On (Random-order) Online Contention Resolution Schemes for the Matching Polytope of (Bipartite) Graphs

Calum MacRury [*] Will Ma [†] Nathaniel Grammel [‡]

Abstract

We present new results for online contention resolution schemes for the matching polytope of graphs, in the random-order (RCRS) and adversarial (OCRS) arrival models. Our results include improved selectability guarantees (i.e., lower bounds), as well as new impossibility results (i.e., upper bounds). By well-known reductions to the prophet (secretary) matching problem, a c-selectable OCRS (RCRS) implies a c-competitive algorithm for adversarial (random order) edge arrivals. Similar reductions are also known for the query-commit matching problem. For the adversarial arrival model, we present a new analysis of the OCRS of Ezra et al. (EC, 2020). We show that this scheme is 0.344-selectable for general graphs and 0.349-selectable for bipartite graphs, improving on the previous 0.337 selectability result for this algorithm. We also show that the selectability of this scheme cannot be greater than 0.361 for general graphs and 0.382 for bipartite graphs. We further show that no OCRS can achieve a selectability greater than 0.4 for general graphs, and 0.433 for bipartite graphs.

For random-order arrivals, we present two attenuation-based schemes which use new attenuation functions. Our first RCRS is 0.474-selectable for general graphs, and our second is 0.476-selectable for bipartite graphs. These results improve upon the recent 0.45 (and 0.456) selectability results for general graphs (respectively, bipartite graphs) due to Pollner et al. (EC, 2022). On general graphs, our 0.474-selectable RCRS provides the best known positive result even for offline contention resolution, and also for the correlation gap. We conclude by proving a fundamental upper bound of 0.5 on the selectability of RCRS, using bipartite graphs.

[*] Department of Computer Science, University of Toronto, Toronto, ON, Canada. cmacrury@cs.toronto.edu

[†] Graduate School of Business, Columbia University, New York. wm2428@gsb.columbia.edu

[‡] Department of Computer Science, University of Maryland, College Park. ngrammel@umd.edu

1 Introduction

Contention resolution schemes provide a useful tool for selecting a subset of elements, originally motivated by constrained submodular optimization [11]. This paper studies contention resolution schemes for selecting a subset of edges forming a matching in a graph $G = (V, E)$. Initially, each edge $e \in E$ is either *active* or not, independently according to a known probability x_e. The scheme must then select a subset of active edges forming a *matching*, which has no two edges incident to the same vertex. The goal of the contention resolution scheme is to select every edge e with probability at least c conditional on it being active, for a constant c as large as possible, over both the randomness in the active edges and any randomness in the algorithm's selection.

Put another way, the scheme must select every edge e with (unconditional) probability at least cx_e. Letting $\partial(v)$ denote the set of edges incident to a vertex v, note that c must be arbitrarily small if $\sum_{e\in\partial(v)} x_e$ can be arbitrarily large, because many $e \in \partial(v)$ will be active while only one of them can be selected. Therefore, contention resolution schemes typically impose the vector $\boldsymbol{x} = (x_e)_{e\in E}$ to be feasible in a fractional relaxation, which in this case means lying in the matching polytope

$$\mathcal{P}_G := \{\boldsymbol{x} \in [0, 1]^E : \textstyle\sum_{e\in\partial(v)} x_e \leq 1 \ \forall v \in V\}. \tag{1.1}$$

A contention resolution scheme is then said to be *c-selectable* if for any graph G and any vector $\boldsymbol{x} \in \mathcal{P}_G$, it selects each edge e with probability at least cx_e, where c is a constant in [0,1].

Fractional constraints akin to (1.1) hold in many contexts: in auctions/pricing, if x_e denotes the ex-ante probability that an optimal mechanism sells to e; in prophet inequalities, if x_e denotes the probability that e has a high enough value worth accepting; and in stochastic probing, if x_e denotes the probability that edge e ends up matched. Broadly speaking, in these contexts x_e represents the decisions of an optimal clairvoyant algorithm, whereas real decisions have to be made sequentially, without knowing the stochastic realizations associated with future elements (but assuming them to be independent). In the language of contention resolution schemes, this translates to the elements e arriving one by one, with their activeness states revealed upon arrival (to 1 with probability x_e and 0 otherwise), and for elements that are active and feasible to select, the algorithm must make an irrevocable decision whether to actually select it. Contention resolution schemes of this sequential nature directly translate to approximation/competitive ratio guarantees for pricing [25], prophet [19, 14], and probing [5, 1, 6, 9] problems, with all of these papers focused on matchings in graphs.

Therefore, improving these sequential contention resolution schemes for graph matchings is of fundamental interest with direct implications, which is the focus of this paper. Two types of schemes have been defined in the literature depending on the order in which elements arrive: *online* contention resolution schemes (OCRS) [15], where this order is chosen by an adversary; and *random-order* contention resolution schemes (RCRS) [2, 23], where this order is chosen uniformly at random. OCRS and RCRS specifically for the matching polytope (1.1) have been considered by [14] and [9, 25] respectively. Like in [14], for OCRS we assume the adversary is *oblivious*, in that it fixes the arrival order based on the algorithm and cannot change the order based on realizations.

1.1 Contributions and Techniques

We improve state-of-the-art OCRS and RCRS for both general graphs and the special case of bipartite [1] graphs. We also derive many new impossibility results, and believe another contribution of this paper to lie in elucidating the limitations of different algorithms or analyses.

To describe our results, we define *selectability* as the maximum value of c for which an OCRS or RCRS is c-selectable, evaluated on the worst case graph G and vector $\boldsymbol{x} \in \mathcal{P}_G$ for the algorithm.

[1] Our OCRS positive result holds more generally for *triangle-free* graphs, which do not contain any 3-cycles.

Without further specification, selectability considers the best possible algorithm and takes a worst case over general graphs, although we also refer to the selectability of a specific algorithm or the selectability taken over bipartite graphs. By definition, the selectability of a specific algorithm is worse (smaller) than that of the best algorithm; the selectability for general graphs is worse than that for bipartite graphs; and the selectability of OCRS is worse than that of RCRS.

Given this understanding, our results are summarized in Table 1. We now describe each result individually, its significance, and the new techniques required to derive it.

Selectability Bounds	General Graphs	Bipartite Graphs
OCRS of [14]	≥ 0.337 [14] $\to \geq$ **0.344** [§2.1] $\leq$ **0.361** [§2.3]	≥ 0.337 [14] $\to \geq$ **0.349** [§2.2] ≤ 0.382 [§2.3, folklore]
Any OCRS	$\leq$ **0.4** [§2.3]	$\leq 4/9$ [19] $\to \leq$ **0.433** [§2.3]
RCRS	≥ 0.45 [25] $\to \geq$ **0.474** [§3.1]	≥ 0.456 [25] $\to \geq$ **0.476** [§3.2] ≤ 0.544 [21] $\to \leq$ **0.5** [§3.3]

Table 1: New results are **bolded**. "$\geq$" refers to lower bounds on c (algorithmic results), "$\leq$" refers to upper bounds (impossibility results), and arrows indicate improvement from state of the art.

Recall that the algorithm must select each edge e with probability at least cx_e. The algorithm is not rewarded for selecting e with probability greater than cx_e, so a common idea behind both OCRS and RCRS is to *attenuate* this probability, by only considering an edge e for selection when its activeness state and another independent random bit A_e both realize to 1. In this case, we say that e "survives", which occurs with a probability that can be calibrated to any value less than x_e. The algorithms we study are all *greedy* with respect to some appropriately-defined attenuation, i.e. they select any surviving edge that is feasible to select at its time of arrival.

Existing c-selectable OCRS. For OCRS the state of the art is a greedy OCRS that calibrates the survival probabilities so that every edge e is selected with probability *exactly* cx_e (the calibration is done by resampling the activeness of past edge arrivals) [14]. For this OCRS to be valid, when any edge $e = (u, v)$ arrives, it must be feasible to select (i.e. neither vertices u, v have already been matched) with probability at least c, so that there is the possibility of selecting e with probability at least cx_e. [14] show that $c = 1/3 \approx 0.333$ easily yields a valid algorithm. Then by arguing that the bad events of u being matched and v being matched cannot be perfectly negatively correlated, or equivalently by providing a non-trivial lower bound on the probability of both u and v being matched (not to each other), [14] show that the improved value of $c = 0.337$ is also valid.

Our improvements to OCRS. We analyze the same OCRS as [14] and provide substantial further improvements. First we show that $c = 0.349$ is valid for bipartite graphs. Note that when edge $e = (u, v)$ arrives, u is guaranteed to be matched if it has a neighbor u' such that: (i) edge (u, u') already arrived and survived; and (ii) no edge incident to u' that arrived before (u, u') survived. A neighbor v' of v satisfying (i)–(ii) can be defined analogously. Our result for bipartite graphs involves using the FKG inequality to show that u having such a neighbor u' is positively correlated with v having such a neighbor v', and moreover, whether two neighbors u_1, u_2 of u satisfy condition (ii) are independent (because there cannot be an edge between u_1 and u_2). Ultimately this reveals that the worst case for the existence of both u' and v' occurs when u, v are surrounded by edges with infinitesimally-small x-values, implying that $c = 0.349$ yields a valid algorithm.

Unfortunately, the preceding argument breaks down for general graphs, both because u' could be the same vertex as v', and because satisfying condition (ii) is no longer independent. To rectify this argument, we take an approach motivated by [14]—u and v will each randomly choose up to one neighbor satisfying (i), and hope that they end up choosing distinct vertices that also satisfy (ii), which would again certify both u and v to be matched. Our choice procedure is quite different[2] from [14], and designed so that the probabilities of two good events (u, v choosing any neighbors at all, and (ii) being satisfied) cannot be simultaneously minimized[3] in a worst-case configuration. Interestingly, this leads to a "hybrid" worst case for general graphs, in which both endpoints u, v of the arriving edge e neighbor a "large" vertex w with $x_{uw} = x_{vw} = 1/2$, but otherwise u, v are surrounded by edges with infinitesimally-small x-values. To prove that this hybrid is the worst case, we bound an infinite-dimensional optimization problem using a finite one with vanishing loss, and solve the finite one numerically. This worst case implies that $c = 0.344$ is valid.

Negative results for OCRS. To complement our positive results, we construct a simple example on which no OCRS can be more than 0.4-selectable, and the OCRS of [14] in particular is no more than 0.361-selectable. This example is related to the worst case from our analysis of general graphs above, in that it has an edge e connected to two "large" vertices w. Performance on this example also demonstrates the shortcoming of the greedy OCRS of [14]—it does not discriminate[4] between different states in which an arriving edge could be feasibly selected. We also derive negative results for OCRS on bipartite graphs—one showing that no OCRS can be more than 0.433-selectable (improving upon the upper bound of $4/9 \approx 0.444$ from [19]), and a simple one showing that the OCRS of [14] is no more than 0.382-selectable.

Existing c-selectable RCRS. For RCRS the state of the art also uses the attenuation framework, with the attenuation bit A_e in this case being set a priori to some value $a(e) \in [0, 1]$, where a is a function of the edge e. The challenge again lies in lower-bounding the probability of an arriving edge e being feasible for selection, in this case by $c/a(x_e)$. [9] lower-bound this probability using a condition similar to (ii) above—when $e = (u, v)$ arrives, if there are no edges incident to u or v that arrived before e and survived (i.e. are active with $A_e = 1$), then e must be feasible to select. We refer to these bad edges incident to u or v as *early*. [9] show for many attenuation functions a, in all of which $a(e)$ depends only on x_e, that the probability of e having no early edges is at least $c/a(x_e)$, with $c = (1 - e^{-2})/2 \approx 0.432$. [25] later identify a barrier of $(1 - e^{-2})/2$ for the analysis method of [9], and overcome it by deriving a lower bound on the probability of e having exactly one early edge, say $f = (u, w)$, but f being *blocked*, in that w was already matched when f arrived. Of course, this lower bound must be 0 if w is only incident to f, so [25] also use a more elaborate a function that heavily attenuates f in this case where $\partial(w) = \{f\}$. Combining these ingredients, [25] derive a 0.45-selectable RCRS, that is 0.456-selectable for bipartite graphs.

Our improvements to RCRS. We provide a substantially improved 0.474-selectable RCRS for general graphs. Our algorithm must slightly deviate from the attenuation framework by running the greedy RCRS on the *1-regularized* version of the graph G, which means that "phantom" edges and vertices are added to make $\sum_{e \in \partial(v)} x_e$ equal to 1 for all v. These phantom edges serve only the purpose of blocking early edges, and allow us to return to simpler attenuation functions based only on x_e (which would have been stuck at $(1 - e^{-2})/2$ without 1-regularity).

Restricting to these simple functions a that map x_e to a probability, our technique is to identify analytical properties of $a : [0, 1] \to [0, 1]$ that lead to characterizable worst-case configurations for the arriving $e = (u, v)$ having early edges and for these edges being blocked. First, conditioning

[2]The procedure in [14] uses a "sampler" under which each vertex u, v has a $1 - c$ probability of not choosing any neighbor at all. We instead design random bits R_{u,u_i} for the neighbors u_i of u, such that R_{u,u_i} implies condition (i), and u only chooses no neighbor in the unlikely event that all of these random bits realize to 0 (see Subsection 2.1).

[3]That is, the probability of random bit R_{u,u_i} realizing to 1 (which would ensure that u chooses a neighbor) is increasing in variable $x_{u_i}(u, u_i)$, while the lower bound on $\mathsf{alone}_{u_i}(u, u_i)$ (corresponding to condition (ii)) is decreasing in $x_{u_i}(u, u_i)$ (see Subsection 2.1).

[4]Contrast this with the tight OCRS for k-uniform matroids [20], which works similarly in that it selects every element e with probability exactly cx_e, but must prioritize selecting e on feasible states with fewer elements selected.

on the only early edge being say $f = (u, w)$, we formulate analytical constraints on function a under which the worst case (minimum probability) for f being blocked arises when w is incident to a *single* edge other than (u, w) and (v, w). Given this worst case for f being blocked, we can formulate further constraints on a under which the worst case for e having zero early edges or one blocked early edge arises when u, v are surrounded by edges f with infinitesimally-small x_f. We show that there exist functions $a : [0,1] \to [0,1]$ satisfying both sets of constraints, and taking the best one yields a 0.474-selectable RCRS for general graphs. Moreover, for bipartite graphs our constraints on a get looser (since the optimization for the worst case is more restricted), allowing us to push the envelope of feasible functions and find one that yields a 0.476-selectable RCRS.

We note that in essence, our 1-regularity reduction achieves the same goals as the elaborate[5] attenuation function from [25], but due to reduction of parameters it allows, we can better "engineer" worst-case configurations through the design of $a : [0,1] \to [0,1]$. We also find it interesting that our technique leads to the best-known RCRS despite using attenuation functions that do not take arrival time into account (as is required in [23, 25]). In fact, our 0.474-selectable RCRS based on these simple a functions improves the state of the art even for *offline* contention resolution schemes and correlation gaps on general graphs (see Subsection 1.2).

Negative result for RCRS. We show that no RCRS can be more than 1/2-selectable, on the complete bipartite graph with n vertices on each side and all edge values equal to $1/n$, as $n \to \infty$. This represents a fundamental barrier for RCRS which requires a non-trivial random graphs analysis, and to our knowledge was missing from the literature (the best existing upper bound implied for RCRS comes from the expected *offline* maximum matching in this complete bipartite graph being less than $0.544n$ as $n \to \infty$ [21]). We also note that no better construction is known for general graphs.

The main challenge lies in quantifying for this complete bipartite graph that the information gained by the RCRS from knowing which edges have already arrived (and hence won't arrive again in the future) has negligible benefit as $n \to \infty$. This allows us to essentially reduce to a problem where each edge is drawn independently *with replacement* uniformly from the n^2 possibilities, on which a greedy policy is optimal, and then through a differential equation based method prove that the selectability is at most 1/2.

1.2 Related Work

Our paper studies both OCRS and RCRS for the matching polytope and provides a comprehensive set of positive and negative results, making improvements on all fronts (see Table 1). The interest in OCRS and RCRS and their applications for auctions/pricing, prophet inequalities, stochastic probing, etc. have already been discussed in the introduction, and we refer to the literature cited there. We now mention some further connections, and other feasibility constraints on the selected subset for which OCRS and RCRS have been studied.

Offline contention resolution and correlation gap for the matching polytope. A more lenient form of contention resolution than OCRS or RCRS is the *offline* setting, where all of the activeness states are revealed before selections have to be made. The best-possible selectability of an offline contention resolution scheme is equal to the *correlation gap* [11], a related concept. Surprisingly, the best known offline contention resolution scheme for general graphs was actually the 0.45-selectable RCRS of [25], which improved upon the selectability of 0.4326 from [10]. Therefore, our 0.474-selectable RCRS represents the state of the art for both offline contention resolution and correlation gap, when it comes to general graphs.

[5] On 1-regular graphs, the term "s_e" from their function which penalizes large neighborhoods always equals x_e.

For bipartite graphs, a 0.4762-selectable offline contention resolution scheme is known, and in fact best-possible if the scheme has to be *monotone* [10]. Our 0.476-selectable RCRS is slightly worse, but the fact that 0.4762 is tight and our analysis could potentially be further improved suggests that RCRS is likely easier than monotone offline contention resolution on bipartite graphs. We should acknowledge that our OCRS and RCRS are generally not monotone, a property of interest in the papers [11, 10, 15] that is relevant for submodular optimization.

Other feasibility constraints. For general matroids, a 1/2-selectable OCRS [22, 23] and $(1-1/e)$-selectable RCRS [13, 23] are known, and both of these selectabilities are tight in the special case of a 1-uniform matroid. The selectability improves in the case of a k-uniform matroid. For $k > 1$, a tight γ_k^*-selectable OCRS was recently derived [20], where γ_k^* is a constant greater than a well-known lower bound of $1 - 1/\sqrt{k+3}$ [3]. Meanwhile, for RCRS the greater selectability of $(1 - e^{-k}k^k/k!)$ is possible [4], which matches the correlation gap constant from [29]. That is, for k-uniform matroids there is no separation between random-order vs. offline contention resolution, but the selectability does degrade under adversarial order.

The knapsack polytope is another well-studied object that captures k-uniform matroids (but is orthogonal to general matroids). For the knapsack polytope, a tight $1/(3+e^{-2}) \approx 0.319$-selectable OCRS [20] is known and no improved RCRS is known.

The matching polytope also happens to (orthogonally) capture k-uniform matroids in the special case of a complete bipartite graph. For general matching polytopes, a distinguishing challenge is that flipping whether a single edge is active can set off a domino effect on which edges get matched, and tight results for OCRS and RCRS are not known. We note however that under the *vertex-arrival* matching model, a tight 1/2-selectable OCRS [14] is known, as is an 8/15-selectable RCRS [18].

