHCi doesn’t know how to display them as strings. This is where type classes come in.

A **type class** is a set of types plus a set of functions that must work on these types, similar to the notion of an abstract interface in object-oriented programming. We say that a type is a **member** of a type class if the type class’ functions have been implemented for that type. Haskell has the built-in type class **Show**, containing the function **show**, which is roughly defined as follows:

```haskell
class Show a where
  show :: a -> String
```

This type class definition contains a type variable `a`, with a slightly different meaning than before. Here, the `a` should be interpreted as “a type `a` belongs to **Show** when there is an implementation of a function **show** that takes a value of type `a` and returns a string.” Most built-in types in Haskell are members of the **Show** type class:

```haskell
> show True
"True"
> show 3
"3"
```

Furthermore, it turns out that when GHCi is asked to evaluate an expression, it tries to call **show** on the resulting value to display the result back to the user, and this only works if the expression’s type is a member of the **Show** type class.

To make our class **Point** a member of **Show**, we need to implement the **show** function for our type, wrapped in a special syntactic form using the **instance** keyword:

```haskell
instance Show Point where
  show (Point x y) = "(" ++ (show x) ++ ", " ++ (show y) ++ ")"
```

Finally, a question: what exactly is **show**? At first this seems silly, as we’ve both referred to it and used it as a function. Yet this is a function whose initial declaration and type signature is decoupled from its implementation, and in fact has multiple implementations scattered throughout the Haskell standard library and in our own **Point** code, which is unlike any function we’ve seen before.

Well, the answer isn’t too complicated, but is worth digging into just a little bit. **show** is a single function with multiple implementations, one for each type that is a member of **Show**. Its type signature is a little different:
This type signature also has a type variable, but this type variable is modified by a new piece of type expression syntax, the `Show a =>`. We call the `Show a` before the `=> a` type class constraint: its meaning is that the type variable `a` can only be instantiated to a member of the `Show` type class.\(^{22}\)

We say that the type variable `a` is a *constrained* type variable, because of the presence of this constraint.

So now our polymorphism detection heuristic is a bit more complicated. If a function’s type signature is concrete (no type variables), it is not polymorphic. If a function’s type signature contains an unconstrained type variable, it is generically-polymorphic (in that variable): the function *must* behave the same way for any type instantiation of that variable. But what does it mean when we see a constrained type variable in a function type signature?

**Ad hoc polymorphism**

In contrast to generic polymorphism, **ad hoc polymorphism** is the ability of an entity to have different behaviours depending on the type context. In Haskell, this is precisely what type classes give us. By making types an instance of a type class, we give different implementations of the same functions for each type separately.\(^{23}\) Then, the presence of type class constraints in a function definition signals the ad hoc polymorphic nature of that function: when we see `Show a =>`, we expect different behaviours for the function for different instantiations of `a`.

A different approach to ad hoc polymorphism is implemented in Java through *method overloading*, which allows the programmer to implement the same method in different ways, giving different parameter signatures for each one.\(^{24}\)

```
public int f(int n) {
    return n + 1;
}

public void f(double n) {
    System.out.println(n);
}

public String f(int n, int m) {
    return "Yup, two parameters.";
}
```

One particular limitation of that technique is that the method’s type does not contain a type class constraint; the only way to determine if a Java method is ad hoc polymorphic is by scanning all of the function signatures in the source

\(^{22}\) A function type can have multiple type constraints, e.g. `Eq a, Ord b =>` ...

\(^{23}\) The term “ad hoc” in the name reflects this: each time we write an implementation for one type, we are free to make the implementation as tailored as possible to that type, without worrying about making it generic.

\(^{24}\) Note: this type of overloading is *not* permitted in Haskell, as it limits the kinds of type-checking that can be performed.
code (or relying on an IDE that does so). This plays back to this chapter’s recurring theme of using types to encode information about a value as being a fundamental purpose of a type system.

**Some built-in type classes**

In this part, we’ll briefly describe some type classes you’ll likely encounter in your study of Haskell. First, here are two type classes for comparing values:

- **Eq**: members of this type class support the \((==)\) and \((/=)\) functions to test for equality. Only \((==)\) needs to be implemented; \((/=)\) is given a default implementation in terms of \((==)\).
- **Ord**: members can be ordered using \(<\), \(\leq\), \(>\), and \(\geq\). Members must also be members of the Eq type class; we say that Ord is a subclass of Eq. Only \(<\) (and \(==\) from Eq) need to be implemented.

And now that we know about type classes, we can finally understand Haskell’s numeric types. Let’s start with addition.

```haskell
1 > :t (+)
2 (+) :: Num a => a -> a -> a
```

You might have come across the `Num a` class constraint before and had this explained to you as saying that `a` is a numeric type, and this is basically correct. More precisely, `Num` is a type class that supports numeric behaviours, and which concrete types like `Integer` and `Float` belong to. There are three main numeric type classes in Haskell:

- **Num** provides basic arithmetic operations such as `+`, `-`, and `*`. Notably, it doesn’t provide a division function, as this is handled differently by its subclasses. It also provides the function `fromInteger`, which we’ll discuss a bit further below.
- **Integral** represents integers, and provides the `div` (integer division) and `mod` functions.

```haskell
1 > div 15 2
2 7
3 > :t div
4 div :: Integral a => a -> a -> a
```

- **Fractional** represents non-integral numbers, and provides the standard `/` operator, among others.

