CSC236 winter 2020, week 12: (non-)regularity

Recommended reading: Chapter 7 Vassos course notes, section 7.6.3—

Colin Morris
colin@cs.toronto.edu
http://www.cs.toronto.edu/~colin/236/W20/

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Reminders

- A3 due Thursday @ 15:00
  - It's short!
  - Extra office hours available by request
- One last tutorial + quiz this Friday
- Also, final Q&A session Wednesday 12:00-14:00
  - These are really worth attending!
- Marking scheme changes
  - Course website will be updated when vote officially closes on Monday
- Exam-like final assessment (worth 20%) to be written April 7-9

Regular languages

A language L is **regular** iff

- L is denoted by a regular expression
- L is accepted by a deterministic FSA
- L is accepted by a non-deterministic FSA

(We now know that all of these criteria are equivalent.)

Proving regularity

A few options to prove that L is regular:

1. Construct an RE, or a DFSA, or an NFSA that matches L.
2. Use closure properties of regular languages. Show that L can be formed by application of union/intersection/complement/Kleene star to some languages that are known to be regular.
3. Use the fact that all finite languages are regular

Example: proving regularity

$L_1 =$ strings over $\{0,1\}$ of length 236. Prove L_1 is regular.

$\{0,1\}^{236}$ is finite, $\therefore$ regular

Example: proving regularity

L_1 = strings over $\{0, 1\}$ of length 236. Prove L_1 is regular.

L_2 = strings over $\{0, 1\}$ where length is a multiple of 236. Prove L_2 is regular.

$L_2 = L_1^*$

L_1 regular, so L_1^* also regular

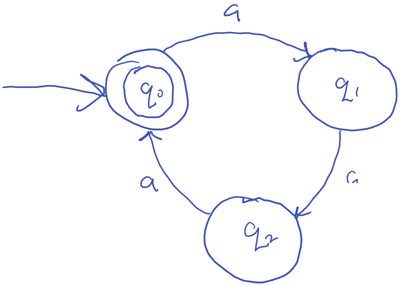
Are all languages regular?

Big if true

$0^n 1^n$

Detour: probing the limits of FSAs

Suppose M is a DFSA such that $\mathcal{L}(M) = \{a^n \mid \exists k \in \mathbb{N}, n = 3k\}$.
What is the minimum number of states M can have?



Proving lower bounds on states

Recall, $\mathcal{L}(M) = \{a^n \mid \exists k \in \mathbb{N}, n = 3k\}$.

Consider

- $\delta^*(s, a) = q_1$
- $\delta^*(s, aa) = q_2$
- $\delta^*(s, aaa) = q_3$

Claim: q_1, q_2, and q_3 are distinct.

Proof: Show that each of the following possibilities leads to a contradiction

- $q_3 = q_1$ — $q_3 \in F$ $\quad q_1 \notin F$
- $q_3 = q_2$
- $q_2 = q_1$ $\quad \delta(q_1, a) \in F$?

If yes, we accept $aa \notin L$

If no, we reject $aaa \in L$

Pigeonhole principle

Figure: 10 pigeons $>$ 9 pigeonholes $\implies$ pigeon cohabitation

Recipe: proving lower bound on DFSA states

To prove that any DFSA M that accepts L must have at least n states

1. Prove that n is *sufficient*, by demonstrating an accepting n-state DFSA
   - (May or may not be necessary, depending on how question is worded)
2. Find n distinct prefixes $x_1, x_2, \ldots x_n$, and matching suffixes[1] $y_1, y_2, \ldots y_n$, such that
   - $x_j y_k \in L \iff j = k$
   - i.e. for each prefix, exactly one of the suffixes can be concatenated to it to form a string in L
3. Prove minimum of n states by contradiction
   - 3.1 Assume, for sake of contradiction, that $|Q| < n$.
   - 3.2 By the pigeonhole principle, there must be two different prefixes, x_j and x_k that go to the same state, q
   - 3.3 So $\delta^*(q, y_j)$ must be accepting and non-accepting. $\Rightarrow\Leftarrow$

y_k

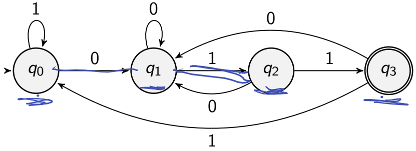
$x_j y_j \in L \qquad x_k y_j \notin L$

[1] It's actually sufficient to find just $n-1$ suffixes, i.e. we can get away with having *one* prefix x that doesn't have a matching suffix. See steps 3.2 and 3.3 for the reason why.

Another (worked out) lower bound example

Find the minimum number of states for a DFSA that accepts $L = \{w \in \{0,1\}^* \mid w \text{ ends with '011'}\}$.

Below, we give a 4-state DFSA for L.



So 4 is sufficient. Is it necessary?

