STA314 Linear Algebra - Part II

Projection, Eigendecomposition, SVD

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Brief Review from Part 1

- **Symmetric Matrix:**

$$A = A^T$$

- **Orthogonal Matrix:**

$$A^T A = AA^T = I \quad \text{and} \quad A^{-1} = A^T$$

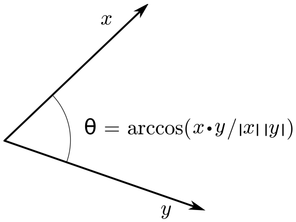
- **L2 Norm:**

$$||x||_2 = \sqrt{\sum_i x_i^2}$$

Angle Between Vectors

Dot product can be written in terms of their L2 norms and the angle θ between them.

$$x^T y = \sum_i x_i y_i = ||x||_2 ||y||_2 \cos(\theta)$$



Orthogonal Vectors: Two vectors x and y are orthogonal to each other if $x^T y = 0$

Vector Projection

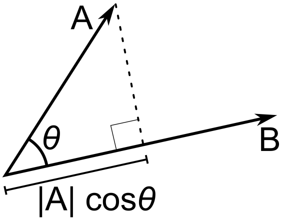
Given two vectors a and b, let

$$\hat{b} = \frac{b}{||b||}$$

be the unit vector in the direction of b.

Then $a_1 = a_1 \hat{b}$ is the orthogonal projection of a onto a straight line parallel to b, where

$$a_1 = ||a|| \cos(\theta) = a \cdot \hat{b} = a \cdot \frac{b}{||b||}$$



Diagonal Matrix

Diagonal matrix has mostly zeros with non-zero entries only in the diagonal, e.g. identity matrix.

A square diagonal matrix with diagonal elements given by entries of vector v is denoted:

$$\mathrm{diag}(v)$$

Determinant

- Determinant of a square matrix is a mapping to a scalar.

$$\det(A) \quad \text{or} \quad |A|$$

- Measures how much multiplication by the matrix expands or contracts the space.
- Determinant of product is the product of determinants:

$$\det(AB) = \det(A)\det(B)$$

- Example:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

List of Equivalencies

The following are all equivalent:

1. A is **invertible**, i.e. A^{-1} exists.
2. $Ax = b$ has a **unique solution**.
3. Columns of A are **linearly independent**.
4. $\det(A) \neq 0$
5. $Ax = 0$ has a unique, trivial solution: $x = 0$.

Zero Determinant

If $\det(A) = 0$, then:

- A is **linearly dependent**.
- $Ax = b$ has **no solution** or **infinitely many solutions**.
- $Ax = 0$ has a **non-zero solution**.

Eigenvectors

An **eigenvector** of a square matrix A is a nonzero vector v such that multiplication by A only changes the scale of v.

$$Av = \lambda v$$

The scalar λ is known as the **eigenvalue**.

If v is an eigenvector of A, so is any rescaled vector sv. Moreover, sv still has the same eigenvalue. Thus, we constrain the eigenvector to be of unit length:

$$||v|| = 1$$

Characteristic Polynomial

Eigenvalue equation of matrix A:

$$\begin{aligned} Av &= \lambda v \\ Av - \lambda v &= 0 \\ (A - \lambda I)v &= 0 \end{aligned}$$

If nonzero solution for v exists, then it must be the case that:

$$\det(A - \lambda I) = 0$$

Unpacking the determinant as a function of λ, we get:

$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda) = 0$$

The $\lambda_1, \lambda_2, ..., \lambda_n$ are roots of the **characteristic polynomial**, and are eigenvalues of A.

Example

Consider the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic polynomial is:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} = 3 - 4\lambda + \lambda^2 = 0$$

It has roots $\lambda = 1$ and $\lambda = 3$ which are the two **eigenvalues** of A.

We can then solve for eigenvectors using $Av = \lambda v$:

$$v_{\lambda=1} = [1, -1]^T \quad \text{and} \quad v_{\lambda=3} = [1, 1]^T$$

Eigendecomposition

Suppose that $n \times n$ matrix A has n linearly independent eigenvectors $\{v^{(1)}, ..., v^{(n)}\}$ with eigenvalues $\{\lambda_1, ..., \lambda_n\}$.

- Concatenate eigenvectors to form matrix V.
- Concatenate eigenvalues to form vector $\lambda = [\lambda_1, ..., \lambda_n]^T$.

The **eigendecomposition** of A is given by:

$$A = V\text{diag}(\lambda)V^{-1}$$

Symmetric Matrices

Every real **symmetric matrix** A can be decomposed into real-valued eigenvectors and eigenvalues:

$$A = Q\Lambda Q^T$$

- Q is an **orthogonal matrix** whose columns are unit eigenvectors of A, and Λ is a **diagonal matrix** of eigenvalues.
- We can think of A as scaling space by λ_j in direction $v^{(j)}$.

Positive Definite Matrix

- A matrix whose eigenvalues are all positive is called **positive definite**.
- If eigenvalues are positive or zero, then matrix is called **positive semidefinite**.

- Positive definite matrices guarantee that:

$$x^T A x > 0 \quad \text{for any nonzero vector } x$$

- Similarly, positive semidefinite guarantees:

$$x^T A x \geq 0$$

Singular Value Decomposition (SVD)

If A is not square, eigendecomposition is **undefined**.

SVD is a decomposition of the form:

$$A = UDV^T$$

- SVD is more general than eigendecomposition.
- **Every real matrix** has a SVD.

SVD Definition (1)

Write A as a product of three matrices: $A = UDV^\top$

- If A is $m \times n$, then U is $m \times m$, D is $m \times n$, and V is $n \times n$.
- U and V are **orthogonal matrices**, and D is a **diagonal matrix** (not necessarily square).

- Diagonal entries of D are called **singular values** of A.
- Columns of U are the **left singular vectors**, and columns of V are the **right singular vectors**.

SVD Definition (2)

SVD can be interpreted in terms of eigendecompostion.

- Left singular vectors of A are the **eigenvectors of AA^T**.
- Right singular vectors of A are the **eigenvectors of A^TA**.
- Nonzero singular values of A are **square roots of eigenvalues of A^TA and AA^T**.