So that’s it for the numeric functions. It turns out that numeric literals are also impacted by type classes, which is a pretty idiosyncratic characteristic of Haskell. Let’s inspect the type of the literal `1`:
Huh, it’s not 

Integer? That’s right, 1 (and all other integer literals) is a polymorphic value, like []: 1 can take on any numeric type, depending on its context. This is the purpose of the fromInteger :: Num a => Integer -> a function in the Num type class, which is called implicitly to convert integer literals into whatever numeric type is necessary for the surrounding context.25

By the way, this should explain why the expression 1 + 2.3 correctly returns 3.3 in Haskell, even though we previously said Haskell does not have any type coercion. When Haskell attempts to infer the type of this expression, it doesn’t automatically assume 1 is an Integer nor 2.3 a Float. Instead, it sees that the type of 1 must be a member of the Num type class, and the type of 2.3 must be a member of the Fractional type class, and that these two types must be equal because (+) takes two arguments of the same type. The Haskell compiler can correctly instantiate the types to satisfy these constraints, and so the type check passes.

Higher-order type classes

Let’s now look at one impressive feature of Haskell’s type class implementation, which goes beyond what you see in most other languages. Consider the following problem: in addition to lists, Haskell has many other “container” types, including sets and vectors. One of the particularly useful higher-order transformations we have seen before is map, which is used to apply some function to each element in a list, producing a new list as a result:

map :: (a -> b) -> [a] -> [b]

While this works for lists, we want such a function for sets and vectors as well:

setMap :: (a -> b) -> Set a -> Set b

vectorMap :: (a -> b) -> Vector a -> Vector b

In Racket, this is achieved by simply implementing different “mapping” functions for each data type, avoiding name collisions by prefixing by the data type name, just as we illustrated above. But this approach has the limitation that we cannot write elegant code that is polymorphic over a container type. For example, consider defining a function that “maps” the “plus 1” function over a collection of elements, that should work regardless of the container type. In Racket, we would do something like the following:

25 The Fractional type class has an analogous function, fromRational, that handles the conversion from fractional literals to fractional types.
As you are probably thinking, the solution in Haskell seems pretty straightforward: why don’t we make map part of a type class, so that we can implement it separately for each container type? And this is indeed what Haskell does, but there is one subtlety: since this type class would represent “container” types, its members are type constructors, not types themselves. Let’s take a look at the definition of the built-in “mappable” type class, which Haskell calls Functor:

```haskell
class Functor f where
  -- the "f" in fmap stands for "Functor"
  fmap :: (a -> b) -> f a -> f b
```

The type variable f represents the member of Functor, but we can tell from the type signature of fmap that f is not a primitive type like Int or Bool, but instead a type constructor like the list ([]), Set, or Vector. And because Haskell makes these different data types instances of Functor, we can implement a “map plus 1” function to work on arbitrary containers:

```haskell
add1All :: Functor f => f Int -> f Int
add1All items = fmap (+1) items

-- Or, using currying:
add1All = fmap (+1)
```

Representing failing computations

Here’s an interesting type that really illustrates the purpose of having multiple data constructors. Suppose you have an initial value, and want to perform a series of computations on this value, but each computation has a chance of failing. We would probably handle this in other programming languages by using exceptions or by checking each step to see if the returned value indicates an error. Both of these strategies often introduce extra runtime risk into our program, because the compiler cannot always determine whether (or what kind of) an exception will be raised, or distinguish between valid and erroneous return values, leaving it to the programmer.

Lest we get too high on our Haskell horse, be reminded that we have seen unsafe functions in Haskell as well: head, tail, and any incomplete pattern matches, though this is mitigated in Java with checked exceptions.
for example. In this section, we will see how to incorporate this possibility of error in a statically-checkable way, using Haskell’s type system. That is, we can indicate in our programs where potentially failing computations might occur, and make this known to the compiler by defining a data type representing the result of a computation that may or may not have been successful.

We start by introducing the fundamental type constructor, `Maybe`, used to represent such values.\(^\text{28}\)

```
data Maybe a = Nothing | Just a
```

Intuitively, the type `Maybe a` represents values that could have two possible states: either they’re a `Nothing`, because they were produced by some computation that failed, or they’re an actual `a` value, because the computation succeeded. Many programming languages encode this idea implicitly through a special value like `None` or `null` or `nil`; Haskell goes a step further, using not just an analogous value `Nothing`, but also encoding this in the very type of the expression: `Maybe Int` is not the same as `Int`, and the compiler can tell the difference!

Here are some simple functions that illustrate this data type; note that the “wrapped” type `a` can be anything, as `Maybe` is a generic type.

```
safeDiv :: Int -> Int -> Maybe Int
safeDiv _ 0 = Nothing
safeDiv x y = Just (x `div` y)
safeHead :: [a] -> Maybe a
safeHead [] = Nothing
safeHead (x:_:_) = Just x
safeTail :: [a] -> Maybe [a]
safeTail [] = Nothing
safeTail (_:xs) = Just xs
assertNonNegative :: Int -> Maybe Int
assertNonNegative n =
  if n >= 0
  then
    Just n
  else
    Nothing
```

The last example is a bit odd, since we aren’t “doing” anything to the input, but instead checking some property. The generalized version is perhaps more suggestive:
assert :: (a -> Bool) -> a -> Maybe a

assert pred x =
  if pred x then
    Just x
  else
    Nothing

However, in order to appreciate how to use this assert function in an interesting way, we need to expand our scope beyond just individual functions that return Maybes, to thinking about how to use such functions in a larger expression.