$x_1 \quad 0 \quad 11$

$x_2 \quad 01 \quad 1$

$x_3 \quad 011 \quad \varepsilon$

$x_4 \quad \varepsilon$

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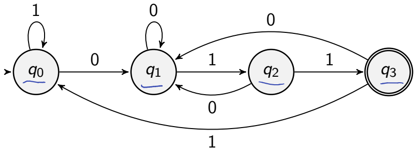

So 4 is sufficient. Is it necessary?

Consider

- $x_0 = \varepsilon$
- $x_1 = 0, y_1 = 11$
- $x_2 = 01, y_2 = 1$
- $x_3 = 011, y_3 = \varepsilon$

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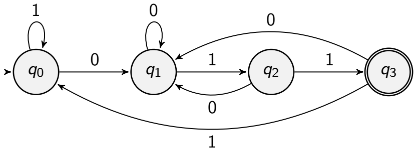

So 4 is sufficient. Is it necessary?
Consider

- $x_0 = \varepsilon$
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By inspection, each suffix y_j has exactly one prefix x_j such that $x_j y_j \in L$.
Suppose FSOC a DFSA with < 4 states accepts L. By the pigeonhole principle, there must be a distinct pair, x_j, x_k, such that $\delta^*(s, x_j) = \delta^*(s, x_k) = q$ for some state q.
WLOG, suppose $j \neq 0$. Then $\delta^*(q, y_j)$ must be accepting. But that would mean we also accept, $x_k y_j \notin L$. $\Rightarrow\Leftarrow$

An infinite flock of pigeons

Prove that $L = \{0^n 1^n \mid n \in \mathbb{N}\}$ is non-regular.

Suppose M is a DFSA s.t. $L(M) = L$

$0, 00, 000, 0000 \ldots$

By pigeonhole principle, $\exists\, n, m \in \mathbb{N},\ n \neq m \wedge \delta^*(s, 0^n) = \delta^*(s, 0^m) = q$, for some state q

$\delta^*(q, 1^n) \in F$?

If yes, we accept $0^m 1^n \Rightarrow \Leftarrow$

If no, we reject $0^n 1^n \Rightarrow \Leftarrow$

Recipe: proving non-regularity via pigeonhole principle

Very similar to recipe for proving lower bound on number of states

To prove that L is non-regular

1. Find a **infinite** family of distinct prefixes $x_1, x_2, \ldots$, and corresponding suffixes $y_1, y_2, \ldots$, such that
   - $x_j y_k \in L \iff j = k$
   - i.e. for each prefix, exactly one of the suffixes can be concatenated to it to form a string in L
2. Prove non-existence of DFSA for L by contradiction
   - 2.1 Assume, for sake of contradiction, that there is a DFSA M such that $\mathcal{L}(M) = L$. Let n be its number of states.
   - 2.2 By the pigeonhole principle, there must be two different prefixes, x_j and x_k that go to the same state, q
   - 2.3 So $\delta^*(q, y_j)$ must be accepting and non-accepting. $\Rightarrow\Leftarrow$

Another approach: the Pumping Lemma

Use whichever approach you prefer. We'll ask you to prove non-regularity, but won't force you to use one approach or the other.

"pumping length"

Let L be a regular language. Then there exists $n \in \mathbb{N}$, such that for every $x \in L$ where $|x| \geq n$, x satisfies the following property:

- $\exists y, v, w \in \Sigma^*, x = uvw \wedge v \neq \varepsilon \wedge |uv| \leq n$, and $uv^k w \in L$ for all $k \in \mathbb{N}$

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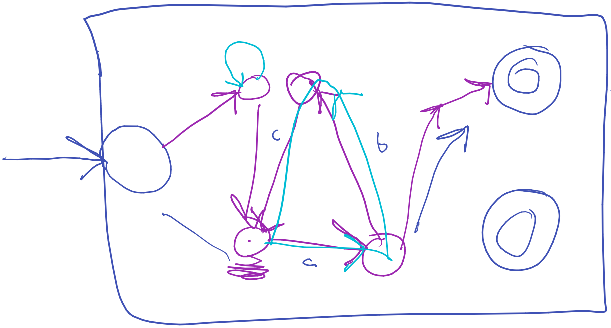
i.e.

If L is regular, then every sufficiently long string in L contains a (non-empty) part that can be repeated ("pumped") any number of times, to keep getting more strings in L.

Pumping Lemma proof sketch

The pigeonhole principle returns



$x = u(abc)w$

$u(abcabc)w$

$u(abc)^*w$

M, n states

$x \in L(M)$, $|x| \geq n$

Using Pumping Lemma to prove non-regularity: example

$10101 \in PAL$

WTS: PAL $= \{x \in \{0,1\}^* \mid x \text{ is a palindrome}\}$ is non-regular.
Assume, for sake of contradiction, that PAL is regular. Then the Pumping Lemma applies for some value $n \in \mathbb{N}$.

Consider $x = 0^n 1 0^n \in L$

$\exists m \in \mathbb{N}^+$, $0^{n+m} 1 0^n \in L$, but this is not a palindrome

$\Rightarrow\Leftarrow$