2 Online Contention Resolution Schemes

Definition 1 (Terminology and Notation)**.** Let $G = (V, E)$ be a graph. An edge $e = (u, v)$ is said to be *incident* to vertices u and v, and v is said to be a *neighbor* of u (and vice versa). For any vertex $v \in V$, let $\partial(v) \subseteq E$ denote the set of edges incident to v, and for any $e = (u, v) \in E$, let $\partial(e) := \partial(u) \cup \partial(v) \setminus \{e\}$. A *matching* M is a subset of edges no two of which are incident to the same vertex, i.e. satisfying $|M \cap \partial(v)| \leq 1$ for all $v \in V$. A vector $\boldsymbol{x} \in [0,1]^E$ lies in the *matching polytope* if $\sum_{e \in \partial(v)} x_e \leq 1$ for all $v \in V$. In this case, we refer to $\boldsymbol{x}$ as a *fractional matching* for G.

Fixing a fractional matching $\boldsymbol{x} = (x_e)_{e \in E}$, each edge e has an *activeness state* X_e that realizes to 1 with probability (w.p.) x_e and 0 w.p. $1 - x_e$, independent of everything else. We denote this random draw as $X_e \sim \mathrm{Ber}(x_e)$, where $\mathrm{Ber}(x)$ represents an independent Bernoulli random variable of mean x for any $x \in [0,1]$. Edges e with $X_e = 1$ are called *active*, and only these edges can be selected, under the additional constraint that the selected subset must form a matching.

At the time an edge $e \in E$ arrives, we say that a vertex $v \in V$ is *matched* if an edge incident to v that has already arrived has been selected. We denote this event using $\mathsf{matched}_v(e)$, noting that it depends on the random active states of edges arriving before e and any randomness in the algorithm. We say that an edge $e = (u, v)$ is *blocked* if either u or v has been matched by the time e arrives, and denote this event using $\mathsf{blocked}(e)$. Blocked edges, even if active, cannot be selected.

Our improved lower bound for OCRS is based on a new analysis of the algorithm of [14], which we restate below using our terminology.

Algorithm 1 OCRS of [14]

Input: $G = (V, E)$, $\boldsymbol{x} = (x_e)_{e \in E}$, and $c \in [0, 1]$ a constant to be determined later
Output: subset of active edges forming a matching $\mathcal{M}$

$\mathcal{M} \leftarrow \emptyset$
for arriving edges e **do**
  Let $\alpha_e := c/\mathbb{P}[\overline{\mathsf{blocked}(e)}]$, where the denominator is the probability that edge e is not blocked, taken over the randomness in the activeness of past edges and the algorithm
  Draw $A_e \sim \mathrm{Ber}(\alpha_e)$
  if e is active, not blocked, and $A_e = 1$ **then**
    $\mathcal{M} \leftarrow \mathcal{M} \cup \{e\}$
return $\mathcal{M}$

Remark 2.1. In Algorithm 1, α_e is a probability over the hypothetical scenarios that could have occurred, based on what the OCRS knows about the edges that have arrived so far. Computing these probabilities exactly requires tracking exponentially many scenarios, but fortunately sampling these scenarios yields an ε loss in selectability given $O(1/\varepsilon)$ runtime [14]. We also note that the values of α_e used by Algorithm 1 are fixed once the graph and order of edge arrival are determined. This is where the assumption that the adversary is *oblivious* comes in—the order, and hence the values of α_e, must be independent of any realizations.

We now define some further concepts specific to Algorithm 1. We say that an edge e *survives* if both X_e and A_e realize to 1, and we let $S_e = X_e A_e$ indicate this event, which is an independent Bernoulli random variable with mean $x_e \alpha_e$. The OCRS of [14] can then be concisely described as "select every surviving unblocked edge". The survival probabilities are calibrated so that

$$\mathbb{P}[e \in \mathcal{M}] = x_e \alpha_e \mathbb{P}[\overline{\mathsf{blocked}(e)}] = c x_e \qquad \forall e \in E \tag{2.1}$$

(by definition of α_e), resulting in a c-selectable OCRS.

However, Algorithm 1 only defines a valid OCRS if α_e is a probability in [0,1] for all $e \in E$. Put another way, constant c must be small enough such that

$$\mathbb{P}[\overline{\mathsf{blocked}(e)}] \geq c \tag{2.2}$$

for every graph G, fractional matching $\boldsymbol{x}$, and arriving edge e (which would ensure that $\alpha_e \leq 1$). Following [14], validity can be inductively established by assuming (2.2) holds for all e under a given G, $\boldsymbol{x}$, and arrival order, and then proving that it also holds for an arbitrary edge $e \notin E$ which could arrive next. Ezra et al. [14] further observe that if this newly arriving edge is $e = (u, v)$, then

$$\begin{aligned} \mathbb{P}[\overline{\mathsf{blocked}(e)}] &= 1 - \mathbb{P}[\mathsf{matched}_u(e) \cup \mathsf{matched}_v(e)] \\ &= 1 - \mathbb{P}[\mathsf{matched}_u(e)] - \mathbb{P}[\mathsf{matched}_v(e)] + \mathbb{P}[\mathsf{matched}_u(e) \cap \mathsf{matched}_v(e)] \\ &= 1 - c\textstyle\sum_{f \in \partial(u)} x_f - c\sum_{f \in \partial(v)} x_f + \mathbb{P}[\mathsf{matched}_u(e) \cap \mathsf{matched}_v(e)] \end{aligned} \tag{2.3}$$

(where the final equality holds by (2.1) and the induction hypothesis). Therefore, the real challenge and intricacy of the problem lies in bounding the term $\mathbb{P}[\mathsf{matched}_u(e) \cap \mathsf{matched}_v(e)]$, which is related to the correlation between u and v being matched (to different partners) in the past.

2.1 New Analysis of Algorithm 1 for General Graphs

We present a new way of analyzing, given a newly arriving edge (u, v), the probability of both u, v being matched. This will allow us to show that Algorithm 1 remains valid for $c = 0.344$.

We consider the following sufficient condition for both u, v being matched. Suppose u inspects all its surviving incident edges, and chooses one (if any exist), and v (independently) does the same. If these chosen edges are (u, u') and (v, v'), where u' and v' are vertices in $V \setminus \{u, v\}$, then we call u' and v' the *candidates* of u and v, respectively. Now, if candidate u' was *alone* in that it had no surviving incident edges at the time of arrival of (u, u'), then this guarantees vertex u to be matched, either to u', or via a surviving incident edge that arrived before (u, u'). A similar argument can be made for candidate v' of vertex v. Therefore, if u' and v' are *distinct* candidates, and both alone at the arrival times of (u, u') and (v, v') respectively, then this guarantees both u and v to be matched.

We note that [14] take a similar approach, but our procedure for choosing candidates is quite different from their "sampler", and generally more likely to choose any candidate at all. Let $u_1, \ldots, u_k$ be vertices in $V \setminus \{u, v\}$ such that $(u, u_1), \ldots, (u, u_k)$ are the edges in E incident to u (recall that E does not include the newly arriving edge (u, v)). If u has multiple surviving edges (u, u_i) it will prioritize choosing the one with the smallest index i; however, it adds some noise to reduce the likelihood that v (after defining an analogous procedure) will choose the same candidate. The ordering of vertices $u_1, \ldots, u_k$ will be specified later based on the analysis.

To add this noise, we define a random bit R_{u,u_i} for each $i = 1, \ldots, k$. We couple R_{u,u_i} with S_{u,u_i} (the random bit for edge (u, u_i) surviving) so that R_{u,u_i} and S_{u,u_i} are *perfectly positively correlated.* Vertex u then chooses u_i as its candidate if i is the smallest index for which R_{u,u_i} realizes to 1. We let $\mathsf{candidate}^u_{u_i}$ denote this event, noting that u can have at most one candidate, and possibly none. Now, although the random bits R_{u,u_i} are coupled with S_{u,u_i}, the bits S_{u,u_i} are independent from everything else, so we can use independence to deduce that

$$\mathbb{P}[\mathsf{candidate}^u_{u_i}] = \mathbb{E}[R_{u,u_i}] \prod_{i'<i} (1 - \mathbb{E}[R_{u,u_{i'}}]) \qquad \forall i = 1, \ldots, k. \tag{2.4}$$

We define an analogous procedure for the edges $(v, v_1), \ldots, (v, v_\ell)$ incident to v. We will specify the means of the random bits R_{u,u_i} and R_{v,v_j} later, after proving some lemmas that bound the probabilities of edges surviving.

Definition 2. Let $e = (u', v')$ be an edge that has already arrived, with u', v' being generic vertices in $V \setminus \{u, v\}$ (not necessarily candidates). Let $x_{u'}(e) := \sum_{f \in \partial(u'): f \prec e} x_f$, where $f \prec e$ indicates that the edge f arrived before e (the sum does not include edge e itself). Similarly, let $x_{v'}(e) := \sum_{f \in \partial(v'): f \prec e} x_f$.

Meanwhile, let $\mathsf{alone}_{u'}(u', v')$ (respectively $\mathsf{alone}_{v'}(u', v')$) denote the event that u' (respectively v') does not have any surviving incident edges at the time of arrival[6] of edge (u', v').

Proposition 2.2. *For any edge $e = (u', v')$, the probability of it surviving satisfies*

$$\frac{cx_e}{1 - c \cdot \max\{x_{u'}(e), x_{v'}(e)\}} \le \mathbb{P}[S_e = 1] \le \frac{cx_e}{1 - cx_{u'}(e) - cx_{v'}(e)}.$$

Proof of Proposition 2.2. The proof follows easily from (2.3): the induction hypothesis implies that

$$1 - cx_{u'}(e) - cx_{v'}(e) \le \mathbb{P}[\overline{\mathsf{blocked}(e)}] \le 1 - \max\{cx_{u'}(e), cx_{v'}(e)\},$$

where we note that $\mathbb{P}[\mathsf{matched}_{u'}(e) \cup \mathsf{matched}_{v'}(e)] \ge \max\{\mathbb{P}[\mathsf{matched}_{u'}(e)], \mathbb{P}[\mathsf{matched}_{v'}(e)]\}$. Recalling that $\mathbb{P}[S_e = 1] = x_e \alpha_e$ with α_e defined to equal $c/\mathbb{P}[\overline{\mathsf{blocked}(e)}]$, this completes the proof. □

[6] We note that [14] use a similar notion in their definition of "witness", but without the qualifier "at time of arrival of (u', v')". We need this qualifier in order to make our subsequent argument.

Proposition 2.3. *For any edge $e = (u', v')$, the probability of a vertex u' being alone satisfies*

$$\mathbb{P}[\mathsf{alone}_{u'}(e)] \geq \frac{1 - c - cx_{u'}(e)}{1 - c}.$$

Proof of Proposition 2.3. Let $(u', w_1), \ldots, (u', w_m)$ be the edges incident to u' arriving before (u', v'), in that order. Note that $x_{u'}(e) = \sum_{i=1}^{m} x_{u',w_i}$. We can use independence to derive

$$\begin{aligned}
\mathbb{P}[\mathsf{alone}_{u'}(e)] &= \prod_{i=1}^{m}(1 - \mathbb{P}[S_{u',w_i}]) \\
&\geq \prod_{i=1}^{m}\left(1 - \frac{cx_{u',w_i}}{1 - cx_{u'}(u', w_i) - cx_{w_i}(u', w_i)}\right) \\
&\geq \prod_{i=1}^{m}\left(1 - \frac{cx_{u',w_i}}{1 - c\sum_{j<i} x_{u',w_j} - c}\right) \\
&= \prod_{i=1}^{m}\frac{1 - c - c\sum_{j\leq i} x_{u',w_j}}{1 - c - c\sum_{j<i} x_{u',w_j}} \\
&= \frac{1 - c - cx_{u'}(e)}{1 - c}
\end{aligned}$$

where the first inequality uses the upper bound in (2.2), and the second inequality uses the definition that $x_{u'}(u', w_i) = \sum_{j<i} x_{u',w_j}$ and the fact that $x_{w_i}(u', w_i) \leq 1$. This leads to the desired result. □

The lower bound in Proposition 2.2 allows us to define the probabilities for the random bits R_{u,u_i}. We would like to ensure that whenever u chooses vertex u_i as its candidate, edge (u, u_i) actually survives. This will be the case whenever $\mathbb{E}[R_{u,u_i}] \leq \mathbb{E}[S_{u,u_i}]$, since the bits R_{u,u_i}, S_{u,u_i} are coupled using perfect positive correlation. By Proposition 2.2, this is ensured if we set

$$\mathbb{E}[R_{u,u_i}] := \frac{cx_{u,u_i}}{1 - cx_{u_i}(u, u_i)} \tag{2.5}$$

for all $i = 1, \ldots, k$, and similarly set $\mathbb{E}[R_{v,v_j}] := \frac{cx_{v,v_j}}{1-cx_{v_j}(v,v_j)}$ for all $j = 1, \ldots, \ell$. These values are set so that if $x_{u_i}(u, u_i)$ is large, which worsens the lower bound of $\frac{1-c-cx_{u_i}(u,u_i)}{1-c}$ on the probability of u_i being alone, then at least we have the consolation prize that $\mathbb{E}[R_{u,u_i}]$ is large, making it more likely that u has a candidate. This will prevent a worst-case configuration from simultaneously minimizing the two good events of u_i being alone and u having a candidate, which is precisely the motivation behind our choice procedure and definition of $\mathsf{alone}_{u_i}(u, u_i)$ that differs from [14].

Having defined these random bits, we are ready to state and prove our main result, which lower-bounds the selectability of Algorithm 1 using an elementary optimization problem.

Definition 3. For any positive integer k and non-negative real number b, let

$$\begin{aligned}
\mathrm{AdvMin}_k(b) := \inf\ & b^2 \left(\sum_{i=1}^{k} \frac{y_i - by_i + by_i^2}{1+by_i} \prod_{i'<i} \frac{1}{1+by_{i'}} \right) \left(\sum_{i=1}^{k} \frac{z_i - bz_i + bz_i^2}{1+bz_i} \prod_{i'<i} \frac{1}{1+bz_{i'}} \right) \\
& - b^2 \sum_{i=1}^{k} \frac{y_i - by_i + by_i^2}{1+by_i} \frac{z_i - bz_i + bz_i^2}{1+bz_i} \prod_{i'<i} \frac{1}{1+by_{i'}} \frac{1}{1+bz_{i'}} \\
\text{s.t.}\ & \sum_{i=1}^{k} y_i = \sum_{i=1}^{k} z_i = 1 \\
& y_i + z_i \leq 1 \qquad \forall i = 1, \dots, k \\
& y_{i-1} \geq y_i \qquad \forall i = 4, \dots, k \\
& z_{i-1} \geq z_i \qquad \forall i = 4, \dots, k \\
& y_i, z_i \geq 0 \qquad \forall i = 1, \dots, k.
\end{aligned}$$

Theorem 2.4.

(i). Algorithm 1 is c-selectable for any c satisfying $1 - 3c + \inf_k \mathrm{AdvMin}_k(\frac{c}{1-c}) \geq 0$.

(ii). $c = 0.3445$ *satisfies* $1 - 3c + \inf_k \mathrm{AdvMin}_k(\frac{c}{1-c}) \geq 0$.

Therefore, Algorithm 1 provides a 0.3445-selectable OCRS for general graphs.

Note that for fixed b, $\mathrm{AdvMin}_k(b)$ is decreasing in k, so $\inf_k \mathrm{AdvMin}_k(b) = \lim_{k\to\infty} \mathrm{AdvMin}_k(b)$.

Remark 2.5. Part (ii) of Theorem 2.4 is proved with the aid of computational verification, after bounding the difference between $\lim_{k\to\infty} \mathrm{AdvMin}_k(b)$ and an optimization problem with $2K$ variables as $O(1/K)$. We then use Non-Linear Programming (NLP) solver COUENNE, modeled with JuMP [12], providing a link to the code. The NLP solver establishes a *provable lower bound* on the infimum value of this finite-dimensional NLP, allowing us to finish the proof. Interestingly, the optimal solution suggested by the solver for a large K is a "hybrid" in which $y_1 = z_1 = 1/2$, and all other values of y_i, z_i are infinitesimally-small.

Proof of Theorem 2.4, (i). Recall from (2.2) and (2.3) that it suffices to show that $\mathbb{P}[\mathsf{matched}_u(u,v) \cup \mathsf{matched}_v(u,v)] \leq 1 - c$ for the newly arriving edge (u,v). Note that if $\sum_{f\in\partial(u)} x_f < 1$, then $\mathbb{P}[\mathsf{matched}_u(u,v) \cup \mathsf{matched}_v(u,v)]$ can only be increased after adding a dummy edge between u and a new vertex that is active with probability $1 - \sum_{f\in\partial(u)} x_f$, which arrives right before (u,v). The same argument can be made if $\sum_{f\in\partial(v)} x_f < 1$. Therefore, we can without loss of generality assume that $\sum_{f\in\partial(u)} x_f = \sum_{f\in\partial(v)} x_f = 1$, which represents the hardest case for $\mathbb{P}[\mathsf{matched}_u(u,v) \cup \mathsf{matched}_v(u,v)] \leq 1 - c$ to be satisfied. Rewriting $\mathbb{P}[\mathsf{matched}_u(u,v) \cup \mathsf{matched}_v(u,v)]$ following (2.3), it suffices for c-selectability to show that

$$0 \leq 1 - 3c + \mathbb{P}[\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)].$$

Therefore, we must show $\mathbb{P}[\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)] \geq \inf_k \mathrm{AdvMin}_k(\frac{c}{1-c})$, where we have assumed that $\sum_{f\in\partial(u)} x_f = \sum_{f\in\partial(v)} x_f = 1$. Recall that $\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)$ occurs whenever all four events $\mathsf{candidate}^u_{u_i}$, $\mathsf{candidate}^v_{v_j}$, $\mathsf{alone}_{u_i}(u,u_i)$, and $\mathsf{alone}_{v_j}(v,v_j)$ occur, for any choice of indices $i \in \{1,\dots,k\}, j \in \{1,\dots,\ell\}$ such that the vertices u_i, v_j do not coincide. This is because $\mathsf{candidate}^u_{u_i}$ implies $R_{u,u_i} = 1$, which implies edge (u,u_i) survives (see (2.5)), and this in conjunction with $\mathsf{alone}_{u_i}(u,u_i)$ ensures that $\mathsf{matched}_u(u,v)$ occurs. An analogous argument ensures that $\mathsf{matched}_v(u,v)$ occurs, assuming u_i is not the same vertex as v_j. Finally, we note that since u and v each choose at most one candidate, the events $\mathsf{candidate}^u_{u_i} \cap \mathsf{candidate}^v_{v_j} \cap \mathsf{alone}_{u_i}(u,u_i) \cap \mathsf{alone}_{v_j}(v,v_j)$ are disjoint across the different combinations of i,j. Therefore, we can derive

$$\mathbb{P}[\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)] \\ \geq \sum_{i,j:u_i \neq v_j} \mathbb{P}[\mathsf{candidate}^u_{u_i} \cap \mathsf{candidate}^v_{v_j} \cap \mathsf{alone}_{u_i}(u,u_i) \cap \mathsf{alone}_{v_j}(v,v_j)]. \quad (2.6)$$

The next step consists in showing that for any combination of i,j such that $u_i \neq v_j$, the probability term on the r.h.s. is lower-bounded by the independent case, i.e.

$$\mathbb{P}[\mathsf{candidate}^u_{u_i} \cap \mathsf{candidate}^v_{v_j} \cap \mathsf{alone}_{u_i}(u,u_i) \cap \mathsf{alone}_{v_j}(v,v_j)] \\ \geq \mathbb{P}[\mathsf{candidate}^u_{u_i}]\mathbb{P}[\mathsf{candidate}^v_{v_j}]\mathbb{P}[\mathsf{alone}_{u_i}(u,u_i)]\mathbb{P}[\mathsf{alone}_{v_j}(v,v_j)]. \quad (2.7)$$

We argue this using the FKG inequality. Consider the bits $\{S_e : e \in E\}$ about the survival of the edges. Note that all four events $\mathsf{candidate}^u_{u_i}, \mathsf{candidate}^v_{v_j}, \mathsf{alone}_{u_i}(u,u_i), \mathsf{alone}_{v_j}(v,v_j)$ are fully determined by these bits, and moreover are increasing in the bits S_{u,u_i}, S_{v,v_j} (they must necessarily be 1 for $\mathsf{candidate}^u_{u_i}, \mathsf{candidate}^v_{v_j}$ to be 1, and note that this does not adversely affect $\mathsf{alone}_{u_i}(u,u_i), \mathsf{alone}_{v_j}(v,v_j)$ since $u_i \neq v_j$), and decreasing in all bits S_e when e is not (u,u_i) or (v,v_j). Since the bits S_e are independent across e, we have that (2.7) holds, for any i,j such that $u_i \neq v_j$.