Lifting pure functions

Recall that we already have an operator, (.), which composes regular functions:

(f . g) = \x -> f (g x)

Let’s start with a simple example: defining a function that takes the first element of a list of numbers, and adds 1 to that number. If we weren’t worried about runtime errors, we could define this function as follows:

add1ToHead :: [Int] -> Int
add1ToHead = (+1) . head

It is tempting to make this function “safe” by swapping in our safeHead from above, and returning a Maybe:

safeAdd1ToHead :: [Int] -> Maybe Int
safeAdd1ToHead = (+1) . safeHead

Unfortunately, this implementation doesn’t type-check. The problem is that (+1) expects a “pure” numeric type (like Int), but we’re giving it a Maybe Int instead. Now, we could define a new function to handle this:

add1Maybe :: Maybe Int -> Maybe Int
add1Maybe Nothing = Nothing
add1Maybe (Just n) = Just (n + 1)

But of course this approach doesn’t scale; we don’t want to have to define “Maybe versions” of every function we want to use. Instead we define a higher-order function that will take any pure a -> b function and turn it into one that works on

\footnote{We deliberately write this in a terser style to emphasize the \textit{composition} of the main functions involved. Make sure you understand this definition before moving on!}
Maybe as instead. We call this operation \texttt{lift}, taking the term from mathematics, usually denoting the transformation of some object to move it into a more abstract or general context.

\begin{verbatim}
1 lift :: (a -> b) -> (Maybe a -> Maybe b)
2 lift _ Nothing = Nothing
3 lift f (Just x) = Just (f x)
\end{verbatim}

With this function, we can complete a correct implementation of \texttt{safeAdd1ToHead}:

\begin{verbatim}
1 safeAdd1ToHead :: [Int] -> Maybe Int
2 safeAdd1ToHead = (lift (+1)) . safeHead
\end{verbatim}

But hang on: doesn’t the type signature for \texttt{lift} look a little familiar? Keep in mind that \texttt{-} is right-associative in type expressions, and so that second pair aren’t necessary:

\begin{verbatim}
1 lift :: (a -> b) -> Maybe a -> Maybe b
\end{verbatim}

That’s right! This expression exactly matches the one for \texttt{fmap}, and this isn’t a coincidence: \texttt{Maybe} is indeed a member of the \texttt{Functor} type class.\footnote{Indeed, we can think about \texttt{Maybe} as a “container” that holds exactly zero or one value of its contained type.}

\begin{verbatim}
1 safeAdd1ToHead :: [Int] -> Maybe Int
2 safeAdd1ToHead = (fmap (+1)) . safeHead
\end{verbatim}

\textbf{Composing failing computations}

Now let’s consider defining a function that returns the second element of a list:

\begin{verbatim}
1 second :: [a] -> a
2 second = head . tail
\end{verbatim}

Our first attempt at making this function safe is to replace the list functions with the safe variants we defined above:

\begin{verbatim}
1 safeSecond :: [a] -> Maybe a
2 safeSecond = safeHead . safeTail
\end{verbatim}

It looks like we have a similar problem as before: \texttt{safeTail} returns a \texttt{Maybe [a]}, while \texttt{safeHead} expects just a \texttt{[a]}. What happens if we try using \texttt{fmap} (or \texttt{lift})?
safeSecond :: [a] -> Maybe a
safeSecond = (fmap safeHead) . safeTail

Unfortunately, this doesn’t quite work: the return type is a Maybe (Maybe a), rather than just a Maybe a. Using lift on a failing computation causes the return type to be a Maybe (Maybe a)! This isn’t what we want, so we’re going to take a different (presumably also higher-order) approach. But what?

To get a sense of the approach Haskell takes, let’s start by writing a correct, if ugly, implementation of safeSecond.\(^\text{31}\)

safeSecond :: [a] -> Maybe a
safeSecond xs =
    let xs' = safeTail xs
    in
    case xs' of
        Nothing -> Nothing
        Just xs'' -> safeHead xs''

To see what’s going on, let’s look at the types:

- xs has type [a], which is the original input to the function
- xs' has type Maybe [a], which is what’s returned by safeTail
- xs'' has type [a]: it is an “unwrapped” version of xs’, which is safe to do inside the case expression because we provide pattern-match rules for both possibilities for xs'.

Unfortunately, using the case expression does not generalize very elegantly, as we compose more and more computations. Suppose we wanted to access the fourth element of a list:

safeFourth :: [a] -> Maybe a
safeFourth xs =
    let xs' = safeTail xs
    in
    case xs' of
        Nothing -> Nothing
        Just xs1 ->
            let xs1' = safeTail xs1
            in
            case xs1' of
                Nothing -> Nothing
                Just xs2 ->
                    let xs2' = safeTail xs2
                    case xs2' of
                        Nothing -> Nothing
                        Just xs3 -> safeHead xs3

\(^{31}\) This also introduces case expressions, which enable pattern-matching in arbitrary expression contexts.
To remove the code duplication, we move the pattern-matching to a helper function. We call our helper `andThen`, suggesting the idea of sequencing these computations.

```haskell
andThen :: Maybe a -> (a -> Maybe b) -> Maybe b
andThen Nothing _ = Nothing
andThen (Just x) f = f x
```

