Tutorial 3

$x_i \sim N(x; \mu, \sigma^2)$ i.i.d.

$\bar{\mu} = \sum_{i=1}^{N} \frac{x_i}{N}$

$x \sim N(x; \mu, \sigma^2)$ mutually indep of x_i

First, note that

$$\mathbb{E}[\bar{\mu}] = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[x_i] = \frac{1}{N}\sum_{i=1}^{N}\mu = \mu$$

1. Bayes error

$$\mathbb{E}[(x-\mu)^2] \overset{\text{def}}{=} \text{Var}(x) = \sigma^2$$

2. Bias

$$(\mathbb{E}[\bar{\mu}] - \mu)^2 = (\mu - \mu)^2 = 0$$

3. Variance

$$\text{Var}(\bar{\mu}) = \text{var}\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right) = \frac{1}{N^2}\text{var}\left(\sum_{i=1}^{N} x_i\right)$$

$$\overset{x_i \text{ indep}}{=} \frac{1}{N^2}\sum_{i=1}^{N}\text{var}\, x_i = \frac{1}{N^2}\sum_{i=1}^{N}\sigma^2 = \frac{\sigma^2}{N}$$

4. squared error risk

$$\mathbb{E}[(x-\bar{\mu})^2] = \mathbb{E}[x^2] - 2\mathbb{E}[x\bar{\mu}] + \mathbb{E}[\bar{\mu}^2]$$

$$= \mathbb{E}[x^2] - 2\mu^2 + \mathbb{E}[\bar{\mu}^2]$$

$$= \underbrace{\mathbb{E}[x^2] - \mu^2}_{= \mathbb{E}[(x-\mu)^2]} \underbrace{- \mu^2 + \mathbb{E}[\bar{\mu}^2]}_{\text{Var}(\bar{\mu})}$$

squared error risk

$$= \mathbb{E}[(x-\mu)^2] + \text{Var}(\bar{\mu}) = \sigma^2 + \frac{\sigma^2}{N}$$

$$= \frac{N+1}{N}\sigma^2$$

$\mathbb{E}[(x-\mu)^2]$: Bayes error

$\text{Var}(\bar{\mu})$: variance