Now, we can use (2.4) and Proposition 2.3 to lower-bound $\mathbb{P}[\mathsf{candidate}^u_{u_i}]$ and $\mathbb{P}[\mathsf{alone}_{u_i}(u,u_i)]$ respectively. Therefore, we derive

$$\begin{aligned} \mathbb{P}[\mathsf{candidate}^u_{u_i}]\mathbb{P}[\mathsf{alone}_{u_i}(u,u_i)] &\geq \frac{1-c-cx_{u_i}(u,u_i)}{1-c} \frac{cx_{u,u_i}}{1-cx_{u_i}(u,u_i)} \prod_{i'<i}(1-\frac{cx_{u,u_{i'}}}{1-cx_{u_{i'}}(u,u_{i'})}) \\ &\geq \frac{1-c-c(1-x_{u,u_i})}{1-c} \frac{cx_{u,u_i}}{1-c(1-x_{u,u_i})} \prod_{i'<i}(1-\frac{cx_{u,u_{i'}}}{1-c(1-x_{u,u_{i'}})}) \\ &= \frac{1-2c+cx_{u,u_i}}{1-c+cx_{u,u_i}} \frac{cx_{u,u_i}}{1-c} \prod_{i'<i} \frac{1-c}{1-c+cx_{u,u_{i'}}} \end{aligned} \quad (2.8)$$

where the second inequality holds because the first expression is decreasing in both $x_{u_i}(u,u_i)$ and $x_{u_{i'}}(u,u_{i'})$, which must satisfy $x_{u_i}(u,u_i) \leq 1-x_{u,u_i}$ and $x_{u_{i'}}(u,u_{i'}) \leq 1-x_{u,u_{i'}}$ respectively.

Combining the derivations in (2.6), (2.7), and (2.8) (and lower bounding the analogous expression $\mathbb{P}[\mathsf{candidate}^v_{v_j}]\mathbb{P}[\mathsf{alone}_{v_j}(v,v_j)]$), we see that $\mathbb{P}[\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)]$ is at least

$$\sum_{i,j:u_i \neq v_j} \left(\frac{1-2c+cx_{u,u_i}}{1-c+cx_{u,u_i}} \frac{cx_{u,u_i}}{1-c} \prod_{i'<i} \frac{1-c}{1-c+cx_{u,u_{i'}}} \right) \left(\frac{1-2c+cx_{v,v_j}}{1-c+cx_{v,v_j}} \frac{cx_{v,v_j}}{1-c} \prod_{j'<j} \frac{1-c}{1-c+cx_{v,v_{j'}}} \right). \quad (2.9)$$

Finally, to relate to $\inf_k \mathrm{AdvMin}_k(\frac{c}{1-c})$, let $U(i)$ denote the first expression in large parentheses in (2.9), and let $V(j)$ denote the second expression in large parentheses in (2.9). We can assume without loss that $k = \ell = |V|-2$, by adding edges with $x_{u,u_i} = 0$ or $x_{v,v_j} = 0$ as necessary, in which case $U(i) = 0$ or $V(j) = 0$ respectively. This allows us to rewrite (2.9) as

$$\sum_{i=1}^k U(i) \sum_{j=1}^k V(j) - \sum_{i,j:u_i=v_j} U(i)V(j). \quad (2.10)$$

This is where we specify the ordering of the vertices $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ in a way that aids our analysis. We specify u_1 so that $x_{u,u_1} = \max_i x_{u,u_i}$, and similarly specify v_1 so that $x_{v,v_1} = \max_j x_{v,v_j}$. We let $v_2 = u_1$, and similarly $u_2 = v_1$; if $u_1 = v_1$ then we instead let $u_2 = v_2$ be any other vertex in $V \setminus \{u, v, u_1\}$. We have completed the specification of u_1, u_2, v_1, v_2 in a way such that $\{u_1, u_2\} = \{v_1, v_2\}$. Hence, both $u_3, \ldots, u_k$ and $v_3, \ldots, v_k$ must be orderings of the vertices in $V \setminus \{u, v, u_1, u_2\}$. We define these orderings in such a way so that $x_{u,u_3} \geq \cdots \geq x_{u,u_k}$ and $x_{v,v_3} \geq \cdots \geq x_{v,v_k}$. This implies $U(3) \geq \cdots \geq U(k)$ and $V(3) \geq \cdots \geq V(k)$.

We have completed the specification of the orderings $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$. Now, consider an adversary trying to design the values of $x_{u,u_1}, \ldots, x_{u,u_k}, x_{v,v_1}, \ldots, x_{v,v_k}$ to minimize expression (2.10), subject to all aforementioned constraints. By the rearrangement inequality, the sum being subtracted is maximized if the largest values of $U(i)$ are paired with the largest values of $V(j)$. That is, the adversary wants $u_i = v_i$ for all $i = 3, \ldots, k$. Moreover, this assignment of vertices is guaranteed to feasibly satisfy $x_{u,u_i} + x_{v,v_i} \leq 1$, since both x_{u,u_i} and x_{v,v_i} must be at most $1/2$ (recall that $x_{u,u_i} \leq x_{u,u_1}$ and $x_{u,u_i} + x_{u,u_1} \leq 1$). Therefore, if we assume that $u_i = v_i$ for all $i = 3, \ldots, k$, then this only provides a lower bound on expression (2.10).

To finish, let $b := \frac{c}{1-c}$. We define shorthand notation $y_i := x_{u,u_i}$ and $z_i := x_{v,v_i}$ for all $i = 3, \ldots, k$, as well as y_1, y_2, z_1, z_2 such that y_1 and z_1 correspond to the same vertex (and y_2 and z_2 correspond to the same vertex). We drop the constraint that at least one of y_1, y_2 must correspond to a maximal value of x_{u,u_i} (and similarly for z_1, z_2). Noting that $U(i)$ can be rewritten as $\frac{1-b+by_i}{1+by_i} by_i \prod_{i'<i} \frac{1}{1+by_{i'}}$ under the new notation (and similarly for $V(j)$), we can express the adversary's optimization problem as minimizing

$$\left(\sum_{i=1}^{k} \frac{1-b+by_i}{1+by_i} by_i \prod_{i'<i} \frac{1}{1+by_{i'}} \right) \left(\sum_{i=1}^{k} \frac{1-b+bz_i}{1+bz_i} bz_i \prod_{i'<i} \frac{1}{1+bz_{i'}} \right)$$
$$- \sum_{i=1}^{k} \frac{1-b+by_i}{1+by_i} by_i \frac{1-b+bz_i}{1+bz_i} bz_i \prod_{i'<i} \frac{1}{1+by_{i'}} \frac{1}{1+bz_{i'}}$$

subject to constraints $\sum_{i=1}^k y_i = \sum_{i=1}^k x_{u,u_i} = 1 = \sum_{i=1}^k z_i = \sum_{i=1}^k x_{v,v_i}$ (recall the assumption that $\sum_{f \in \partial(u)} x_f = \sum_{f \in \partial(v)} x_f = 1$), constraint $y_i + z_i \leq 1$ for all $i = 1, \ldots, k$, and constraints $y_3 \geq \cdots \geq y_k, z_3 \geq \cdots \geq z_k$ (due to the ordering chosen by the algorithm) as well as non-negativity constraints. This is a lower bound on the original expression in (2.9), and is exactly the $\mathrm{AdvMin}_k(b)$ optimization problem. Therefore, $\mathbb{P}[\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)] \geq \mathrm{AdvMin}_k(b)$ where k denotes the number of vertices in $V \setminus \{u, v\}$. To ensure $1 - 3c + \mathbb{P}[\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)] \geq 0$, it suffices to ensure $1 - 3c + \inf_k \mathrm{AdvMin}_k(b)$, completing the proof of Theorem 2.4, part (i).

□

Proof of Theorem 2.4, (ii). Fix a large positive integer "cutoff" K and consider any $k \geq K$. Since any term subtracted in the latter sum in the objective of $\mathrm{AdvMin}_k(b)$ also appears when the first two large parentheses are expanded, the objective can only be reduced if we reduce the term

$$\frac{y_i - by_i + by_i^2}{1+by_i} \prod_{i'<i} \frac{1}{1+by_{i'}} \tag{2.11}$$

for any index i. To reduce this term, note that $\frac{y_i - by_i + by_i^2}{1+by_i} = \frac{1-b+by_i}{1+by_i} y_i \geq (1-b)y_i$ and $\frac{1}{1+by_{i'}} \geq 1 - by_{i'}$, which allows us to reduce (2.11) to $(1-b)y_i \prod_{i'<i}(1 - by_{i'})$. We can similarly lower bound $\frac{z_i - bz_i + bz_i^2}{1+bz_i} \prod_{i'<i} \frac{1}{1+bz_{i'}}$ by $(1-b)z_i \prod_{i'<i}(1 - bz_{i'})$. Therefore, the objective of $\mathrm{AdvMin}(k)$ is

lower-bounded by the following:

$$
\begin{aligned}
& b^2 \left(\sum_{i=1}^{K} \frac{y_i - by_i + by_i^2}{1 + by_i} \prod_{i' < i} \frac{1}{1 + by_{i'}} + \sum_{i > K} (1-b) y_i \prod_{i' < i} (1 - by_{i'}) \right) \cdot \\
& \left(\sum_{i=1}^{K} \frac{z_i - bz_i + bz_i^2}{1 + bz_i} \prod_{i' < i} \frac{1}{1 + bz_{i'}} + \sum_{i > K} (1-b) z_i \prod_{i' < i} (1 - bz_{i'}) \right) \\
& - b^2 \sum_{i=1}^{K} \frac{y_i - by_i + by_i^2}{1 + by_i} \frac{z_i - bz_i + bz_i^2}{1 + bz_i} \prod_{i' < i} \frac{1}{1 + by_{i'}} \frac{1}{1 + bz_{i'}} - b^2 \sum_{i > K} (1-b)^2 y_i z_i \prod_{i' < i} (1 - by_{i'})(1 - bz_{i'}) \\
& \geq b^2 \left(\sum_{i=1}^{K} \frac{y_i - by_i + by_i^2}{1 + by_i} \prod_{i' < i} \frac{1}{1 + by_{i'}} + \frac{1-b}{b} \prod_{i'=1}^{K} (1 - by_{i'}) \sum_{i=K+1}^{k} by_i \prod_{i'=K+1}^{i-1} (1 - by_{i'}) \right) \\
& \cdot \left(\sum_{i=1}^{K} \frac{z_i - bz_i + bz_i^2}{1 + bz_i} \prod_{i' < i} \frac{1}{1 + bz_{i'}} + \frac{1-b}{b} \prod_{i'=1}^{K} (1 - bz_{i'}) \sum_{i=K+1}^{k} bz_i \prod_{i'=K+1}^{i-1} (1 - bz_{i'}) \right) \\
& - b^2 \sum_{i=1}^{K} \frac{y_i - by_i + by_i^2}{1 + by_i} \frac{z_i - bz_i + bz_i^2}{1 + bz_i} \prod_{i' < i} \frac{1}{1 + by_{i'}} \frac{1}{1 + bz_{i'}} - b^2 \sum_{i > K} (1-b)^2 \frac{1}{(i-2)^2} \\
& \geq b^2 \left(\sum_{i=1}^{K} \frac{y_i - by_i + by_i^2}{1 + by_i} \prod_{i' < i} \frac{1}{1 + by_{i'}} + \frac{1-b}{b} \prod_{i=1}^{K} (1 - by_i) \left(1 - \prod_{i=K+1}^{k} (1 - by_i) \right) \right) \\
& \cdot \left(\sum_{i=1}^{K} \frac{z_i - bz_i + bz_i^2}{1 + bz_i} \prod_{i' < i} \frac{1}{1 + bz_{i'}} + \frac{1-b}{b} \prod_{i=1}^{K} (1 - bz_i) \left(1 - \prod_{i=K+1}^{k} (1 - bz_i) \right) \right) \\
& - b^2 \sum_{i=1}^{K} \frac{y_i - by_i + by_i^2}{1 + by_i} \frac{z_i - bz_i + bz_i^2}{1 + bz_i} \prod_{i' < i} \frac{1}{1 + by_{i'}} \frac{1}{1 + bz_{i'}} - b^2 (1-b)^2 \int_{K-2}^{\infty} \frac{1}{x^2} dx \\
& \geq \left(\sum_{i=1}^{K} by_i \left(1 - \frac{b}{1 + by_i} \right) \prod_{i' < i} \frac{1}{1 + by_{i'}} + \frac{1-b}{b} \prod_{i=1}^{K} (1 - by_i) \left(1 - \exp(-b(1 - \sum_{i=1}^{K} y_i)) \right) \right) \qquad (2.12) \\
& \cdot \left(\sum_{i=1}^{K} bz_i \left(1 - \frac{b}{1 + bz_i} \right) \prod_{i' < i} \frac{1}{1 + bz_{i'}} + \frac{1-b}{b} \prod_{i=1}^{K} (1 - bz_i) \left(1 - \exp(-b(1 - \sum_{i=1}^{K} z_i)) \right) \right) \qquad (2.13) \\
& - \sum_{i=1}^{K} \left(1 - \frac{b}{1 + by_i} \right) \left(1 - \frac{b}{1 + bz_i} \right) \prod_{i' < i} \frac{1}{1 + by_{i'}} \frac{1}{1 + bz_{i'}} - \frac{b^2 (1-b)^2}{K-2}. \qquad (2.14)
\end{aligned}
$$

We explain each inequality. The first inequality rewrites terms in the first two lines and applies the bounds $y_i \leq \frac{1}{i-2}$ and $z_i \leq \frac{1}{i-2}$ on the final subtracted term, which hold because $\sum_{i=1}^{k} y_k = 1$ and $y_3 \geq \cdots \geq y_k \geq 0$ (and similarly for the z_i's). For the second inequality, note that $\sum_{i=K+1}^{k} by_i \prod_{i'=K+1}^{i-1} (1 - by_{i'})$ is equivalent to the probability that at least one of independent Bernoulli random variables with means by_i for $i = K+1, \ldots, k$ realizes to 1 (similarly for the z_i's). Moreover, we have $\sum_{i>K} \frac{1}{(i-2)^2} \leq \int_{K-2}^{\infty} \frac{1}{x^2} dx$ by Riemann sums. For the final inequality, we have applied the fact $1 - by_i \leq \exp(-by_i)$ and the constraint that $\sum_{i=1}^{K} y_i = 1$ (similarly for the z_i's) and evaluated the integral.

Since this holds for all $k \geq K$, we have proven that for any positive integer $K > 2$, $\inf_k \mathrm{AdvMin}_k(b)$

is lower-bounded by the auxiliary optimization problem defined by

$$
\begin{aligned}
\mathrm{AdvMinAux}_K(b) := \inf\ & (2.12)\text{–}(2.14) \\
\text{s.t.}\ & \sum_{i=1}^{K} y_i \le 1 \\
& \sum_{i=1}^{K} z_i \le 1 \\
& y_i + z_i \le 1 && \forall i = 1, \dots, K \\
& y_i, z_i \ge 0 && \forall i = 1, \dots, K
\end{aligned}
$$

(note that we have relaxed the constraints $y_3 \ge \cdots \ge y_K$ and $z_3 \ge \cdots \ge z_K$ on the adversary). That is, we have $1 - 3c + \inf_k \mathrm{AdvMin}_k(\frac{c}{1-c}) \ge 1 - 3c + \mathrm{AdvMinAux}_K(\frac{c}{1-c})$. The proof of Theorem 2.4, part (ii) is then completed by computationally verifying[7] that for $c = 0.3445$ and $K = 80$ (a finite optimization problem), $1 - 3c + \mathrm{AdvMinAux}_K(\frac{c}{1-c}) \ge 0$. □

2.2 Improvement for Bipartite Graphs

We improve the analysis of Algorithm 1 in the special case where $G = (V, E)$ is a bipartite graph. Adopting the same proof skeleton and terminology, our goal is to lower-bound, given a newly arriving edge $(u, v) \notin E$, the probability that vertices u and v have both been matched.

In Subsection 2.1, we analyzed the probability of the sufficient condition that u and v "randomly chose" distinct candidates who were alone. In this subsection, we can analyze the easier-to-satisfy condition of u and v both *having* candidates who are alone. The reason for this is twofold: the neighbors $u_1, \dots, u_k$ of u (i.e. the potential candidates) are clearly distinct from the neighbors of v, because edge (u, v) cannot form a 3-cycle; and, the neighbors $u_1, \dots, u_k$ being alone are independent events, because there cannot be any edges between them (which again would form a 3-cycle).

By lower-bounding the probability of this easier-to-satisfy condition, we show that Algorithm 1 is 0.349-selectable for all graphs without a 3-cycle (which includes all bipartite graphs), improving upon the earlier guarantee of 0.344 for general graphs.