Like `lift`, the implementation of `andThen` is very simple; really, the only difference is that because the input function `f` returns a `Maybe`, we do not wrap the second line’s body in a `Just` (logically, we shouldn’t assume that calling `f` will succeed!). Let’s use this helper to see how to simplify `safeSecond`:

```haskell
safeSecond :: [a] -> Maybe a
safeSecond xs =
  let xs' = safeTail xs
  in
      andThen xs' safeHead
```

However, this isn’t that readable. The reason we chose the name `andThen` only becomes apparent when we use backticks to call `andThen` from an infix position:

```haskell
safeSecond :: [a] -> Maybe a
safeSecond xs =
  let xs' = safeTail xs
  in
      xs' `andThen` safeHead
```

-- Or simply,
```haskell
safeSecond xs = safeTail xs `andThen` safeHead
```

With this use of `andThen`, we can entirely remove all of the boilerplate from `safeFour` as well, resulting in a truly beautiful implementation.

```haskell
safeFourth :: [a] -> Maybe a
safeFourth xs =
  safeTail xs `andThen`
      safeTail `andThen`
      safeTail `andThen`
      safeHead
```
**Error reporting**

One limitation of using `Maybe` is that chained computations don’t preserve information about where or why the computation failed. A `Nothing` conveys a failure, but nothing else.

In practice, when error reporting is important the `Either` type constructor is used instead:

```haskell
data Either a b = Left a | Right b
```

`Either` is a versatile type constructor that can be used to represent an alternative between two different types (e.g., “this function returns either an `Int` or a `String`”), but one of its most common uses is to use the `Left` constructor to represent a failure with some error information, and a `Right` to represent a success, analogous to `Just`. For example, a bare-bones approach might store an error message as a string:

```haskell
safeDiv2 :: Int -> Int -> Either String Int
safeDiv2 _ 0 = Left "Division by 0 error"
safeDiv2 x y = Right (x `div` y)
```

More sophisticated systems might use a custom data type to store more elaborate error information as the “left” value.

**Modeling mutation in pure functional programming**

One of the central tenets of functional programming we’ve had in the course so far was avoiding the concept of `state`; that is, writing code without mutation. Though we hope you have seen the simplicity gained by not having to keep track of changing variables, it is often the case that mutable state is the most natural way to model problem domains. So the question we’ll try to answer in this section is, “How can we simulate changing state in Haskell, which does not even allow re-binding of identifiers?”

**Postorder tree labeling**

Consider the following problem: given a binary tree, label each node with its position in a postorder traversal of that tree.

Here is some code in Python to solve this problem, using a `global mutable variable` to keep track of the “current” position in the traversal.

```python
i = 0
```
This code is very elegant, but the reason for its elegance is the use of this global variable $i$ that is mutated through the recursive calls to `post_order_label`. Our goal is to implement this same algorithm in Haskell.

### Stateful computations

Believe it or not, we have already seen the basic “mutating state” technique in this course. Recall the basic loop iteration pattern:

```python
acc = init
for x in lst:
    acc = update(x, acc)
```

We achieved the same affect using `foldl`, and the key idea was that the update function passed to `foldl` was interpreted to “mutate” the accumulator. Concretely, the function took as input the old state of the accumulator, and returned its new state. By chaining together such functions—each taking the result of the previous—we can simulate a mutating “accumulated state” over many different computations.

For any kind of mutable state, the two primitive operations are `read` and `write`, which here we’ll label `get` and `put`. We could implement each one as follows:

```haskell
get :: s -> s
get state = state
put :: s -> s -> s
put item state = item
```

This is pretty weird code, so let’s unpack our intentions.

- `get` takes in the current state and simply returns it; this simulates “reading” from the state.
• put takes an item and the current state, and returns the item, representing the new state (note that the old state is completely discarded).

But the above implementation is very confusing, and the key reason for this is that it is impossible to tell from the return types of these two functions that one of them returns a lookup value (but leaves the state unchanged), and the other actually modifies the stored state. To resolve this issue, we unify their return types by using a tuple, where the first element of the tuple represents what value (if any) is “produced” by the computation, and the second element represents the new state.

```
1 get :: s -> (s, s)
2 get state = (state, state)
3 4 put :: s -> s -> (() , s)
5 put item state = (() , item)
```

Now let’s try to interpret this new code:

• get takes the old state and returns its value as both elements of the tuple. The first element represents what’s being produced (the value is looked up), while the second represents the new state, which is actually exactly the same as the old state.
• put updates the state, which is why the second element in the returned tuple is item—the old state has been replaced. The () in the first position is called unit in Haskell, and is used to denote the fact that the put doesn’t produce a value. This is analogous to a void return type in Java or C.

That’s better, but we’ll go one step further and use Haskell’s built-in State data type, which essentially wraps around the “updating state” function type:

```
data State s a = State (s -> (a, s))
```

Warning: this is actually a pretty confusing name. In Haskell, this State type constructor doesn’t refer to the state itself, but rather to a stateful computation that takes in and returns state, as we described above. This is a pretty unfortunate historical name, so please pay attention to the name when we use it through this section!

Our final step is to tweak our get and put implementations to use this type:

```
1 get :: State s s
2 get = State (\state -> (state, state))
3 4 put :: s -> State s ()
5 put item = State (\state -> (() , item))
```

Somewhat confusingly, () is both a value and a type, both called “unit”.
This should match with our intuitions about these computations: get is a stateful computation that produces whatever was stored in the state without changing it, and put takes an item, and then the result is a stateful computation that produces nothing but replaces the existing state with that item.