Theorem 2.6. *On bipartite graphs, Algorithm 1 provides a c-selectable OCRS for any value of* $c \in [0, 1/2]$ *satisfying* $1 - 3c + \left(1 - \exp(-\frac{c(1-2c)}{(1-c)^2})\right)^2 \ge 0$. *Therefore, Algorithm 1 is 0.349-selectable.*

Proof of Theorem 2.6. By the same argument as in the start of the proof of Theorem 2.4 part (i), we can without loss of generality assume that $\sum_{f \in \partial(u)} x_f = \sum_{f \in \partial(v)} x_f = 1$, after which it suffices to show that $1 - 3c + \mathbb{P}[\mathsf{matched}_u(u, v) \cap \mathsf{matched}_v(u, v)] \ge 0$. We will show that $\mathbb{P}[\mathsf{matched}_u(u, v) \cap \mathsf{matched}_v(u, v)] \ge \left(1 - \exp(-\frac{c(1-2c)}{(1-c)^2})\right)^2$. To do so, recall that $\mathsf{matched}_u(u, v) \cap \mathsf{matched}_v(u, v)$ occurs whenever u and v both have a neighbor that survives (i.e. can be a candidate) and is alone. Letting $u_1, \dots, u_k$ be the vertices in $V \setminus \{u, v\}$ such that $\{(u, u_i) : i = 1, \dots, k\} = \partial(u)$ are the edges in E incident to u, and respectively $v_1, \dots, v_\ell$ be the vertices (which are distinct from $u_1, \dots, u_k$)

[7] Code can be found at `https://github.com/Willmasaur/OCRS_matching/blob/main/ocrs.jl`, which uses the JuMP [12] and COUENNE packages.

such that $\{(v, v_j) : j = 1, \dots, \ell\} = \partial(v)$, we have that

$$\mathbb{P}[\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)] \geq \mathbb{P}\left[\left(\bigcup_i (S_{u,u_i} \cap \mathsf{alone}_{u_i}(u,u_i))\right) \bigcap \left(\bigcup_j (S_{v,v_j} \cap \mathsf{alone}_{v_j}(v,v_j))\right)\right].$$

We argue that the r.h.s. of the preceding inequality is lower-bounded by the independent case, i.e.

$$\mathbb{P}[\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)] \geq \mathbb{P}\left[\bigcup_i (S_{u,u_i} \cap \mathsf{alone}_{u_i}(u,u_i))\right] \mathbb{P}\left[\bigcup_j (S_{v,v_j} \cap \mathsf{alone}_{v_j}(v,v_j))\right], \tag{2.15}$$

again using the FKG inequality. To see this, consider the bits $\{S_e : e \in E\}$, and note that the events $S_{u,u_i} \cap \mathsf{alone}_{u_i}(u,u_i)$ and $S_{v,v_j} \cap \mathsf{alone}_{v_j}(v,v_j)$ are fully determined by these bits, and moreover are increasing in the bits $\{S_e : e \in \partial(u) \cup \partial(v)\}$ (such bits affect only S_{u,u_i} and S_{v,v_j}) and decreasing in the bits $\{S_e : e \notin \partial(u) \cup \partial(v)\}$ (such bits affect only $\mathsf{alone}_{u_i}(u,u_i)$ and $\mathsf{alone}_{v_j}(v,v_j)$). Since the bits S_e are independent across e, we have that (2.15) holds.

Now, we can derive that

$$\begin{aligned}
\mathbb{P}\left[\bigcup_i (S_{u,u_i} \cap \mathsf{alone}_{u_i}(u,u_i))\right] &= 1 - \prod_i (1 - \mathbb{P}[S_{u,u_i}]\mathbb{P}[\mathsf{alone}_{u_i}(u,u_i)]) \\
&\geq 1 - \prod_i \left(1 - \frac{cx_{u,u_i}}{1 - cx_{u_i}(u,u_i)} \frac{1 - c - cx_{u_i}(u,u_i)}{1-c}\right) \\
&\geq 1 - \prod_i (1 - \frac{c(1-2c)}{(1-c)^2} x_{u,u_i}) \\
&\geq 1 - \exp\left(-\frac{c(1-2c)}{(1-c)^2} \sum_i x_{u,u_i}\right).
\end{aligned}$$

To explain the equality, note that event $\mathsf{alone}_{u_i}(u,u_i)$ depends only on the independent bits $\{S_e : e \in \partial(u_i) \setminus (u,u_i)\}$, which must be disjoint from $\{S_e : e \in \partial(u_{i'}) \setminus (u,u_{i'})\}$ for any $i' \neq i$, since otherwise u_i and $u_{i'}$ would form a 3-cycle with u. Therefore, the $2k$ events $S_{u,u_1}, \cdots, S_{u,u_k}$, $\mathsf{alone}_{u_1}(u,u_1), \dots, \mathsf{alone}_{u_k}(u,u_k)$ are mutually independent, allowing us to decompose the probability $\mathbb{P}[\bigcup_i (S_{u,u_i} \cap \mathsf{alone}_{u_i}(u,u_i))]$ into the product in the first line. After that, the first inequality holds by Propositions 2.2 and 2.3, the second inequality holds because $x_{u_i}(u,u_i) \leq 1$ and $c \leq 1/2$, and the final inequality holds elementarily. Finally, applying the assumption that $\sum_{i=1}^k x_{u,u_i} = 1$, we conclude that $\mathbb{P}[\bigcup_i (S_{u,u_i} \cap \mathsf{alone}_{u_i}(u,u_i))] \geq 1 - \exp(-\frac{c(1-2c)}{(1-c)^2})$.

After an analogous lower bound for $\mathbb{P}\left[\bigcup_j (S_{v,v_j} \cap \mathsf{alone}_{v_j}(v,v_j))\right]$ and substituting into (2.15), we have shown that $\mathbb{P}[\mathsf{matched}_u(u,v) \cap \mathsf{matched}_v(u,v)] \geq (1 - \exp(-\frac{c(1-2c)}{(1-c)^2}))^2$. It can be numerically verified that $c = 0.349$ satisfies $1 - 3c + (1 - \exp(-\frac{c(1-2c)}{(1-c)^2}))^2 \geq 0$, completing the proof that Algorithm 1 is 0.349-selectable. □

2.3 Impossibility Results for OCRS

We present the following construction here which is new. The other constructions are deferred to the proofs since they are standard, although the analysis is still new.

Example 2.7. *Let G be a complete graph on vertices $V = \{1, 2, 3, 4\}$, and consider the fractional matching whose edge values are $x_{12} = x_{23} = x_{34} = x_{41} = (1-\varepsilon)/2$ along a 4-cycle and $x_{13} = x_{24} = \varepsilon$ on the diagonals. ε is a small positive constant that we will take to 0. The arrival order of edges, known in advance, is: $(1, 2), (3, 4)$ (a diametrically opposite pair of edges), followed by $(2, 3), (4, 1)$ (another diametrically opposite pair), followed by $(1, 3), (2, 4)$ (the diagonal edges).*

Proposition 2.8. *On the $G, \boldsymbol{x}$ given in Example 2.7, any OCRS is no more than 0.4-selectable.*

Proof of Proposition 2.8. Since edge (3,4) comes after (1,2), the probability of it being selected conditional on (1,2) being selected is at most $x_{34} = \frac{1-\varepsilon}{2}$. That is, $\mathbb{P}[(1, 2) \in \mathcal{M} \cap (3, 4) \in \mathcal{M}] \leq \frac{1-\varepsilon}{2}\mathbb{P}[(1, 2) \in \mathcal{M}]$. Thus,

$$\begin{aligned}\mathbb{P}[(1, 2) \in \mathcal{M} \cup (3, 4) \in \mathcal{M}] &\geq \mathbb{P}[(1, 2) \in \mathcal{M}] + \mathbb{P}[(3, 4) \in \mathcal{M}] - \frac{1-\varepsilon}{2}\mathbb{P}[(1, 2) \in \mathcal{M}] \\ &= \frac{1+\varepsilon}{2}\mathbb{P}[(1, 2) \in \mathcal{M}] + \mathbb{P}[(3, 4) \in \mathcal{M}]. \\ &\geq \left(\frac{1+\varepsilon}{2} + 1\right) c\frac{1-\varepsilon}{2}\end{aligned}$$

where the final inequality must hold if we were to have a c-selectable OCRS. We can similarly derive that $\mathbb{P}[(2, 3) \in \mathcal{M} \cup (4, 1) \in \mathcal{M}] \geq \frac{3+\varepsilon}{2}c\frac{1-\varepsilon}{2}$. Now, note that $(1, 2) \in \mathcal{M} \cup (3, 4) \in \mathcal{M}$ and $(2, 3) \in \mathcal{M} \cup (4, 1) \in \mathcal{M}$ are disjoint events. Hence, the probability that any of the edges (1,2),(2,3),(3,4),(4,1) is selected is at least $\frac{(3+\varepsilon)(1-\varepsilon)}{2}c$. If any such edges are selected, then the diagonal edges (1,3),(2,4) cannot be selected. Therefore, the probability that (1,3) can be selected is at most $(1 - \frac{(3+\varepsilon)(1-\varepsilon)}{2}c)\varepsilon$, which must be at least $c\varepsilon$ in order to have a c-selectable OCRS. Consequently we have $1 - \frac{(3+\varepsilon)(1-\varepsilon)}{2}c \geq c$, and taking $\varepsilon \to 0$ implies $c \leq 0.4$. ☐

Proposition 2.9. *On Example 2.7, the OCRS of [14] is no more than 0.361-selectable.*

Proof of Proposition 2.9. First, note that when the first two edges $(1, 2)$ and $(3, 4)$ arrive, they cannot be blocked. Therefore, $\mathbb{P}[\overline{\mathsf{blocked}(1, 2)}] = \mathbb{P}[\overline{\mathsf{blocked}(3, 4)}] = 1$. Therefore, $\alpha_{(1,2)} = \alpha_{(3,4)} = c$, and matched with probability $c(1 - \epsilon)/2$ (and independently of each other).

Next, each of the next two edges (i.e., $(2, 3)$ and $(4, 1)$) are blocked if *either* of the first two edges was matched. We have:

$$\begin{aligned}\mathbb{P}[\overline{\mathsf{blocked}(2, 3)}] = \mathbb{P}[\overline{\mathsf{blocked}(4, 1)}] &= \mathbb{P}[(1, 2) \notin \mathcal{M} \cap (3, 4) \notin \mathcal{M}] \\ &= (1 - c(1 - \epsilon)/2)^2\end{aligned}$$

which further gives that $\alpha_{(2,3)} = \alpha_{(4,1)} = c(1 - c(1 - \epsilon)/2)^{-2}$.

Finally, consider the final two arrivals, the diagonal edges. Edge $(1, 3)$ is not blocked as long as none of the previous arrivals was matched. That is, both of edges $(1, 2)$ and $(3, 4)$ must have been left unmatched (each with probability $1-c(1-\epsilon)/2$), and then each of the next two edges (i.e., $(2, 3)$ and $(4, 1)$) must have failed to survive (which occurs with probability $1-c(1-\epsilon)(1-c(1-\epsilon)/2)^2/2$). The same holds for edge $(2, 4)$. This gives us:

$$\mathbb{P}[\overline{\mathsf{blocked}(1, 3)}] = \mathbb{P}[\overline{\mathsf{blocked}(2, 4)}] = \left(1 - c\frac{1-\epsilon}{2}\right)^2 \left(1 - \frac{c(1-\epsilon)}{2(1 - c(1-\epsilon)/2)^2}\right)^2$$

As per (2.2), we want this probability to be at least c. As $\epsilon \to 0$, we get

$$\left(1 - \frac{c}{2}\right)^2 \left(1 - \frac{c}{2(1 - c/2)^2}\right)^2 \geq c$$

which is satisfied only if $c \leq 0.3602$. Thus, on this graph, we must have $c < 0.361$, and the OCRS cannot be better than 0.361-selectable. □

Proposition 2.10. *Any OCRS is no more than c-selectable for bipartite graphs, where $c \leq 0.433$ is the real number satisfying $1 - c - e^{-(1-c)} = 0$.*

Proof of Proposition 2.10. Given $n \geq 2$, let $G = (U, V, E)$ be a bipartite graph with $|U| = |V| = n$, and $E = U \times V$. Set $x_e := 1/n$ for all $e \in E$. Given a vertex ordering $v_1, \ldots, v_n$, we assume that the states of the edges incident to v_i are presented to the OCRS before that of v_{i+1} for each $i = 1, \ldots, n-1$. We consider the asymptotic setting when $n \to \infty$.

Fix an arbitrary OCRS with selectability $c \geq 0$. Let $\mathcal{M}_i$ be the matching constructed by the OCRS after vertices $v_1, \ldots, v_i$ arrive. First observe that since the OCRS is c-selectable, $\mathbb{E}[|\mathcal{M}_{n-1}|] \geq c(n-1)$, and so

$$m_n := \frac{\mathbb{E}[|\mathcal{M}_{n-1}|]}{n-1} \geq c. \tag{2.16}$$

Consider now the final arriving vertex v_n. Observe that v_n can only be included in $\mathcal{M}_n$ if it has an active edge adjacent to some vertex $u \in U$ *not* matched by $\mathcal{M}_{n-1}$. Thus, after conditioning on $\mathcal{M}_{n-1}$,

$$\begin{aligned}\mathbb{P}[v_n \in \mathcal{M}_n \mid \mathcal{M}_{n-1}] &\leq 1 - \left(1 - \frac{1}{n}\right)^{n - |\mathcal{M}_{n-1}|} \\ &= (1 + o(1))\left(1 - \exp\left(\frac{|\mathcal{M}_{n-1}|}{n} - 1\right)\right).\end{aligned}$$

Observe that $z \to (1 - \exp(z-1))$ is concave on $[0,1]$, so after taking expectations and applying Jensen's inequality,

$$\mathbb{P}[v_n \in \mathcal{M}_n] \leq (1 + o(1))\left(1 - \exp\left(m_n \cdot \frac{n-1}{n} - 1\right)\right). \tag{2.17}$$

Yet, $\mathbb{P}[v_n \in \mathcal{M}_n] \geq c$, so after applying (2.16) and (2.17) and taking $n \to \infty$, it follows that

$$c \leq 1 - \exp(m_n - 1) \leq 1 - \exp(c - 1).$$

Thus $1 - \exp(c-1) - c \geq 0$, completing the proof. □

Proposition 2.11. *The OCRS of [14] is no more than 0.382-selectable for bipartite graphs.*

Proof of Proposition 2.11. Let G be a graph on vertices $V = \{1, 2, 3, 4\}$ that is a path of three edges (1,2),(2,3),(3,4), and consider the fractional matching whose edge values are $x_{12} = 1 - \varepsilon, x_{23} = \varepsilon, x_{34} = 1 - \varepsilon$, with ε being a small positive constant. The arrival order of edges is (1,2),(3,4),(2,3), where the middle edge arrives last.

Notice that the first two edges, $(1,2)$ and $(3,4)$ cannot be blocked and so $\mathbb{P}[\overline{\mathsf{blocked}(1,2)}] = \mathbb{P}[\overline{\mathsf{blocked}(3,4)}] = 1$. This means $\alpha_{(1,2)} = \alpha_{(3,4)} = c$. Each of these edges will therefore be matched with probability $c(1-\epsilon)$. When the middle edge arrives, then, the probability it is not blocked is $(1 - c(1-\epsilon))^2$. Applying (2.2), we get

$$(1 - c(1-\epsilon))^2 \geq c$$

and for $\epsilon \to 0$ this gives $c \leq 0.3819$. This means that the OCRS cannot be better than 0.382-selectable for this graph. □

3 Random-order Contention Resolution Schemes

We reuse the terminology and notation about graphs and matching polytopes defined at the start of Section 2 for OCRS, and add the following definitions below.

Definition 4 (Terminology and Notation for RCRS)**.** Suppose the edges of $G = (V, E)$ arrive uniformly at random. In our analysis, we will treat each edge e as having an *arrival time* Y_e drawn independently and uniformly from $[0, 1]$. Edges then arrive in increasing order of arrival times.

Also, if $\boldsymbol{x} = (x_e)_{e \in E}$ satisfies constraints $\sum_{e \in \partial(v)} x_e \leq 1$ for all $v \in V$ as equality, then we then say that G is 1-*regular* (with respect to $\boldsymbol{x}$), and refer to $(G, \boldsymbol{x})$ as a 1-*regular input*.

We first argue that when designing an RCRS, it suffices to only consider 1-regular inputs. The proof is similar to that of [18], and so we defer it to Appendix A.

Lemma 3.1 (Reduction to 1-Regular Inputs)**.** *If there exists a c-selectable RCRS for all* 1*-regular inputs, then there exists a c-selectable RCRS for all inputs via a reduction to a* 1*-regular input. Moreover, this reduction can be computed efficiently, and preserves bipartiteness.*

Let us now fix an arbitrary *attenuation function* $a : [0, 1] \to [0, 1]$. Consider the following template RCRS, which is presented the edges of a graph $G = (V, E)$ in random order.

Algorithm 2 Attenuate-ROM

Input: Graph $G = (V, E)$ and a fractional matching $\boldsymbol{x} = (x_e)_{e \in E}$.
Output: subset of active edges forming a matching $\mathcal{M}$.

```
M ← ∅.
for arriving edges e ∈ E do
    Draw A_e ~ Ber(a(x_e)) independently.          ▷ attenuate with probability a(x_e)
    if e is active, not blocked and A_e = 1 then
        M ← M ∪ {e}.
return M
```

We consider Algorithm 2 with the quadratic attenuation function $a_1(x) := (1 - (3 - e)x)^2$ when working with general graphs, and a new attenuation function, $a_2(x) := (1 - x)^4/(e^x - ex)^2$ for $x \in [0, 1)$ where $a_2(1) := \lim_{x \to 1^-} a(x) = 4/e^2$, when working with bipartite graphs.

Theorem 3.2 (General graphs)**.** *If $a(x) = a_1(x)$, where $a_1(x) := (1 - (3 - e)x)^2$, then Algorithm 2 is $\frac{e^2 - 4e^3 + e^4 + 20e - 22}{4e^2} \geq 0.474035$ selectable for* 1*-regular general graphs.*

Theorem 3.3 (Bipartite graphs)**.** *If $a(x) = a_2(x)$, where $a_2(x) := (1 - x)^4/(e^x - ex)^2$, then Algorithm 2 is $\frac{e^2 + e^4 - 10}{2e^4} \geq 0.476089$ selectable for* 1*-regular bipartite graphs.*

Remark 3.4. Our RCRS for attaining the positive results as claimed in Table 1 hold due to the reduction of Lemma 3.1, which is computationally efficient. Note that for bipartite graphs, the reduction to a 1-regular instance is done in a way that preserves bipartiteness.

As in the adversarial order setting, we define $S_e := X_e \cdot A_e$ and say that e *survives* (the attenuation function a) if $S_e = 1$. Observe that each edge e survives independently with probability $q(x_e) := x_e a(x_e)$. We say that $f \in \partial(e)$ is *early* (for e), provided $Y_f < Y_e$ and f survives. Otherwise, f is *late*. Denote the early edges of e by $\mathcal{F}_e$. Observe that if e survives and $\mathcal{F}_e = \emptyset$, then e is selected by Algorithm 2 (note that the latter event is equivalent to $\mathsf{alone}_u(e) \cap \mathsf{alone}_v(e)$

in our OCRS terminology). In [9], Brubach et al. use a different attenuation function to argue that $\mathbb{P}[\mathcal{F}_e = \emptyset] \geq (1 - e^{-2})/2 \geq 0.432$, and it is not hard to see that their analysis is tight. Our improvement comes from restricting to 1-regular inputs, as this allows e to be matched when $\mathcal{F}_e \neq \emptyset$, yet none of the edges of $\mathcal{F}_e$ were matched.