**Running and combining stateful computations**

One of the side effects of using the `State` type to represent state computations is that the underlying function is now wrapped, and can’t be called directly. For example, calling `get 5` produces an error rather than the expected `(5, 5)`. To resolve this, Haskell has a built-in function `runState` whose only purpose is to return the wrapped function:

```haskell
runState :: State s a -> (s -> (a, s))
runState (State op) = op
```

The beauty of this and Haskell’s support for higher-order functions means that we can call `runState` on both a `State` and a state value, essentially “running the stateful computation on the given initial state.”

```
> runState get 5
(5,5)
```

With this in mind, we can chain together stateful computations as well. As an example, let’s translate the following Java code:

```java
public String f() {
    int x;
    x = 10;
    x = x * 2;
    return str(x);
}
```

The idea is pretty simple once we get the hang of it: except for the variable declaration and initialization, every other line consist of a `get`, `put` or pure computation (like `*2`). Our approach is to perform each of these computations in turn, using `let` to bind the intermediate states so that each `State` computation gets as input the state from the previous one.33

```haskell
f :: State Int String
f = State (\state0 ->
    let (_, state1) = runState (put 10) state0
        (x, state2) = runState get state1
        (_, state3) = runState (put (x + 2)) state2
        (x', state4) = runState get state3
    in str(x'))
```

33 To make the code a little nicer, we use pattern-matching on the left-hand side of the `let` bindings.
This code is certainly easy to understand, and its size may not even suffer (much) in comparison to other languages. But hopefully it seems at least somewhat inelegant to have to manually keep track of this state. This is analogous to the clumsy use of pattern-matching to distinguish between Nothing and Just values in the previous section. So, as before we’ll define a higher-order function to abstract away the composition of two State operations. For reasons that will become clear shortly, the main function we’ll implement is andThen. Here’s a preliminary (not final) version:

```
-- Perform two state computations and return the second one's value.
andThen :: State s a -> State s b -> State s b
andThen op1 op2 = State (\state0 ->
  let (x1, state1) = runState op1 state0
      (x2, state2) = runState op2 state1
  in
    (x2, state2)
)
```

This is a good start, but it is not quite general enough for our purposes. The problem is that the “produced” value \(x_1\) of the first computation is unused by the second computation. But in general, when we sequence two stateful computations, we want the second to be able to use the result of the first. So what we need to do is define a function that will perform two State operations in sequence, but have the second depend on the result of the first one. On the face of it, this sounds quite strange—but remember that closures give languages a way of dynamically creating new functions with behaviours depending on some other values.

So rather than take two fixed State operations, andThen will take a first State operation, and then a function that takes the result of the first State operation and returns a new State operation. Think about this second parameter as an “incomplete” State function, which gets completed by taking the value produced by the first function. This might seem pretty abstract, but should become clearer when we look at the new type signature and implementation:

```
andThen :: State s a -> (a -> State s b) -> State s b
andThen op1 opMaker s = State (\state0 ->
  let (x1, state1) = runState op1 state0
      op2 = opMaker x1
      (x2, state2) = runState op2 state1
  in
    (x2, state2)
)
```
That is, `andThen` does the following:

1. Run its first `State` computation (`op1`).
2. Use the produced result (`x1`) and its second argument (`opMaker`) to create a second `State` computation.
3. Run the newly-created `State` computation, and return the result (and updated state).

With this function in hand, we can now greatly simplify our example function:

```haskell
f :: State Int String
f = put 10 `andThen` (\_ ->
  get `andThen` (\x ->
    put (x + 2) `andThen` (\_ ->
      get `andThen` (\x' ->
        State (\state -> (show x', state)))))
```

Well, almost. We have used indentation and parentheses to delimit the bodies of each lambda expression, which causes our code to look a little ugly. It turns out that because the `->` keyword has very low precedence, the parentheses are unnecessary. Moreover, indentation has no syntactic significance here, so we can vertically-align each operation. These two changes clean up the code dramatically:

```haskell
f :: State Int String
f = put 10 `andThen` (\_ ->
  get `andThen` (\x ->
    put (x + 2) `andThen` (\_ ->
      get `andThen` (\x' ->
        State (\state -> (show x', state)))))
```

Finally, we note that the final `State` value is special, since it simply “produces” a value (`show x'`) without any interaction with the underlying state at all. We’ll call this type of computation a “pure” one, and use a helper function to simplify the code further:

```haskell
pureValue :: a -> State s a
pureValue item = State (\state -> (item, state))

f :: State Int String
f = put 10 `andThen` (\_ ->
  get `andThen` (\x ->
    put (x + 2) `andThen` (\_ ->
      get `andThen` (\x' ->
        pureValue (show x'))))
```
Back to post-order labeling

It turns out that we have everything we need to implement a post-order binary tree labeling in Haskell. First, here is the corresponding data type we’ll use to represent binary trees in Haskell.\cite{note:binary-tree-definition}

```haskell
data BTree a = Empty
              -- The Int parameter is the label
              | Node a Int (BTree a) (BTree a)
```

Now let’s recall the mutating Python version:

```python
i = 0

def post_order_label(tree):
    global i

    if tree.is_empty():
        return
    else:
        post_order_label(tree.left)
        post_order_label(tree.right)
        tree.root.label = i
        i += 1
```

To keep things relatively immutable, we won’t mutate a BTree directly, but rather return a new BTree with all nodes labelled properly. To accomplish this using the same algorithm as shown in the Python code, we’ll need one Int of mutable state, and so our type signature will be:

```haskell
postOrderLabel :: BTree a -> State Int (BTree a)
```

The base case is pretty easy to implement, and can be done without any statefulness at all:

```haskell
postOrderLabel Empty = pureValue Empty
```

As a first pass for the recursive step, let’s just try translating the two recursive calls in Python into the equivalent chained Haskell code, using our `andThen`.\cite{note:chaining-haskell-calls}

```haskell
postOrderLabel (Node item _ left right) =
    postOrderLabel left `andThen` \newLeft ->
    postOrderLabel right
```

\cite{note:chaining-haskell-calls} Note that we ignore the original node label, since the goal here is to set the label ourselves.

\cite{note:binary-tree-definition} Note the similarity to our recursive list definition!
Not too bad, although these two recursive calls aren’t really doing anything (other than spawning other useless calls). Let’s first increment our “i”:

```haskell
postOrderLabel (Node item _ left right) =
  postOrderLabel left `andThen` \newLeft ->
  postOrderLabel right `andThen` \newRight ->
  get `andThen` \i ->
  put (i + 1) `andThen` _ ->
  pureValue (Node item i newLeft newRight)
```

That version almost compiles, but unfortunately ends with a State value of the wrong type: State Int () instead of State Int (BTree a). To fix this, let’s actually consider what we want to return: a new BTree with all the nodes labeled correctly. Through the power of recursion, we can assume that both newLeft and newRight (produced by the recursive calls) are indeed correctly labeled. So then to put everything together, we need to create and return a new root node, containing the original item, an updated label, and new subtrees:

```haskell
postOrderLabel (Node item _ left right) =
  postOrderLabel left `andThen` \newLeft ->
  postOrderLabel right `andThen` \newRight ->
  get `andThen` \i ->
  put (i + 1) `andThen` _ ->
  pureValue (Node item i newLeft newRight)
```

And that’s it! Not too shabby indeed—and quite close to what we’d write in Python if we had to return a new tree rather than mutate the original.

**Performing I/O**

In addition to mutating state, another common type of impure function is one that interacts with some entity external to the program: standard input or output, a graphical display, the computer filesystem, or a server thousands of kilometres away. Even though we’ve been programming without explicit mention of I/O since the beginning of the term, it’s always been present: any REPL requires functions that take in user input, and that actually display the output to the user as well. If Haskell had no I/O capabilities whatsoever, we would never know if a program actually worked or not, since we would have no way to observe its output!

In this section, we’ll briefly study how Haskell programs can interact directly with the standard input and standard output provided by the operating system. While the primitives we’ll use in this section might look underwhelming, all of the other I/O operations we listed above behave exactly the same. In fact, even more is true: it turns out that all I/O follows our approach to mutable state from the previous section. Even though both I/O and mutation are side-effectful
behaviours, they seem quite different at first glance, and so this might come as a surprise. Let’s take a closer look at what’s going on.

Our State s a values represented computations with two behaviours: a “pure” part, the computation performed to produce a value, and an impure part, a “mutation” operation, representing an interaction with the underlying context of the computation. We can think of pure functions as those that have no context (e.g., external mutable state) they need to reference, and impure functions as those that do.

Even though our post-order labeling example used a State Int representing just one integer’s worth of mutable state, the beauty of the State type constructor is that it can be parameterized on the type of state s being used, and so in more complex programs we might have State String or State [Int] or even State (Map String Expr). But even this view is too narrow: there is no need to limit our computational context to the program’s heap-allocated memory. By widening the scope of this context to user interactions, files, and faraway servers, we can model any type of I/O computation at all.

Standard I/O primitives

Recall that we built complex State computations using two primitives, get and put. With standard I/O there are two built-in values that we’re going to cover: putStrLn, which prints a newline-terminated string to standard output, and getLine, which reads a string from standard input. A natural question to ask when in an unfamiliar environment is about the types of the values. Let’s try it:

```
1 > :t getLine
2 getLine :: IO String
3 > :t putStrLn
4 putStrLn :: String -> IO ()
```

Here, IO plays an analogous role to State s, except rather than indicating that the function changes the underlying state context, it indicates that the function has some sort of interaction with an external entity. IO is a unary type constructor, where the type IO a represents an I/O action that produces a value of type a. So getLine has type IO String because it is an I/O action that produces a string, and putStrLn has type String -> IO () because it takes a string, prints it out, but produces nothing.

We can now define a function which prints something to the screen:

```
1 greet :: IO ()
2 greet = putStrLn "Hello, world!"
3 > greet
4 Hello, world!
```
Now suppose we want to print out two strings. In almost every other programming language, we would simply call a printing function twice, in sequence. But in our pure functional programming setting, we work with expressions and not statements, and so can’t simply write one expression after the other.41

What we want is to chain together two I/O actions, a way to say “do this computation, and then do that one.” This hopefully rings a bell from the previous section, when we chained together two State computations using two different versions of andThen. And this is where things get interesting: we actually have both versions as infix operators for I/O:

```
-- andThen, but discard the result produced by the first
-- By convention, called "then"
(>>) :: IO a -> IO b -> IO b

-- andThen, using the first result to create the second
-- By convention, called "bind"
(>>=) :: IO a -> (a -> IO b) -> IO b
```