Definition 5. Fix $e = (u, v) \in E$, and suppose that $f \in \partial(e)$ has vertex w not in e. We say that $h \in \partial(w) \setminus \{(u, w), (v, w)\}$ is a *simple-blocker* for f, denoted $\mathsf{sblocker}_f(h)$, if:

1. h arrives before f (i.e., $Y_h < Y_f$).
2. Each $h' \in \partial(h) \setminus \partial(e)$ is late for h.

We denote the event in which f has *some* simple-blocker by $\mathsf{blocker}_f$. If $\mathsf{blocker}_f$ occurs for each $f \in \mathcal{F}_e$, then we say that $\mathcal{F}_e$ is *safe* (for e).

Observe the following basic properties of the simple-blocker definition:

Proposition 3.5. *For any $f \in \partial(e)$:*

1. *f has at most one simple-blocker.*
2. *The event $\mathsf{blocker}_f$ is independent from the random variables S_f and $(Y_g, S_g)_{g \in \partial(e) \cup \{e\} \setminus \{f\}}$.*

Moreover, if $\mathcal{F}_e$ is safe, then each edge $f \in \mathcal{F}_e$ cannot get selected, as its endpoint not in (u, v) must already have been matched. The following thus trivially holds:

Proposition 3.6. *If $\mathcal{F}_e$ is safe, and e survives, then $e \in \mathcal{M}$.*

3.1 Proving Theorem 3.2

Throughout this section, we analyze Algorithm 2 when executed with the quadratic attenuation function $a(x) = (1 - (3 - e)x)^2$. However, we are careful to isolate the required analytic properties of a as we proceed through the argument (see Propositions 3.8, 3.10 and 3.12, which we prove in Appendix A).

We consider the case when there is at most one early edge; that is, $|\mathcal{F}_e| \leq 1$. Observe first that by Proposition 3.6,

$$\mathbb{P}[e \in \mathcal{M} \mid S_e = 1] \geq \mathbb{P}[|\mathcal{F}_e| = 0] + \mathbb{P}[\mathcal{F}_e \text{ is safe and } |\mathcal{F}_e| = 1]. \tag{3.1}$$

In order to lower bound the r.h.s. of (3.1), it will be convenient to first condition on $Y_e = y$ for an arbitrary $y \in [0, 1]$. The expression $\mathbb{P}[|\mathcal{F}_e| = 0 \mid Y_e = y]$ is then easy to control, since $|\mathcal{F}_e|$ is distributed as $\sum_{g \in \partial(e)} \mathrm{Ber}(yq(x_g))$ where the Bernoulli's are independent, and so

$$\mathbb{P}[|\mathcal{F}_e| = 0 \mid Y_e = y] = \prod_{g \in \partial(e)} \ell(x_g, y), \tag{3.2}$$

where $\ell(x_g, y) := 1 - yq(x_g)$ is the probability that g is late. We focus on lower bounding $\mathbb{P}[\mathcal{F}_e \text{ is safe and } |\mathcal{F}_e| = 1 \mid Y_e = y]$. In order to do so, we fix $f \in \partial(e)$ with vertex w not in e, and derive a lower bound on the likelihood that $h \in \partial(w) \setminus \{(u, w), (v, w)\}$ is a simple-blocker for f, conditional on $f \in \mathcal{F}_e$. Note that if $f = (w, u)$ (or $f = (w, v)$), then we define $f^c := (w, v)$ (respectively, $f^c := (w, u)$) to be the *pair* of f in the triangle $\{(u, v), (w, v), (w, u)\}$.

Lemma 3.7 (First-order minimization). *If f has vertex w not in e, then for each $h \in \partial(w)\setminus\{f, f^c\}$,*

$$\mathbb{P}[\mathsf{sblocker}_f(h) \mid \{f \in \mathcal{F}_e\}, Y_e = y] \geq \frac{q(x_h)}{z_h}\left(1 - \frac{e^{-z_h}}{z_h y}\right), \tag{3.3}$$

where $z_h = 2(1 - x_h) - x_f - x_{f^c}$.

In order to prove Lemma 3.7, we show that the minimum probability of the event $\mathsf{sblocker}_f(h)$ corresponds to when all the edges $h' \in \partial(w) \setminus \{f, f^c\}$ have vanishing edges values. This is implied by the following analytic properties of the attenuation function a:

Proposition 3.8 (First-order minimization). *For each $x, y \in [0, 1]$, the function $x \to \ln \ell(x, y)$ is convex. Moreover, $a(0) = 1$, and a is continuous and decreasing on $[0, 1]$.*

Proof of Lemma 3.7. Let us assume that $f = (u, w)$, and $h \in \partial(w) \setminus \{f, f^c\}$. We then condition on $Y_e = y$, $S_f = 1$, $Y_f = y_f$, and $Y_h = y_h$, where $y_f, y_h \in [0, y]$ satisfy $y_h < y_f$. Our goal is to first derive a lower bound on $\mathbb{P}[\mathsf{sblocker}_f(h) \mid Y_h = y_h, Y_f = y_f]$. Observe that since $y_h < y_f$, h is a simple-blocker for f if and only if each $h' \in \partial(h) \setminus \partial(e)$ is late for h. Thus,

$$\mathbb{P}[\mathsf{sblocker}_f(h) \mid Y_h = y_h, Y_f = y_f, Y_e = y] = x_h a(x_h) \prod_{h' \in \partial(h)\setminus\partial(e)} \ell(x_{h'}, y_h) \tag{3.4}$$

where we recall that $\ell(x_{h'}, y_h) := 1 - y_h q(x_{h'})$. Now, (3.4) is when minimized when $\partial(h) \setminus \partial(e)$ has as many edges as possible, so we hereby assume w.l.o.g. that $\partial(h) \cap \partial(e) = \{f, f^c\}$. In order to minimize (3.4), we analyze

$$\sum_{h' \in \partial(h)\setminus\{f, f^c\}} \log \ell(x_{h'}, y_h), \tag{3.5}$$

subject to $\sum_{h' \in \partial(h)\setminus\{f,f^c\}} x_{h'} = 2 - 2x_h - x_f - x_{f^c} =: z_h$. The convexity of $x_{h'} \to \log \ell(x_{h'}, y_h)$ guaranteed by Proposition 3.8 allows us to conclude that (3.5) is minimized when $\max_{h' \in \partial(h)\setminus\{f,f^c\}} x_{h'} = o(1)$ and $|\partial(h) \setminus \{f, f^c\}| \to \infty$. Thus,

$$\mathbb{P}[\mathsf{sblocker}_f(h) \mid Y_h = y_h, Y_f = y_f, Y_e = y] \geq x_h a(x_h) \exp\left(-z_h y_h\right). \tag{3.6}$$

(We provide the full details in Appendix A, as this part of the argument is due to [9]). Using (3.6), we integrate over $y_h \in [0, y_f]$, followed by $y_f \in [0, y]$, to get that

$$\begin{aligned}\mathbb{P}[Y_f \leq y \text{ and } \mathsf{sblocker}_f(h) \mid Y_e = y] &\geq x_h a(x_h) \int_0^y \int_0^{y_f} \exp\left(-z_h y_h\right)\, dy_h\, dy_f \\ &= \frac{x_h a(x_h)}{z_h^2}\left(z_h y + \exp(-z_h y) - 1\right).\end{aligned}$$

Finally, after dividing both sides by $\mathbb{P}[Y_f \leq y] = y$, the proof is complete. ☐

Next, we lower bound the probability that f has a simple-blocker, conditional on $f \in \mathcal{F}_e$.

Lemma 3.9 (Second-order minimization). *For each $f \in \partial(e)$,*

$$\mathbb{P}[\mathsf{blocker}_f \mid \{f \in \mathcal{F}_e\}, Y_e = y] \geq \frac{q(1 - x_f - x_{f^c})}{x_f + x_{f^c}}\left(1 - \frac{e^{-(x_f + x_{f^c})y}}{(x_f + x_{f^c})y}\right) =: T(x_f + x_{f^c}, y).$$

We prove Lemma 3.9 by characterizing the minimum probability of the event $\mathsf{blocker}_f$. This minimum occurs when w of $f = (w, u)$ has a single neighbor (other than u and w), and its corresponding edge value is $1 - x_f - x_{f^c}$. Note that this is the *opposite* worst-case in comparison to Lemma 3.7. Our proof relies on the following property of a:

Proposition 3.10 (Second-order minimization)**.** *For all* $x \in [0,1]$, $\frac{a'(x)}{a(x)} + \frac{4}{1-x} - \frac{2(1-\exp(x-1))}{\exp(x-1)-x} \leq 0$.

Proof of Lemma 3.9. Let us assume that $f = (w, u)$ for some $w \in N(u) \setminus \{v\}$. We shall assume that $1 - x_f - x_{f^c} < 1$, as otherwise the statement follows immediately. In this case, $\sum_{h \in \partial(w) \setminus \{f, f^c\}} x_h > 0$ since w has fractional degree 1 (as G is 1-regular). Observe that by definition, $\mathsf{blocker}_f$ occurs if and only if $\cup_{h \in \partial(w) \setminus \{f, f^c\}} \mathsf{sblocker}_f(h)$ occurs. On the other hand, f has at most one simple-blocker by Proposition 3.5. Thus, after applying Lemma 3.7, if $z_h := 2(1 - x_h) - x_f - x_{f^c}$, then

$$\mathbb{P}[\mathsf{blocker}_f \mid f \in \mathcal{F}_e, Y_e = y] \geq \sum_{h \in \partial(w) \setminus \{f, f^c\}} \frac{x_h a(x_h)}{z_h} \left(1 - \frac{e^{-z_h}}{z_h y}\right), \tag{3.7}$$

subject to the constraint, $\sum_{h \in \partial(w) \setminus \{f, f^c\}} x_h = 1 - x_f - x_{f^c}$. Fix y, x_f and x_{f^c}. We claim that the worst-case for (3.7) occurs when $|\partial(w) \setminus \{f, f^c\}| = 1$, and the single edge h within this set satisfies $x_h = 1 - x_f - x_{f^c}$. In this case, the r.h.s of (3.7) is $T(x_f + x_{f^c}, y)$ so this will complete the proof.

Define $A(x_h) := \frac{x_h a(x_h)}{z_h} \left(1 - \frac{e^{-z_h}}{z_h y}\right)$. Observe that if we can show that $A(x_h)$ is decreasing as a function of x_h on the interval $[0, 1 - x_f - x_{f^c}]$, then this will imply the claimed worst-case. Now, setting $B(x_h) := \log A(x_h)$, we have that

$$B(x_h) = \log a(x_h) - 2 \log z_h + \log(z_h y + e^{-y z_h} - 1),$$

and so after differentiating B with respect to x_h,

$$B'(x_h) = \frac{a'(x_h)}{a(x_h)} + \frac{4}{z_h} - \frac{2y(1 - \exp(-y z_h)}{z_h y + \exp(-y z_h - 1)}. \tag{3.8}$$

Our goal is to show that $B'(x_h) \leq 0$ for all $x_h \in [0,1]$. First, since $z_h \in [0,2]$, the function

$$y \to \frac{2y(1 - \exp(-y z_h))}{z_h y + \exp(-y z_h - 1)}$$

is decreasing, and so (3.8) is minimized at $y = 1$, when it is equal to

$$\frac{a'(x_h)}{a(x_h)} + \frac{4}{z_h} - \frac{2(1 - \exp(-z_h))}{z_h + \exp(-z_h - 1)}.$$

Similarly, the function

$$z_h \to \frac{4}{z_h} - \frac{2(1 - \exp(-z_h))}{z_h + \exp(-z_h - 1)}$$

is decreasing in z_h, and thus *increasing* in x_h (as $z_h = 2 - 2x_h - x_f - x_{f^c}$). Its maximum therefore occurs at $x_h = 1 - x_f - x_{f^c}$, and so (3.8) is upper-bounded by

$$\frac{a'(x_h)}{a(x_h)} + \frac{4}{1 - x_h} - \frac{2(1 - \exp(x_h - 1))}{\exp(x_h - 1) - x_h},$$

which is at most 0 by Proposition 3.8. Thus, $B'(x_h) \leq 0$ for all $x_h \in [0,1]$, and so $B(x_h)$ is decreasing as a function of x_h. By exponentiating, the same statement is true for $A(x_h)$, and so the proof is complete.

□

Recall that by Proposition 3.5, for each $f \in \partial(e)$ the event $\mathsf{blocker}_f$ is independent from random variables $(S_g, Y_g)_{g \in \partial(e) \setminus \{f\}}$. We can therefore apply Lemma 3.9, to get that

$$\mathbb{P}[\mathcal{F}_e \text{ is safe and } |\mathcal{F}_e| = 1 \mid Y_e = y] \geq \sum_{f \in \partial(e)} T(x_f + x_{f^c}, y) \cdot q(x_f) y \prod_{g \in \partial(e) \setminus \{f\}} \ell(x_g, y).$$

Thus, combined with (3.2), (3.1) implies that $\mathbb{P}[e \in \mathcal{M} \mid S_e = 1]$ is lower bounded by

$$\mathrm{obj}(G, \boldsymbol{x}, e) := \int_0^1 \prod_{g \in \partial(e)} \ell(x_g, y) + \sum_{f \in \partial(e)} T(x_f + x_{f^c}, y) \cdot q(x_f) y \prod_{g \in \partial(e) \setminus \{f\}} \ell(x_g, y)\, dy. \tag{3.9}$$

In order to prove the theorem, we must identify the infimum of the function obj over graphs which contain e, and whose fractional matching assigns x_e to e. We claim that no matter the value of x_e, the infimum occurs when $|\partial(e)| \to \infty$ and $\max_{f \in \partial(e)} x_f = o(1)$ (i.e., the *Poisson regime*). In order to prove this, we apply a *vertex-splitting procedure*. Specifically, fix any $k \geq 1$, and replace an *arbitrary* vertex $w \in N(u) \cup N(v) \setminus \{u, v\}$ with k copies of itself, say $w_1, \ldots, w_k$. Let $G' = (V', E')$ be the resulting graph. We define a new fractional matching $\boldsymbol{x}'$ for G' where we split the values of the edges incident to w uniformly amongst $w_1, \ldots, w_k$, and keep the remaining edge values the same. That is, $x'_{w_i, r} := x_{w,r}/k$ for each $i \in [k]$ and $r \in V \setminus \{w_1, \ldots, w_k\}$, and $x'_f := x_f$ for all other $f \in E$. We lower bound $\mathrm{obj}(G, \boldsymbol{x}, e)$ by the limiting value of $\mathrm{obj}(G', \boldsymbol{x}', e)$ as $k \to \infty$.

Lemma 3.11 (Vertex Splitting)**.** *$obj(G, \boldsymbol{x}, e) \geq \lim_{k \to \infty} obj(G', \boldsymbol{x}', e)$.*

Lemma 3.11 relies on the following properties of a:

Proposition 3.12 (Vertex splitting)**.** *For all $x_1, x_2, y \in [0, 1]$, define*

$$Q(x_1, x_2, y) := T(x_1 + x_2, s)(yq(x_1)\ell(x_2, y) + yq(x_2)\ell(x_1, y)). \tag{3.10}$$

Then,

1. $\ell(x_1, y)\ell(x_2, y) - \exp(-(x_1 + x_2)y) \geq 0$.
2. *The function $y \to \ell(x_1, y)\ell(x_2, y) + Q(x_1, x_2, y) - e^{-(x_1+x_2)y}\left(1 + \frac{(x_1+x_2)a(1)y^2}{2}\right)$ is initially non-negative on $[0, 1]$, and changes sign at most once. Moreover,*

$$\int_0^1 \ell(x_1, s)\ell(x_2, s) + Q(x_1, x_2, s) - e^{-(x_1+x_2)s}\left(1 + \frac{(x_1 + x_2)a(1)s^2}{2}\right) ds \geq 0.$$

We also make use of the following elementary lower bound on the integral of the product of two functions.