It shouldn’t be a big surprise that we can use the former to sequence two or more I/O actions:

```
greet2 :: IO ()
greet2 = putStrLn "Hello, world!" >> putStrLn "Second line time!"

prompt :: IO ()
prompt = putStrLn "What's your name?" >> getline >> putStrLn "Nice to meet you, ____!"
```

How do we fill in the blank? We need to take the string produced by the getline and use it to define the next I/O action. Again, this should be ringing lots of bells, because this is exactly what we did with our final version of andThen, corresponding to the I/O operator (>>=):

```
prompt2 :: IO ()
prompt2 = putStrLn "What's your name?" >> getline >>= \name -> putStrLn "Nice to meet you, " ++ name ++ "!"

> prompt2
What's your name?
David -- This line is user *input*
Nice to meet you, David!
```
Aside: standalone Haskell programs

Though we’ve mainly used GHCi, Haskell programs can also be compiled and run.\footnote{Say, by using the command `runhaskell myfile.hs.`} Such programs require a `main` function with type `IO ()`, which is executed when the program is run. The type of `main` is very suggestive: the program as a whole is an `IO` action that can interact with the outside world, using `main` as its entry point. In this course we’ve been mainly using the REPL as our entry point, but in practice any kind of I/O behaviour would start under `main`.

Types as constraints

You have now seen two examples of impure values in Haskell: the state-“mutating” `State` computations, and the external-world-interacting `IO`. Despite our prior insistence that functional programming is truly different, it seems possible to simulate imperative programs using these functions and others like them. This is technically correct, but it misses a very important point: unlike other languages, Haskell’s type system enforces a strict separation between pure and impure code.

The separation of concerns principle is one of the most well-known in computer science, and you’ve probably studied this extensively in the context of designing classes, libraries, and larger application architectures like MVC. However, in most languages, is easy for impure and pure code to be tightly wound together, because side-effecting behaviour like mutation and I/O is taught to us from our very beginnings as programmers. But as we’ve discussed earlier, this coupling makes programs difficult to reason about, and often in our refactoring we expend great effort understanding and simplifying such side-effects.

In stark contrast, such mixing is explicitly forbidden by Haskell’s type system. Think back to `IO`. We only used functions that combine values of `IO` types (like `(>>=)`), or functions that returned a value of these types. But once the `IO` was introduced, there was no way to get rid of it: there is no function `IO a -> a`.\footnote{This isn’t strictly true, but the functions that do exist are used rarely and with great care in Haskell production.} This means that any Haskell function that uses standard I/O must have a `IO` in its type.

It might feel restrictive to say that every time you want to print something in a function, you must change the fundamental type (and hence structure) of that function. However, as a corollary of this, any function that doesn’t have `IO` in its type is guaranteed not to interact with the terminal, and this guarantee is provided at compile time, just through the use of types! This is perhaps the best illustration of this chapter’s central principle: a type signature in Haskell tells us not just how to use a function and what it returns, but what it cannot do as well.

One last abstraction: monads

When we first started our study of types and Haskell, you might have thought we would focus on the syntax of creating new types in Haskell. But even though
we did introduce quite a few new keywords for Haskell’s type system, our ambitions were far grander than recreating a classical object-oriented system in Haskell. Rather than focusing on types representing concrete models or actors in our programs, we have focused on using types to encode generic modes of computation: failing computations, stateful computations, and I/O computations.

But defining type constructors like Maybe, State, and IO to represent elements of these computations was only a small part of what we did. After defining Maybe and State, we could have performed any computation we wanted to with these types simply by using let bindings, and called it a day. The novel part was really our use of higher-order functions to abstract away even how we computed on these types, allowing us to elegantly compose primitive values of these types to express complex computations. And this, of course, is the true spirit of functional programming.

But there is one additional layer of abstraction that we’ve only hinted at by our choices of names for our operators. Though the contexts we studied were quite different, we built the same operators to work for each one. To make this explicit, let’s look at the type signatures for the one function common to all three:

```
andThen :: Maybe a -> (a -> Maybe b) -> Maybe b
andThen :: State s a -> (a -> State s b) -> State s b
(>>=) :: IO a -> (a -> IO b) -> IO b
```

And though the implementations differed for each context, their spirit of composing context-specific computations did not. As always when we see similarities in definitions and behaviour, we look to precisely identify the commonalities, which presents an opportunity for abstraction.

In fact, this is exactly what happens in Haskell: the similarity between Maybe, State, and IO here is abstracted into a single type class, called Monad. We misled you earlier in our discussion of I/O: if you query the type of either (>>) or (>>=), you’ll see a type class constraint that generalizes IO:

```
> :: t (>>)
(>>) :: Monad m => m a -> m b -> m b
> :: t (>>=)
(>>=) :: Monad m => m a -> (a -> m b) -> m b
```

It is no coincidence that these functions were the ones we have explored in great context; they are two of the three functions which are part of the Monad definition:

```
class Monad m where
  return :: a -> m a
  (>>) :: Monad m => m a -> m b -> m b
  (>>=) :: Monad m => m a -> (a -> m b) -> m b
```

44 We (basically) already did this in Racket.

45 We don’t even know how (>>=) is implemented for IO!

46 You’ll find that the other function is also very familiar, especially if you’ve been keeping up with the exercises.
You should have some intuition for these functions already based on how we’ve used them in the various contexts, but here is a brief description of each one:

- **return**: take some pure value, and “wrap” it in the monadic context.
- **(>>)**: chain together two monadic values; the result cannot depend on the result of the first one.
- **(>>=)**: chain together two monadic values; the result can depend on the result of the first one.