Lemma 3.13. *Suppose that $\lambda, \phi : [0, 1] \to \mathbb{R}$ are integrable, $\lambda \geq 0$, and λ is non-increasing. Moreover, assume that there exists $0 \leq s_c \leq 1$ such that $\phi(s) \geq 0$ for all $s \in [0, s_c]$, and $\phi(s) \leq 0$ for all $s \in [s_c, 1]$. Then,*

$$\int_0^1 \phi(s)\lambda(s)\, ds \geq \lambda(s_c) \int_0^1 \psi(s)\, ds$$

Proof of Lemma 3.11. Recall that $w \in N_G(u) \cup N_G(v) \setminus \{u, v\}$ is the vertex which is copied $k \geq 1$ times in the construction G'. For convenience, we define $\widetilde{\partial}(e) := \partial_G(e) \setminus \partial_G(w)$. We first define $\psi_k(y)$ to be integrand of $\mathrm{obj}(G', \boldsymbol{x}', e)$. To write out this function, it will be convenient to use

$$T(x_1 + x_2, y) = \frac{q(1 - x_1 - x_2)}{x_1 + x_2}\left(1 - \frac{e^{-(x_1+x_2)y}}{(x_1 + x_2)y}\right),$$

and $Q(x_1,x_2,y) = T(x_1+x_2,y)(yq(x_1)\ell(x_2,y)+yq(x_2)\ell(x_1,y))$, where the latter function is defined in (3.10) of Proposition 3.12. Observe then that $\psi_k(y)$ is equal to

$$\sum_{f\in\widetilde{\partial}(e)} T(x_f + x_{f^c}, y)\cdot yq(x_f)(\ell(x_{w,u}/k,y)\ell(x_{w,v}/k,y))^k \prod_{g\in\widetilde{\partial}(e)\setminus\{f\}} \ell(x_g,y)$$
$$+kQ(x_{u,w}/k, x_{v,w}/k, y)\cdot(\ell(x_{w,u}/k,y)\ell(x_{w,v}/k,y))^{k-1} \prod_{g\in\widetilde{\partial}(e)} \ell(x_g,y)$$
$$+(\ell(x_{w,u}/k,y)\ell(x_{w,v}/k,y))^{k} \prod_{g\in\widetilde{\partial}(e)} \ell(x_g,y)$$

Instead of working directly with $\psi_k(y)$, we consider its point-wise limit as $k\to\infty$. First, using the continuity of a, and the fact that $a(0)=1$,

$$\lim_{k\to\infty}(\ell(x_{w,u}/k,y)\ell(x_{w,v}/k,y))^k = \lim_{k\to\infty}(\ell(x_{w,u}/k,y)\ell(x_{w,v}/k,y))^{k-1} = e^{-(x_{w,u}+w_{w,v})y},$$

and

$$\lim_{k\to\infty} k(yq(x_{u,w}/k)\ell(x_{v,w}/k,y) + yq(x_{v,w}/k)\ell(x_{u,w}/k,y)) = (x_{u,w}+x_{v,w})y.$$

Moreover, it is not hard to show that $\lim_{x\to 0^+} T(x,y)$ exists, and is equal to $a(1)y/2$. Thus,

$$\lim_{k\to\infty} kQ(x_{u,w}/k, x_{v,w}/k, y) = \frac{a(1)(x_{u,w}+x_{v,w})y^2}{2}.$$

By combining all these expressions, $\lim_{k\to\infty}\psi_k(y)$ is equal to

$$\sum_{f\in\widetilde{\partial}(e)} T(x_f + x_{f^c}, y)\cdot q(x_f)ye^{-(x_{w,u}+w_{w,v})y} \prod_{g\in\widetilde{\partial}(e)\setminus\{f\}} \ell(x_g,y)$$
$$+\left(\frac{a(1)(x_{u,w}+x_{v,w})y^2}{2}+1\right)e^{-(x_{w,u}+w_{w,v})y} \prod_{g\in\widetilde{\partial}(e)} \ell(x_g,y).$$

Let us compare $\lim_{k\to\infty}\psi_k(y)$ with the integrand of $\text{obj}(G,\boldsymbol{x},e)$:

$$\sum_{f\in\widetilde{\partial}(e)} T(x_f+x_{f^c},y)\cdot q(x_f)y \prod_{g\in\partial(e)\setminus\{f\}} \ell(x_g,y) + \prod_{g\in\partial(e)} \ell(x_g,y) + Q(x_f,x_{f^c},y) \prod_{g\in\widetilde{\partial}(e)} \ell(x_g,y).$$

Define $D_1(y)$ to be the difference of each expression's first term:

$$D_1(y) := \sum_{f\in\widetilde{\partial}(e)} \left(\ell(x_{u,w},y)\ell(x_{v,w},y) - e^{-(x_{w,u}+w_{w,v})y}\right) T(x_f+x_{f^c},y)yq(x_f) \prod_{g\in\widetilde{\partial}(e)\setminus\{f\}} \ell(x_g,y).$$

Similarly, let $D_2(y)$ be the difference of each expression's remaining terms:

$$\left(\ell(x_{u,w},y)\ell(x_{v,w},y) + Q(x_f,x_{f^c},y) - \left(1+\frac{a(1)(x_{u,w}+x_{v,w})y^2}{2}\right)e^{-(x_{w,u}+w_{w,v})y}\right) \prod_{g\in\widetilde{\partial}(e)} \ell(x_g,y).$$

Observe now that we can exchange the order of integration and point-wise convergence so that

$$\lim_{k\to\infty} \text{obj}(G',\boldsymbol{x}',e) = \lim_{k\to\infty}\int_0^1 \psi_k(y)\,dy = \int_0^1 \lim_{k\to\infty}\psi_k(y)\,dy.$$

Thus, to complete the proof it suffices to show that $\int_0^1 D_i(y)\,dy \geq 0$ for each $i \in [2]$. We start with D_1. Observe that $\ell(x_{u,w}, y)\ell(x_{v,w}, y) - e^{-(x_{w,u}+w_{w,v})y} \geq 0$ for all $y \in [0,1]$ by the first property of Proposition 3.12. Moreover, the remaining terms in each summand of D_1 are non-negative, so $\int_0^1 D_1 \geq 0$. Consider now D_2. Observe that the function $y \to \prod_{g \in \widetilde{\partial}(e)} \ell(x_g, y)$ is non-increasing in y, as $\ell(x_g, y) := 1 - yq(x_g)$, and $q(x_g) \in [0,1]$ for $x_g \in [0,1]$. Moreover, by the second property of Proposition 3.12, the function

$$y \to \left(\ell(x_{u,w}, y)\ell(x_{v,w}, y) + Q(x_f, x_{f^c}, y) - \left(1 + \frac{a(1)(x_{u,w} + x_{v,w})y^2}{2}\right) e^{-(x_{w,u}+w_{w,v})y}\right),$$

is initially non-negative, changes sign at most once, and has a non-negative integral. Thus, we can apply Lemma 3.13 (with λ as the first function, and ϕ as the second), to conclude that $\int_0^1 D_2 \geq 0$. ☐

Given any $\varepsilon > 0$, we can apply Lemma 3.11 to each vertex of $N_G(u) \cup N_G(v) \setminus \{u, v\}$ to get a graph $G^* = (V^*, E^*)$, and a fractional matching $\boldsymbol{x}^*$ of G^*, such that $x^*_e = x_e$, and $x^*_f \leq \varepsilon$ for all $f \in \partial(e)$. Moreover, $\text{obj}(G^*, \boldsymbol{x}^*, e) \leq \text{obj}(G, \boldsymbol{x}, e) + \varepsilon$. Since this holds for each $\varepsilon > 0$, the infimum of obj for a fixed edge e with fractional value x_e occurs provided $|\partial(e)| \to \infty$ and $\max_{f \in \partial(e)} x_f = o(1)$. In particular, suppose G'_k is the graph formed after splitting each vertex of $N_G(u) \cup N_G(v) \setminus \{u, v\}$ into $k \geq 1$ copies. Let $\phi_k(y)$ be the integrand of obj when evaluated on G'_k, its corresponding fractional matching, and the edge e. By the above discussion,

$$\text{obj}(G, \boldsymbol{x}, e) \geq \lim_{k \to \infty} \int_0^1 \phi_k(y)\,dy.$$

On the other hand, for each $y \in [0,1]$,

$$\phi_k(y) = \prod_{g \in \partial(e)} (\ell(x_g/k, y))^k + \sum_{f \in \partial(e)} T((x_f + x_{f^c})/k, y) \cdot q(x_f/k) y \prod_{g \in \partial(e) \setminus \{f\}} (\ell(x_g/k, y))^k. \tag{3.11}$$

Now, by taking $k \to \infty$, and applying the same asymptotic computations from the proof of Lemma 3.11,

$$\lim_{k \to \infty} \phi_k(y) = \left(1 + \sum_{f \in \partial(e)} \frac{a(1) x_f y^2}{2}\right) e^{-\sum_{f \in \partial(e)} x_f y}.$$

Thus, since $\sum_{f \in \partial(e)} x_f = 2(1 - x_e)$, $\lim_{k \to \infty} \phi_k(y) = \exp(-2y(1 - x_e))\,(1 + a(1)(1 - x_e)y^2)$, and so

$$\text{obj}(G, \boldsymbol{x}, e) \geq \int_0^1 e^{-2(1-x_e)y} \left(1 + a(1)(1 - x_e)y^2\right)\,dy, \tag{3.12}$$

where we have once again exchanged the order of integration and point-wise convergence. The proof of Theorem 3.2 now follows immediately.

Proof of Theorem 3.2. First, observe that after multiplying (3.9) by $a(x_e)$, $\mathbb{P}[e \in \mathcal{M} \mid X_e = 1] \geq a(x_e) \cdot \text{obj}(G, \boldsymbol{x}, e)$. Now, after applying (3.12),

$$a(x_e) \cdot \text{obj}((x_f)_{f \in \partial(e)}) \geq a(x_e) \int_0^1 e^{-2y(1-x_e)}(1 + a(1)(1 - x_e)y^2)\,dy. \tag{3.13}$$

Upon evaluating the integral in (3.13), we get a function of x_e whose minimum occurs at $x_e = 0$ when it takes on the value $\frac{e^2 - 4e^3 + e^4 + 20e - 22}{4e^2} \geq 0.474035$. The proof is thus complete. ☐

3.2 Proving Theorem 3.3

We now consider when Algorithm 2 is executed on a bipartite graph $G = (V, E)$ using the attenuation function $a(x) = (1-x)^4/(e^x - ex)^2$. Note that the precise bipartition of G is not important, and the theorem in fact holds when G is only triangle-free[8].

Our proof follows the same structure as the general graph case in that after fixing $e \in E$, we consider at most one early edge (i.e., $|\mathcal{F}_e| \leq 1$). Observe that our new attenuation function satisfies the same analytic properties stated in Propositions 3.8 and 3.10, and so we can apply the argument from the previous section to get that

$$\mathbb{P}[e \in \mathcal{M} \mid S_e = 1] \geq \int_0^1 \prod_{g \in \partial(e)} \ell(x_g, y) + \sum_{f \in \partial(e)} T(x_f + x_{f^c}, y) \cdot q(x_f) y \prod_{g \in \partial(e) \setminus \{f\}} \ell(x_g, y)\, dy. \quad (3.14)$$

(See Appendix A for a verification of the claimed analytic properties of a). Unfortunately, the attenuation function does *not* satisfy the properties of Proposition 3.12, and so we cannot conclude that the r.h.s. of (3.14) is minimized for vanishing edge values. Instead, we use the lack of triangles in G to first simplify this expression before identifying its infimum. Specifically, observe that $x_{f^c} = 0$ for each $f \in \partial(e)$. Thus, $T(x_f + x_{f^c})$ becomes $T(x_f)$, and so the r.h.s. is

$$\int_0^1 \prod_{g \in \partial(e)} \ell(x_g, y) + \sum_{f \in \partial(e)} T(x_f, y) \cdot q(x_f) y \prod_{g \in \partial(e) \setminus \{f\}} \ell(x_g, y)\, dy$$

Let us denote the above equation by $\mathrm{obj}(G, \boldsymbol{x}, e)$, and consider the same vertex splitting procedure as before for $k \geq 1$. Note that the resulting graph G' is triangle-free, as G is triangle-free. By restricting obj to triangle-free inputs, we are able to prove an analogous version of Lemma 3.11. We rely on the following properties of a, which we note are a special case (i.e, weakening) of those presented in Proposition 3.12.

Proposition 3.14 (Triangle-free vertex splitting)**.** *For all $x, y \in [0,1]$,*

1. *$\ell(x, y) - \exp(-xy) \geq 0$.*
2. *The function $y \to \ell(x,y) + yq(x)T(x,s) - e^{-xy}\left(1 + \frac{xa(1)y^2}{2}\right)$ is initially non-negative on $[0,1]$, and changes sign at most once. Moreover,*

$$\int_0^1 \ell(x,s) + yq(x)T(x,s) - e^{-xs}\left(1 + \frac{xa(1)s^2}{2}\right) ds \geq 0.$$

Lemma 3.15 (Triangle-free vertex splitting)**.** *$obj(G, \boldsymbol{x}, e) \geq \lim_{k \to \infty} obj(G', \boldsymbol{x}', e)$.*

The proof of Lemma 3.15 is identical to that of Lemma 3.11, but with a different attenuation function substituted in.

Proof of Theorem 3.3. We can use Lemma 3.15 to conclude that no matter the value of x_e, the infimum of obj occurs as $|\partial(e)| \to \infty$ and $\max_{f \in \partial(e)} x_f = o(1)$. Moreover, the same asymptotic computation used in (3.12) can be applied to get that

$$\mathrm{obj}(G, \boldsymbol{x}, e) \geq \int_0^1 e^{-2(1-x_e)y}\left(1 + a(1)(1-x_e)y^2\right) dy, \quad (3.15)$$

[8] We cannot claim a selectability lower bound of 0.476 for *all* triangle-free graphs, as our 1-regular reduction may create triangles when G is not bipartite.

where the only difference is that the r.h.s. of (3.15) now has $a(1) = 4/e^2$. Thus,

$$\mathbb{P}[e \in \mathcal{M} \mid X_e = 1] \geq a(x_e) \int_0^1 e^{-2y(1-x_e)}(1 + a(1)(1-x_e)y^2)\, dy.$$

After evaluating the above integral, we get a function of x_e whose minimum occurs at $x_e = 0$ when it takes on the value $\frac{e^2+e^4-10}{2e^4} \geq 0.476089$. The proof is thus complete. □

3.3 Impossibility Result for RCRS

Theorem 3.16. *No RCRS is better than* $1/2$*-selectable on bipartite graphs.*

In order to prove Theorem 3.16, we again analyze the complete 1-regular bipartite graph with $2n$ vertices and uniform edge values, except instead of adversarially chosen edge arrivals, we work with random order edge arrivals. Let $G = (U_1, U_2, E)$ where $E = U_1 \times U_2$, and $|U_1| = |U_2| = n$ for $n \geq 1$, and set $x_e = 1/n$ for all $e \in E$. Once again, we work in the asymptotic setting as $n \to \infty$. We say that a sequence of events $(\mathcal{E}_n)_{n \geq 1}$ occurs *with high probability* (w.h.p.), provided $\mathbb{P}[\mathcal{E}_n] \to 1$ as $n \to \infty$.

Lemma 3.17. *For any RCRS which outputs matching* M *on* G*,* $\mathbb{E}[|M|] \leq \frac{(1+o(1))n}{2}$.

Assuming Lemma 3.17, Theorem 3.16 then follows immediately.

Proof of Theorem 3.16. Suppose that we fix an arbitrary RCRS which is c-selectable for $c \geq 0$, and let M be the matching it creates when executing on G. Clearly, $\mathbb{E}[|M|]/n \geq c$ by definition of c-selectability. By applying Lemma 3.17, and taking $n \to \infty$, we get that $c \leq 1/2$. □

To prove Lemma 3.17, we consider an algorithm for maximizing $\mathbb{E}[|M|]$ and show that the cardinality of the matching cannot exceed $\frac{(1+o(1))n}{2}$ in expectation. Without loss of generality, we can assume such an algorithm is deterministic, even though we will refer to it colloquially as an "RCRS".

For each $1 \leq t \leq n^2$, let F_t be the t^{th} edge of G presented to the RCRS, and denote its state by X_{F_t} (clearly, $X_{F_t} \sim \mathrm{Ber}(1/n)$). Observe that if $E_t := \{F_1, \ldots, F_t\}$, then conditional on E_t, F_{t+1} is distributed u.a.r. amongst $E \setminus E_t$ for $0 \leq t \leq n^2 - 1$. If $\mathcal{M}_t$ is the matching constructed by the RCRS after t rounds, then since the RCRS is deterministic, $\mathcal{M}_t$ is a function of $(F_i, X_{F_i})_{i=1}^t$. Thus, $\mathcal{M}_t$ is measurable with respect to $\mathcal{H}_t$, the sigma-algebra generated from $(F_i, X_{F_i})_{i=1}^t$ (here $\mathcal{H}_0 := \{\emptyset, \Omega\}$, the trivial sigma-algebra). We refer to $\mathcal{H}_t$ as the *history* after t steps. It will be convenient to define $W(t) := n \cdot |\mathcal{M}_t|$ for each $0 \leq t \leq n^2$. We can think of $W(t)$ as indicating the weight of the matching $\mathcal{M}_t$, assuming each edge of G has weight n.

Let $w(s) := s/(1+s)$ for each real $s \geq 0$. Note that w is the unique solution to the differential equation $w' = (1-w)^2$ with initial condition $w(0) = 0$. Roughly speaking, we will prove that w.h.p., the random variable $W(t)/n^2$ is upper bounded by $(1+o(1))w(t/n^2)$ for each $0 \leq t \leq n^2$.

Proposition 3.18. *For each constant* $0 \leq \delta < 1$*, with probability at least* $1 - o(1/n^2)$*,*

$$W(t) \leq (1+o(1))w(t/n^2)n^2$$

for all $0 \leq t \leq \delta n^2$*.*

We emphasize that in Proposition 3.18, a constant δ is fixed first, and n is taken to ∞ afterward. As we take constant δ to be arbitrarily close to 1, the $1/2$ upper bound in Lemma 3.17 is established. We now provide the proof of this fact. The rest of this section is then devoted to proving Proposition 3.18.

Proof of Lemma 3.17 using Proposition 3.18. Fix $0 \leq \delta < 1$. Observe that Proposition 3.18 implies

$$\mathbb{E}[W(\delta n^2)] \leq (1 - o(1/n^2))(1 + o(1))w(\delta)n^2 + o(1/n^2)\delta n^2 = (1 + o(1))w(\delta)n^2, \tag{3.16}$$

where the $o(1/n^2)\delta n^2$ term uses the bound that $W(\delta n^2)$ cannot exceed the expected weight of active edges up to time δn^2, which is δn^2. Moreover, the same bound yields $\mathbb{E}[W(n^2)-W(\delta n^2)] \leq (1-\delta)n^2$. Thus,

$$\mathbb{E}[W(n^2)] \leq (1 + o(1))\frac{\delta}{1+\delta}n^2 + (1 - \delta)n^2,$$

and so after dividing by n^2, $\mathbb{E}[|\mathcal{M}_{n^2}|]/n \leq (1+o(1))\frac{\delta}{1+\delta}+(1-\delta)$. Since this holds for each $0 \leq \delta < 1$, and $\frac{\delta}{1+\delta} + (1 - \delta) \to 1/2$ as $\delta \to 1$, we get that

$$\frac{\mathbb{E}[|\mathcal{M}_{n^2}|]}{n} \leq (1 + o(1))\frac{1}{2}.$$

As $\mathbb{E}[|\mathcal{M}_{n^2}|]$ is an upper bound on the expected size of any matching created by an RCRS, the proof is complete. □

In order to prove Proposition 3.18, for each constant $0 \leq \delta < 1$, and $0 \leq t \leq \delta n^2$, we first upper bound the expected one-step changes of $W(t)$, conditional on the current history $\mathcal{H}_t$. More formally, we upper bound $\mathbb{E}[\Delta W(t) \mid \mathcal{H}_t]$, where $\Delta W(t) := W(t+1) - W(t)$. Our goal is to show that

$$\mathbb{E}[\Delta W(t) \mid \mathcal{H}_t] \leq (1 + o(1)) \left(1 - \frac{W(t)}{n^2}\right)^2.$$

It turns out that this upper bound only holds for *most* instantiations of the random variables $(F_i)_{i=0}^{t}$ (upon which the history $\mathcal{H}_t$ depends). We quantify this by defining a sequence of events, $(Q_t)_{t=0}^{\delta n^2}$, which occur w.h.p., and which help ensure the upper bound holds.

Fix a pair of vertex subsets (S_1, S_2), where $S_j \subseteq U_j$ for $j = 1, 2$. We say that (S_1, S_2) is *large*, provided $|S_j| \geq n/2$ for $j = 1, 2$. Given $0 \leq t \leq n^2$, we say that (S_1, S_2) is *well-controlled* at time t, provided

$$|L_t \cap S_1 \times S_2| \leq (1 + n^{-1/3})|S_1||S_2| \left(1 - \frac{t}{n^2}\right), \tag{3.17}$$

where $L_t := E \setminus E_t$ denotes the edges which have yet to arrive after t rounds. We define the event Q_t to occur, provided *each* pair of *large* vertex subsets is well-controlled at time t. Observe that the event Q_t is $\mathcal{H}_t$-measurable.

Lemma 3.19. *For any constant* $0 \leq \delta < 1$, $\mathbb{P}[\cap_{i=0}^{\delta n^2} Q_i] \geq 1 - o(1/n^2)$.

Proof. We shall prove that for each $0 \leq i \leq \delta n^2$, Q_i holds with probability at least $1 - o(1/n^4)$. Since there are $\delta n^2 \leq n^2$ rounds, this will imply that $\mathbb{P}[\cap_{i=0}^{\delta n^2} Q_i] \geq 1 - o(1/n^2)$ after applying a union bound.

Observe first that $L_i = E \setminus E_i$ is a uniformly random subset of E of size $n^2 - i$. Thus, $|L_i \cap S_1 \times S_2|$ is distributed as a hyper-geometric random variable on a universe of size n^2 with success probability $|S_1||S_2|/(n^2 - i)$ (we denote this by $|L_i \cap S_1 \times S_2| \sim \text{Hyper}(n^2, |S_1||S_2|, n^2 - i)$. Now, the distribution $\text{Hyper}(n^2, |S_1||S_2|, n^2 - i)$ is at least as concentrated about its expectation as the binomial distribution, $\text{Bin}(n^2, |S_1||S_2|/(n^2 - i))$ (see Chapter 21 in [17] for details). As such, standard Chernoff bounds ensure that if $\mu := |S_1||S_2| \left(1 - \frac{i}{n^2}\right)$, then for each $0 < \lambda < 1$,

$$\mathbb{P}[|L_i \cap S_1 \times S_2| \geq (1 + \lambda)\mu] \leq \exp\left(\frac{-\lambda^2 \mu}{3}\right).$$

By assumption, $|S_1||S_2| \geq n^2/4$. Thus, since $0 \leq i \leq \delta n^2$, $\mu \geq n^2(1-\delta)/4$. By taking $\lambda = n^{-1/3}$, we get that

$$|L_i \cap S_1 \times S_2| \geq (1+\lambda)|S_1||S_2|\left(1 - \frac{i}{n^2}\right)$$

with probability at most $\exp\left(-\frac{n^{4/3}(1-\delta)}{12}\right)$ which is $\exp\left(-\Omega(n^{4/3})\right)$ because $\delta < 1$ is a constant. Now, after union bounding over at most 4^n subsets, we get that Q_t does *not* occur with probability at most $4^n \exp(-\Omega(n^{4/3})) = o(1/n^4)$. The proof is thus complete.

□

Let us define $\varepsilon(t) := 4/n^{1/3} + 2t/n^{7/3}$ for each $0 \leq t \leq n^2$. Upon conditioning on the history $\mathcal{H}_t$ for $0 \leq t \leq \delta n^2$, if Q_t occurs and $W(t) \leq (1+\varepsilon(t))w(t/n^2)n^2$, then we can upper bound $\mathbb{E}[\Delta W(t) \mid \mathcal{H}_t]$.

Lemma 3.20. *For each $0 \leq t \leq \delta n^2$, if Q_t occurs and $W(t) \leq (1+\varepsilon(t))w(t/n^2)n^2$, then*

$$\mathbb{E}[\Delta W(t) \mid \mathcal{H}_t] \leq (1+n^{-1/3})\left(1 - \frac{W(t)}{n^2}\right)^2. \tag{3.18}$$

Remark 3.21. Note that the assumption $W(t) \leq (1+\varepsilon(t))w(t/n^2)n^2$ is what we are trying to prove in Proposition 3.18. This is a common feature of Wormald's dfferential equation method [28] (see [Lemma 8, [27]] for an explicit statement), where the *deterministic* behaviour of the function $w(t/n^2)$ guides the assumptions one places on the random variable $W(t/n^2)$. We shall see that the proof structure of Proposition 3.18 is amicable to this approach.

Proof of Lemma 3.20. Suppose $0 \leq t \leq \delta n^2$ is such that Q_t occurs and $W(t) \leq (1+\varepsilon(t))w(t/n^2)n^2$. Observe that since $W(t) = n|\mathcal{M}_t|$, it suffices to show that

$$\mathbb{E}[|\mathcal{M}_{t+1}| - |\mathcal{M}_t| \mid \mathcal{H}_t] \leq \frac{1}{n}(1+n^{-1/3})\left(1 - \frac{|\mathcal{M}_t|}{n}\right)^2.$$

For $j = 1, 2$, let $U_{j,t}$ denote the vertices of U_j which are *not* selected by the RCRS after edges $E_t = \{F_1, \dots, F_t\}$ arrive, where $U_{j,0} := U_j$. Since the graph is bipartite, we have $|U_{j,t}| = n - |\mathcal{M}_t|$. Observe that a necessary condition for the RCRS to match F_{t+1} is that it must be an edge of $U_{1,t} \times U_{2,t}$. On the other hand, conditional on $\mathcal{H}_t$, F_{t+1} is distributed u.a.r. amongst $L_t := E \setminus E_t$. Thus,

$$\mathbb{P}[F_{t+1} \in U_{1,t} \times U_{2,t} \mid \mathcal{H}_t] = \frac{|(U_{1,t} \times U_{2,t}) \cap L_t|}{|E \setminus L_t|} = \frac{|(U_{1,t} \times U_{2,t}) \cap L_t|}{n^2 - t}, \tag{3.19}$$

where the equality follows since $|E \setminus L_t| = n^2 - t$. In order to simplify (3.19), we make use the upper bound on $W(t)$, and the occurrence of the event Q_t. First, $W(t) \leq (1+\varepsilon(t))w(t/n^2)n^2$, where we note that $w(t/n^2) \leq w(\delta) = \frac{\delta}{1+\delta} < 1/2$, and hence for a sufficiently large n we have $W(t) \leq n^2/2$ and hence $|\mathcal{M}_t| \leq n/2$. Thus, $|U_{j,t}| = (n - |\mathcal{M}_t|) \geq n/2$, and so we can apply (3.17) to subsets $U_{1,t}$ and $U_{2,t}$ to ensure that

$$|(U_{1,t} \times U_{2,t}) \cap L_t| \leq (1+n^{-1/3})(n - |\mathcal{M}_t|)^2\left(1 - \frac{t}{n^2}\right).$$

Combined with (3.19), this implies that

$$\mathbb{P}[F_{t+1} \in U_{1,t} \times U_{2,t} \mid \mathcal{H}_t] \leq \frac{(1+n^{-1/3})(n - |\mathcal{M}_t|)^2}{n^2} = (1+n^{-1/3})\left(1 - \frac{|\mathcal{M}_t|}{n}\right)^2. \tag{3.20}$$

Now, a second necessary condition for the RCRS to match F_{t+1} is that F_{t+1} must be active (i.e., $X_{F_{t+1}} = 1$). This event occurs with probability $1/n$, independently of the event $F_{t+1} \in U_{1,t} \times U_{2,t}$ and the history $\mathcal{H}_t$. By combining both necessary conditions, and (3.20),

$$\mathbb{E}[|\mathcal{M}_{t+1}| - |\mathcal{M}_t| \mid \mathcal{H}_t] \leq \frac{1}{n}(1 + n^{-1/3}) \left(1 - \frac{|\mathcal{M}_t|}{n}\right)^2,$$

and so the proof is complete.

□

Observe that since $\cap_{t=0}^{\delta n^2} Q_t$ holds w.h.p., by scaling both t and $W(t)$ by n^2, (3.18) suggests the following differential inequality satisfied for each real $s \geq 0$:

$$r'(s) \leq (1 - r(s))^2, \tag{3.21}$$

where $r(0) = 0$ (it is useful to think of $s = t/n^2$). When (3.21) is replaced with equality, recall that $w(s) := s/(1+s)$ is then the unique solution to the corresponding differential equation. A classical result due to Petrovitsch [24] implies that any solution $r(s)$ to (3.21) is dominated by the function $w(s)$; that is, $r(s) \leq w(s)$ for all $0 \leq s \leq \delta$ (this can also be seen as an application of Gronwall's inequality). The analogous statement holds when working with the (scaled) random variables $(W(t)/n^2)_{t=0}^{\delta n^2}$, and this is exactly the statement of Proposition 3.18.

If we were working with the greedy RCRS, then (3.18) of Lemma 3.20 would be replaced with equality. In this case, Wormald's differential equation method [28, 27] immediately implies that w.h.p. $W(t) = (w(t/n^2) + o(1))n^2$ for all $0 \leq t \leq \delta n^2$. Unfortunately, we are not aware of a similar "black-box" theorem statement which allows for the inequality in Lemma 3.20. Proposition 3.18 can instead be proven using the *critical interval* approach, which was first used in [26], and whose terminology is due to [8]. The proof follows the same structure as in [Section 3.3, [7]], with a minor modification needed to account for the event $\cap_{i=0}^{\delta n^2} Q_i$. We also need to use Freedman's inequality [16] instead of the Azuma–Hoeffding inequality. The details of the proof are included below for completeness.

Proof of Proposition 3.18. Fix $0 \leq \delta < 1$. For each $0 \leq t \leq n^2$, recall that $\varepsilon(t) := 4/n^{1/3} + 2t/n^{7/3}$. We then define the *critical interval* $I_t := [(w(t/n^2) + \varepsilon(t)/2)n^2, (w(t/n^2) + \varepsilon(t))n^2]$. The benefit of working with I_t is that if $W(t/n^2)$ enters I_t, then it is tightly bounded around $w(t/n^2)$. We can then use the analytic properties of w to show that $W(t/n^2)$ is unlikely to exceed $(w(t/n^2) + \varepsilon(t))n^2$. We prove this using a supermartingale argument in conjunction with Freedman's inequality; see [Section 2, [7]] for the exact form of the inequality we invoke.

Given $j \leq t \leq \delta n^2$, we define $\mathcal{E}_{j,t}$ to occur, provided the following events occur:

1. $W(i) \leq (w(i/n^2) + \varepsilon(i))n^2$ for all $0 \leq i \leq t$,
2. $W(i) \in I_i$ for all $j \leq i \leq t$,
3. $\cap_{i=0}^{t} Q_i$ occurs.

Thus, if $\mathcal{E}_{j,t}$ occurs, then $W(i)$ lies within the critical interval for steps, $j \leq i \leq t$, and also does not exceed the right-hand endpoint of I_i at any point $0 \leq i \leq t$. The final condition ensures that the rare event $\neg Q_i$ does not occur at any time step $0 \leq i \leq t$.

For each $0 \le j \le \delta n^2$, we define a sequence of random variables, $(N_j(t))_{t=j}^{\delta n^2}$, where

$$N_j(t) = \begin{cases} W(t) - (w(t/n^2) + \varepsilon(t)) & \text{if } t = j, \text{ or } t > j \text{ and } \mathcal{E}_{j,t-1} \text{ occurs.} \\ N_j(t-1) & \text{otherwise.} \end{cases}$$

Note that we can view $N_j(t)$ as being *frozen* or *stopped*, at the first time $t \ge j$ when $\mathcal{E}_{j,t-1}$ fails to occur.

In order to see why we have defined these random variables, let us suppose that $W(t) > (w(t/n^2) + \varepsilon(t))n^2$ for some $0 \le t \le \delta n^2$, and t is the first step at which this occurs. First observe that $t > 0$, since $W(0) = 0$. Now, if we assume that $\cap_{i=0}^{\delta n^2} Q_i$ also occurs, then $\mathcal{E}_{j,t-1}$ must occur for some $0 \le j \le t-1$. Let j be the *minimum* such time. Again, $j > 0$, as $W(0) = 0 < \varepsilon(0)n^2/2$ (and so $W(0) \notin I_0$). Moreover, since $\cap_{i=0}^{j} Q_j$ occurs and $W(j) \le (w(j/n^2) + \varepsilon(j))n^2$, it must be that $W(j-1) < (w((j-1)/n^2) + \varepsilon(j-1))n^2$. Thus, j is the first step the random variables $(W(i))_{i=0}^{\delta n^2}$ enter the critical interval.

Our goal is to show that if the above t exists, then the difference between $N_j(\delta n^2)$ and $N_j(j)$ must be large. First observe that $N_j(\delta n^2) = N_j(t) > 0$, by the assumption on t. Now, $W(j) \le W(j-1) + n$ (as at most one edge is added in a round), and $W(j-1) < (w((j-1)/n^2) + \varepsilon(j-1))n^2$ (as already argued). Thus, $W(j) \le (w((j-1)/n^2) + \varepsilon(j-1)/2)n^2 + n$. Since $w((j-1)/n^2) \le w(j/n^2)$ and $\varepsilon(j-1) \le \varepsilon(j)$, we get that

$$\begin{aligned} N_j(j) &= W(j) - (w(j/n^2) + \varepsilon(j))n^2 \\ &\le (w((j-1)/n^2) + \varepsilon(j-1)/2)n^2 + n - (w(j/n^2) + \varepsilon(j))n^2 \\ &= (w((j-1)/n^2) + \varepsilon(j-1)/2 - w(j/n^2) - \varepsilon(j))n^2 + n \\ &\le -\frac{\varepsilon(j)n^2}{2} + n \le -\frac{\varepsilon(0)n^2}{2} + n \le -n^{5/3}, \end{aligned}$$

where the last inequalities follow since $\varepsilon(0) \le \varepsilon(j)$, and $\varepsilon(0)n^2 = 4n^{5/3}$. Putting everything together, we get that

$$\begin{aligned} &\mathbb{P}[\cap_{i=0}^{\delta n^2} Q_i \text{ and } \exists\, 0 \le t \le \delta n^2 : W(t) > (w(t/n^2) + \varepsilon(t))n^2] \\ &\le \mathbb{P}[\cap_{i=0}^{\delta n^2} Q_i \text{ and } \exists\, 0 < j \le \delta n^2 : N_j(\delta n^2) - N_j(j) > n^{5/3}] \\ &\le \sum_{j=1}^{\delta n^2} \mathbb{P}[N_j(\delta n^2) - N_j(j) > n^{5/3}]. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{P}[\exists\, 0 \le t \le \delta n^2 : W(t) > (w(t/n^2) + \varepsilon(t))n^2] &\le \sum_{j=0}^{\delta n^2} \mathbb{P}[N_j(\delta n^2) - N_j(j) > n^{5/3}] + \mathbb{P}[\cup_{i=0}^{\delta n^2} \neg Q_i] \\ &\le \sum_{j=0}^{\delta n^2} \mathbb{P}[N_j(\delta n^2) - N_j(j) > n^{5/3}] + o(1/n^2), \end{aligned}$$

where the final line follows by Lemma 3.19. Thus, in order to complete the proof, it suffices to show that $\mathbb{P}[N_j(\delta n^2) - N_j(j) > n^{5/3}] \le o(1/n^4)$ for each $0 < j \le \delta n^2$.

Fix $0 < j \le \delta n^2$. Using Lemma 3.20, we shall first show that $(N_j(t))_{t=j}^{\delta n^2}$ forms a supermartingale. To see this, let us assume that $\mathcal{E}_{j,t}$ occurs for $t \ge j$, as the other case is easy. Under this assumption, the events of Lemma 3.20 hold, and so we get that

$$\begin{aligned}\mathbb{E}[\Delta W(t) \mid H_t] &\leq (1+n^{-1/3})\left(1-\frac{W(t)}{n^2}\right)^2 \\ &\leq \big(1-w(t/n^2)\big)^2+n^{-1/3},\end{aligned}$$

where the second inequality follows since $\frac{W(t)}{n^2} \geq (w(t/n^2)+\varepsilon(t)/2) \geq w(t/n^2)$ (as $W(t)/n^2$ is within the critical interval I_t). By applying this inequality, and $\varepsilon(t+1)-\varepsilon(t) = 2n^{-7/3}$, we get that

$$\begin{aligned}\mathbb{E}[\Delta N_j(t) \mid H_t] &= \mathbb{E}[\Delta W(t) \mid H_t] - (w((t+1)/n^2)-w(t/n^2))n^2 - (\varepsilon(t+1)-\varepsilon(t))n^2 \\ &\leq \big(1-w(t/n^2)\big)^2 - (w((t+1)/n^2)-w(t/n^2))n^2 - n^{-1/3},\end{aligned}$$

Now, $(w((t+1)/n^2)-w(t/n^2))n^2 = w'(t/n^2)+O(1/n^2)$ by Taylor's theorem, and we also know that $w'(t/n^2) = (1-w(t/n^2))^2$. Thus, combined with the above, $\mathbb{E}[\Delta N_j(t) \mid H_t] \leq 0$ for n sufficiently large. It follows that $(N(t))_{t=j}^{\delta n^2}$ forms a supermartingale.

In order to apply Freedman's inequality, we must control the one-step changes in the supermartingale. Recall that $|\Delta W(t)| \leq n$, and so,

$$|\Delta N_j(t)| \leq n + O(1/n^2) - n^{-1/3} \leq n$$

for n sufficiently large. Similarly,

$$\mathbb{E}[|\Delta N_j(t)| \mid \mathcal{H}_t] \leq \frac{1}{n}n + O(1/n^2) - n^{-1/3} \leq 1,$$

for n sufficiently large. By combining these upper bounds, we can control the conditional variance of $\Delta N(t)$ in the following way:

$$\mathbf{Var}[\Delta N(t) \mid H_t] \leq \mathbb{E}[|\Delta N(t)|^2 \mid H_t] \leq n.$$

Using the form of Freedman's inequality from [Section 2, [7]], it follows that for any $\lambda > 0$,

$$\mathbb{P}[\exists\, 0 \leq t \leq \delta n^2 : N_j(t) - N_j(j) \geq \lambda] \leq \exp\left(-\frac{\lambda^2}{2(\delta n^3 + n\lambda)}\right).$$

In particular, setting $\lambda = n^{5/3}$,

$$\mathbb{P}[N_j(\delta n^2) - N_j(j) \geq n^{-5/3}] \leq \exp\left(-\Theta(n^{1/3})\right).$$

Since $\exp\left(-\Theta(n^{1/3})\right) = o(1/n^4)$, the desired upper bound holds, and so the proof is complete. □

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References

[1] Marek Adamczyk, Fabrizio Grandoni, and Joydeep Mukherjee. Improved approximation algorithms for stochastic matching. In Nikhil Bansal and Irene Finocchi, editors, *Algorithms - ESA 2015 - 23rd Annual European Symposium, Patras, Greece, September 14-16, 2015, Proceedings*, volume 9294 of *Lecture Notes in Computer Science*, pages 1–12. Springer, 2015.

[2] Marek Adamczyk and Michał Włodarczyk. Random order contention resolution schemes. In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 790–801. IEEE, 2018.

[3] Saeed Alaei, MohammadTaghi Hajiaghayi, and Vahid Liaghat. Online prophet-inequality matching with applications to ad allocation. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, EC '12, page 18–35, New York, NY, USA, 2012. Association for Computing Machinery.

[4] Nick Arnosti and Will Ma. Tight guarantees for static threshold policies in the prophet secretary problem. In *Proceedings of the 23rd ACM Conference on Economics and Computation*, EC '22, page 242, New York, NY, USA, 2022. Association for Computing Machinery.

[5] Nikhil Bansal, Anupam Gupta, Jian Li, Julián Mestre, Viswanath Nagarajan, and Atri Rudra. When LP is the cure for your matching woes: Improved bounds for stochastic matchings. *Algorithmica*, 63(4):733–762, 2012.