Of course, generic terms like “chain” and “contents” change meaning depending on the context, and this is reflected when we actually make instances of `Monad`. For example, we can describe these functions in the `IO` context as follows:

- **return**: create an I/O action which does nothing except produces the given value.
- **(>>)**: combine two I/O actions into a new I/O action which performs the two in sequence, producing the result of the second.
- **(>>=)**: create a new I/O action which performs the first one, takes the result and uses it to create a second action, and then performs that second action.

`Maybe` is also an instance of `Monad`, with the following definition:

```haskell
instance Monad Maybe where
    return = Just
    Nothing >> _ = Nothing
    (Just _) >> x = x
    Nothing >>= _ = Nothing
    (Just x) >>= f = f x
```

While the “bind” and “return” should look familiar, the “then” operator is new, although it makes sense: it returns the second `Maybe` only when the first one is a success.

We’ll leave it as an exercise to make `State s` an instance of `Monad`, and to describe precisely what the three operations would mean in that context.

**Why monads?**

Because these contexts are so different, you might wonder what purpose it serves to unite them under a single type class at all. It turns out that the common set of operations—`return`, `(>>)` and `(>>=)`—are powerful enough to serve as building blocks for much more complex, generic operations. Consider, for example, the following situations:
• Take a list of results of possibly-failing computations, and return a list of the success values if they were all successes, otherwise fail.
• Take a list of I/O actions, perform them all, and return the list of the produced results.
• Take a list of stateful operations, perform them all in sequence on the same initial state, and return a list of the results.

All of these actions have essentially the same type: \( [m \ a] \rightarrow m \ [a] \), where \( m \) is either \( \text{Maybe} \), \( \text{State s} \), or \( \text{IO} \). Moreover, all of them involve the type of chaining that is captured in the monadic operations, so this should be a hint that we can define a single function to accomplish all three of these. Because operating with the monadic functions is relatively new to us, let us first try to implement this—you guessed it—recursively.

```haskell
seqM :: Monad m => [m a] -> m [a]
seqM [] = return []
seqM (m1:ms) =
  let rest = seqM ms
  in
    -- extract the first value
    m1 >>= \x ->
      -- extract the other values
    rest >>= \xs ->
    return (x:xs)
```

“But wait,” you say. “What happened to all of the context-specific stuff, like handling \( \text{Nothing} \) or mutating the state?” The brilliance of this is that all of that is handled entirely by the implementation of \((>>=)\) and \(\text{return}\) for the relevant types, and \(\text{seqM}\) itself doesn’t need to know any of it!
> seqM [Just 10, Just 300, Just 15]
> Just [10, 300, 15]
> (seqM [putStrLn "Hello", putStrLn "Goodbye", putStrLn "CSC324!!!"])
> Hello
> Goodbye
> CSC324!!!
> [(),(),()]

Now, this one example doesn’t mean that there aren’t times when you want, say, Maybe-specific code. Abstraction is a powerful concept, but there is a limit to how far you can abstract while still accomplishing what it is you want to do. The Monad type class is simply a tool that you can use to design far more generic code than you probably have in other languages. But what a tool it is.

Exercise Break!

3.1 Implement seqM using foldl.
4 In Which We Say Goodbye

If all you have is a hammer, everything looks like a nail.

Law of the instrument

When teaching introductory computer science, we often de-emphasize the idiosyncrasies and minutiae of our programming languages, focusing instead on universal theoretical and practical concepts like data structures, modularity, and testing. Programming languages are to computer science what screwdrivers are to carpentry: useful tools, but a means to an end. If you have a cool idea for the Next Big Thing, there are dozens, if not hundreds, of programming languages and application frameworks\(^1\) to choose from.

It is easy to get lost in the myriad technical and non-technical details that we use to compare languages. Though these are very important—and don’t get us wrong, we covered several in this course—we must not lose sight of the forest for the trees. At its core, the study of programming languages is the study of how we communicate to computers: how language constructs and features inform the properties of the code we can express.

Our first big idea was exploring how first-class functions allow us to write programs in a pure functional style, achieving greater abstraction through higher-order functions and writing elegant code without side-effects. We then turned to using macros to add two new constructs to Racket: class-based objects through closures, and backtracking through declarative choice expressions using continuations. Finally, we saw how to use a strong static type system in Haskell to encode common language features like errors, mutation, and I/O in an explicit and compile-time checkable way.

We hope you got the sense from this course that programming language research is a truly exciting and fascinating field. If you are interested in learning more, **CSC\textit{465}** studies a rigorous, mathematical approach for understanding the semantics of (classical imperative) programming languages. **CSC\textit{410}** studies how to specify and ensure the correctness of programs, using a combination of static and dynamic analysis techniques. And **CSC\textit{488}**, our compilers course, is fundamentally the study of translation: taking a human-readable, high-level language and turning it into the sequence of primitive operations directly understandable by a computer.

\(^1\) Large, monolithic frameworks are essentially specialized languages themselves.
Newer, modern programming languages extend the functional programming and static analysis ideas in interesting ways. Two languages of note that draw inspiration from other domains are *Elm*, which uses the declarative *functional reactive programming* paradigm to build web applications, and *Rust*, which uses strong static analysis to add powerful compile-time safety guarantees at the low level of systems programming. And of course, if you would like to talk about any of the material in this course or beyond, don’t be afraid to ask! That’s what we’re here for.