[6] Alok Baveja, Amit Chavan, Andrei Nikiforov, Aravind Srinivasan, and Pan Xu. Improved bounds in stochastic matching and optimization. *Algorithmica*, 80(11):3225–3252, Nov 2018.

[7] Patrick Bennett and Andrzej Dudek. A gentle introduction to the differential equation method and dynamic concentration, 2020.

[8] Tom Bohman, Alan Frieze, and Eyal Lubetzky. Random triangle removal. *Advances in Mathematics*, 280:379–438, 2015.

[9] Brian Brubach, Nathaniel Grammel, Will Ma, and Aravind Srinivasan. Improved guarantees for offline stochastic matching via new ordered contention resolution schemes. *Advances in Neural Information Processing Systems*, 34:27184–27195, 2021.

[10] Simon Bruggmann and Rico Zenklusen. An optimal monotone contention resolution scheme for bipartite matchings via a polyhedral viewpoint. *Math. Program.*, 191(2):795–845, 2022.

[11] Chandra Chekuri, Jan Vondrák, and Rico Zenklusen. Submodular function maximization via the multilinear relaxation and contention resolution schemes. *SIAM Journal on Computing*, 43(6):1831–1879, 2014.

[12] Iain Dunning, Joey Huchette, and Miles Lubin. Jump: A modeling language for mathematical optimization. *SIAM review*, 59(2):295–320, 2017.

[13] Soheil Ehsani, MohammadTaghi Hajiaghayi, Thomas Kesselheim, and Sahil Singla. Prophet secretary for combinatorial auctions and matroids. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '18, page 700–714, USA, 2018. Society for Industrial and Applied Mathematics.

[14] Tomer Ezra, Michal Feldman, Nick Gravin, and Zhihao Gavin Tang. Prophet matching with general arrivals. *Mathematics of Operations Research*, 47(2):878–898, 2022.

[15] Moran Feldman, Ola Svensson, and Rico Zenklusen. Online contention resolution schemes with applications to bayesian selection problems. *SIAM Journal on Computing*, 50(2):255–300, 2021.

[16] David A. Freedman. On Tail Probabilities for Martingales. *The Annals of Probability*, 3(1):100 – 118, 1975.

[17] Alan Frieze and Michał Karoński. *Introduction to Random Graphs.* Cambridge University Press, 2015.

[18] Hu Fu, Zhihao Gavin Tang, Hongxun Wu, Jinzhao Wu, and Qianfan Zhang. Random Order Vertex Arrival Contention Resolution Schemes for Matching, with Applications. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, *48th International Colloquium on Automata, Languages, and Programming (ICALP 2021)*, volume 198 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 68:1–68:20, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[19] Nikolai Gravin and Hongao Wang. Prophet inequality for bipartite matching: Merits of being simple and non adaptive. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, pages 93–109, 2019.

[20] Jiashuo Jiang, Will Ma, and Jiawei Zhang. Tight guarantees for multi-unit prophet inequalities and online stochastic knapsack. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1221–1246. SIAM, 2022.

[21] Richard M Karp and Michael Sipser. Maximum matching in sparse random graphs. In *22nd Annual Symposium on Foundations of Computer Science (sfcs 1981)*, pages 364–375. IEEE, 1981.

[22] Robert Kleinberg and S Matthew Weinberg. Matroid prophet inequalities and applications to multi-dimensional mechanism design. *Games and Economic Behavior*, 113:97–115, 2019.

[23] Euiwoong Lee and Sahil Singla. Optimal Online Contention Resolution Schemes via Ex-Ante Prophet Inequalities. In Yossi Azar, Hannah Bast, and Grzegorz Herman, editors, *26th Annual European Symposium on Algorithms (ESA 2018)*, volume 112 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 57:1–57:14, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[24] M. Michel Petrovitch. Sur une manière d'étendre le théorème de la moyenne aux équations différentielles du premier ordre. *Mathematische Annalen*, 54:417–436.

[25] Tristan Pollner, Mohammad Roghani, Amin Saberi, and David Wajc. Improved online contention resolution for matchings and applications to the gig economy. In *Proceedings of the 23rd ACM Conference on Economics and Computation*, EC '22, New York, NY, USA, 2022. Association for Computing Machinery.

[26] András Telcs, Nicholas Wormald, and Sanming Zhou. Hamiltonicity of random graphs produced by 2-processes. *Random Struct. Algorithms*, 31:450–481, 12 2007.

[27] Lutz Warnke. On wormald's differential equation method. *CoRR*, abs/1905.08928, 2019.

[28] Nicholas C Wormald et al. The differential equation method for random graph processes and greedy algorithms. *Lectures on approximation and randomized algorithms*, 73:155, 1999.

[29] Qiqi Yan. Mechanism design via correlation gap. In *Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms*, pages 710–719. SIAM, 2011.

A Deferred Proofs from Section 3

Below we state a detailed version of Lemma 3.1 for general graphs. Note that this construction does not preserve bipartiteness; for bipartite graphs, we give another reduction (Lemma A.2) which does preserve the bipartiteness.

Lemma A.1 (Reduction to 1-Regular Inputs – Long Version)**.** *Given $G = (V, E)$ with fractional matching $\boldsymbol{x} = (x_e)_{e \in E}$, for each $k \geq |V|$, there exists $G' = (V', E')$ and $\boldsymbol{x}' = (x'_e)_{e \in E'}$ with the following properties:*

1. *$|V' \setminus V| = k$, $|E' \setminus E| = k(|V| + k - 1)$, and $(G', \boldsymbol{x}')$ can be computed in poly(nk) time.*

2. *$(G', \boldsymbol{x}')$ is 1-regular.*

3. *If there is an α-selectable RCRS for G' and $\boldsymbol{x}'$, then there exists an α-selectable RCRS for G and $\boldsymbol{x}$.*

4. *G is a subgraph of G', with $x'_e = x_e$ for all $e \in E$, and $x'_e \leq 1/|V|$ for all $e \in E' \setminus E$.*

Proof of Lemma A.1. The graph G' is constructed as follows. We create a k-clique $K_k = (V_k, E_k)$ of *dummy vertices*. Let $E_{G,k} = \{(u, v_k) \mid u \in V \wedge v_k = V_k\}$ be a set of edges connecting every vertex in V to each of the vertices of the dummy K_k. Then, $V' = V \cup V_k$ and $E' = E \cup E_k \cup E_{G,k}$. Then, $\mathbf{x}'$ is given by setting $x'_e = x_e$ for every $e \in E$ and $x'_e = (1 - x_u)/k$ for $e = (u, v) \in E_{G,k}$, where $x_u := \sum_{v \in V} x_{u,v}$. Finally, for $e = (v, v') \in E_k$, let $x'_e = (1 - \sum_{e \in E} \frac{x_e}{k})/k$;

Certainly, for $e \in E_{G,k} \cup E_k$, we have $x'_e \leq 1/k \leq 1/|V|$.

Clearly, $|V' \setminus V| = |V_k| = k$, and $|E' \setminus E| = |E_k| + |E_{G,k}| = k(k-1) + k|V|$. If $n = |V|$, we can certainly therefore construct G' in time $O(k + nk) = poly(nk)$, and similarly each x'_e can be computed in time $poly(nk)$. The construction also clearly ensures that for each $u \in V$, $\sum_{v \in V'} x'_{uv} = x_u + \sum_{v \in V_k} x_{uv} = 1$. We can similarly verify that, for $v \in V_k$, we have $\sum_{v' \in V'} x_{v,v'} = 1$.

Now consider an α-selectable RCRS for $(G', \mathbf{x}')$. We now show how to produce an α-selectable RCRS for $(G, \mathbf{x})$. For each edge $e \in E' \setminus E$, generate a uniformly random arrival time $Y'_e \in [0, 1]$. Then, we run the RCRS for G', allowing the edges of $E' \setminus E$ to arrive in the order of Y'_e (letting Y'_e for an edge $e \in E$ be equal to its original arrival time in G).

Clearly, the arrivals of E' are uniformly random, so the RCRS for G' selects each edge $e \in E'$ with probability at least αx_e (since the RCRS is α-selectable). Since the RCRS further processes the edges of E in their original (random) order, this is also an RCRS for G, and it is clearly α-selectable. □

Next we show a similar reduction for the case when G is bipartite.

Lemma A.2 (Reduction to 1-Regular Inputs for Bipartite Graphs – Long Version)**.** *Given bipartite $G = (U \cup V, E)$ with fractional matching $\boldsymbol{x} = (x_e)_{e \in E}$, there exists bipartite $G' = (U' \cup V', E')$ and $\boldsymbol{x}' = (x'_e)_{e \in E'}$ with the following properties:*

1. *$(G', \boldsymbol{x}')$ can be computed in poly(nk) time.*

2. *$(G', \boldsymbol{x}')$ is 1-regular.*

3. *If there is an α-selectable RCRS for G' and $\boldsymbol{x}'$, then there exists an α-selectable RCRS for G and $\boldsymbol{x}$.*

4. *G is a subgraph of G', with $x'_e = x_e$ for all $e \in E$, and $x'_e \leq 1/|V|$ for all $e \in E' \setminus E$.*

Proof. The graph G' is constructed as follows. First, assume w.l.o.g. that $|U| = |V|$ (we can do this by creating dummy vertices on the smaller side with edge values of 0). Let $n := |U|$. We create a biclique $K_{n,n} = (U_K \cup V_K, E_K)$ of *dummy vertices*. Let $E_{G,K} = (U \times V_K) \cup (U_K \cup V)$ be a set of edges connecting every vertex in G to each of the vertices of the dummy $K_{n,n}$. Let $U' = U \cup U_K$, $V' = V \cup V_K$, and $E' = E \cup E_K \cup E_{G,K}$.

Then, $\mathbf{x}'$ is given by setting $x'_e = x_e$ for every $e \in E$ and $x'_e = (1 - x_u)/n$ for $e = (u, v) \in E_{G,K}$, where $x_u := \sum_{v \in V} x_{u,v}$. Clearly, for $u \in U$, we have $\sum_{v \in V'} x'_{uv} = 1$ and similarly for $v \in V$, $\sum_{u \in U'} x'_{uv} = 1$.

Finally, for $e = (u, v) \in E_K$, set $x'_{uv} := \frac{1}{n^2} \sum_{v \in V} x_v$. Note that by the handshaking lemma, we have $\sum_{v \in V} x_v = \sum_{u \in U} x_u$ so $x'_{uv} = \frac{1}{n^2} \sum_{u \in U} x_u$.

Therefore, for $u \in U_K$, we have:

$$x'_u = \sum_{v \in V'} x'_{uv} = \sum_{v \in V} x'_{uv} + \sum_{v \in V_K} x'_{uv} = \frac{1}{n} \sum_{v \in V} (1 - x_v) + \sum_{v_K \in V_K} \frac{1}{n^2} \sum_{v \in V} x_v = \frac{1}{n} \sum_{v \in V} (1 - x_v) + \frac{1}{n} \sum_{v \in V} x_v = 1$$

and similarly, for $v \in V_K$:

$$x'_v = \sum_{u \in U} x'_{uv} + \sum_{u \in U_K} x'_{uv} = \frac{1}{n} \sum_{u \in U} (1 - x_u) + \sum_{u_K \in U_K} \frac{1}{n^2} \sum_{u \in U} x_u = \frac{1}{n} \sum_{u \in U} (1 - x_v) + \frac{1}{n} \sum_{u \in U} x_u = 1$$

Certainly, for $e \in E_K \cup E_{G,K}$, we have $x'_e \le 1/n \le 1/|V|$.

We have $|U'| = 2|U|$, $|V'| = 2|V|$, and $|E' \setminus E| = |U|^2|V|^2$. We can certainly therefore construct G' in time *poly*(n), and similarly each x'_e can be computed in time *poly*(n). By the same argument as in the proof of Lemma A.1, an α-selectable RCRS for $(G', \mathbf{x}')$ can be used to get an α-selectable RCRS for $(G, \boldsymbol{x})$. □

Proof of Equation 3.6 from Lemma 3.7. In order to prove (3.6), we will show that

$$\prod_{h' \in \partial(h) \setminus \{f, f^c\}} \ell(x_{h'}, y_h) \ge \exp\left(-z_h y_h\right), \tag{A.1}$$

where we recall that $\ell(x_{h'}, y_h) := 1 - y_h q(x_{h'})$, and $z_h := 2 - 2x_h - x_f - x_{f^c}$. To begin, we exponentiate the left-hand side of (A.1) to get

$$\exp\left(\sum_{h' \in \partial(h) \setminus \{f, f^c\}} \log \ell(x_{h'}, y)\right). \tag{A.2}$$

Our goal is then to minimize

$$\sum_{h' \in \partial(h) \setminus \{f, f^c\}} \log \ell(x_{h'}, y_h), \tag{A.3}$$

subject to $\sum_{h' \in \partial(h) \setminus \{f, f^c\}} x_{h'} = 2 - 2x_h - x_f - x_{f^c} = z_h$. Observe that by Proposition 3.8, $\log \ell(x_{h'}, y_h)$ is convex as a function $x_{h'}$ for each $h' \in \partial(h) \setminus \{f, f^c\}$. Thus, if $k := |\partial(h) \setminus \{f, f^c\}|$, then (A.3) is minimized when $x_{h'} = z_h / |\partial(h) \setminus \{f, f^c\}|$ for each $h' \in \partial(h) \setminus \{f, f^c\}$, and $k \to \infty$. By applying the first part, we get that

$$\sum_{h' \in \partial(h) \setminus \{f, f^c\}} \log \ell(x_{h'}, y_h) \ge \log \left(1 - y_h q\left(\frac{z_h}{k}\right)\right)^k.$$

and so,

$$\exp\left(\sum_{h'\in\partial(h)\setminus\{f,f^c\}} \log \ell(x_{h'},y)\right) \geq \exp\left(\log\left(1-y_h q\left(\frac{z_h}{k}\right)\right)^k\right)$$
$$= \left(1-y_h q\left(\frac{z_h}{k}\right)\right)^k$$

Now, since $a(0)=1$ (and a is continuous), $\lim_{k\to\infty} k\cdot q\left(\frac{z_h}{k}\right) = \lim_{k\to\infty} z_h a\left(\frac{z_h}{k}\right) = z_h$. Thus,

$$\lim_{k\to\infty}\left(1-y_h q\left(\frac{z_h}{k}\right)\right)^k = \lim_{k\to\infty}\exp\left(-k y_h q\left(\frac{z_h}{k}\right)\right)$$
$$= \exp\left(-z_h y_h\right).$$

Since the minimum of (A.3) occurs as $k\to\infty$, (A.1) holds.

□

We give an outline for proving the properties of Proposition 3.8 for both attenuation functions $a_1(x)$ and $a_2(x)$.

Proof of Proposition 3.8. First observe that $a_1(0)=a_2(0)=1$, and a_1 and a_2 are continuous and decreasing on $[0,1]$ (by definition, we extended a_2 to be continuous at $x=1$). We follow the same approach for checking the remaining properties of a_1 and a_2.

Consider the second derivative $\frac{d^2}{dx^2}\ln\ell(x,y)$; minimizing this over all $x\in[0,1], y\in[0,1]$ gives a minimum of 0 at $x=y=0$. Thus, if we fix any $y\in[0,1]$, the function $x\mapsto \frac{d^2}{dx^2}\ell(x,y)$ is nonnegative for $x\in[0,1]$, implying that for this fixed value of y, $x\mapsto\ln\ell(x,y)$ is convex on the interval $[0,1]$. □

We give an outline for verifying the properties of Proposition 3.10 for both a_1 and a_2.

Proof of Proposition 3.10. Begin with $a_1(x)=(1-(3-e)x)^2$. The function

$$x\mapsto \frac{a_1'(x)}{a_1(x)} + \frac{4}{1-x} - \frac{2(1-\exp(x-1))}{\exp(x-1)-x}$$

is decreasing on the interval $[0,1]$, as can be seen by examining its first derivative. Thus, it is maximized at $x=0$, where it takes on value 0.

For a_2, observe that $a_2(x)=(1-x)^4/(e^x-ex)^2$ is in fact the unique solution to the first-order differential equation

$$\frac{y'(x)}{y(x)} + \frac{4}{1-x} - \frac{2(1-\exp(x-1))}{\exp(x-1)-x} = 0,$$

with initial condition $y(0)=1$. Thus, the required property holds by definition.

□

Next, we give an outline for proving Proposition 3.12 for attenuation function a_1.

Proof of Proposition 3.12. The first property can be checked fairly easily: the minimum value of $\ell(x_1,y)\ell(x_2,y)-\exp(-(x_1+x_2)y)$ over $x_1,x_2,y\in[0,1]$ occurs either when of $x_1=x_2=0$ or $y=0$, for which $\ell(x_1,y)\ell(x_2,y)-\exp(-(x_1+x_2)y)=0$.

The second property is more complicated to verify. We provide here an outline of the verification. Set $I(x_1, x_2) := \int_0^1 \ell(x_1, s)\ell(x_2, s) + Q(x_1, x_2, s) - e^{-(x_1+x_2)s}\left(1 + \frac{(x_1+x_2)a(1)s^2}{2}\right) ds$. This function has a closed form, and its minimum occurs when $x_1 = x_2 = 0$,

Next, let $F_{x_1,x_2}(s) := \ell(x_1, s)\ell(x_2, s) + Q(x_1, x_2, s) - e^{-(x_1+x_2)s}\left(1 + \frac{(x_1+x_2)a(1)s^2}{2}\right)$. It can be observed that for $x_1, x_2 \in [0, 1]$, $F_{x_1,x_2}(0) = 0$, $F'_{x_1,x_2}(0) \geq 0$, and $F''_{x_1,x_2}(s) \leq 0$ for all $s \in (0, 1)$. These can be checked easily by e.g. numerically minimizing $F'_{x_1,x_2}(0)$ over $x_1, x_2 \in [0, 1]$ and numerically maximizing $F''_{x_1,x_2}(s)$ over $x_1, x_2, s \in [0, 1]$ (we find that the minimum of $F'_{x_1,x_2}(0)$ is $F'_{x_1,x_2}(0) = 0$, and the maximum of $F''_{x_1,x_2}(s)$ occurs for $x_1 = x_2 = 0$ where $F''_{0,0}(s) = 0$ for all s). Thus, $F_{x_1,x_2}(s)$ is initially nonnegative and increasing.

Now, suppose for sake of contradiction that for a given x_1, x_2, there exist two points $s_1 < s_2$ for which F_{x_1,x_2} changes sign, and without loss of generality, assume these are the first two such points. Since F_{x_1,x_2} is initially nonnegative and increasing, it must be the case that F_{x_1,x_2} is positive on the interval $(0, s_1)$ with $F'_{x_1,x_2}(s_1) < 0$ and negative on the interval (s_1, s_2) with $F'_{x_1,x_2}(s_2) > 0$. Therefore, it must be the case that $F'_{x_1,x_2}(s)$ is increasing on the interval (s_1, s_2), but this means that $F''_{x_1,x_2}(s) > 0$ somewhere on this interval; however, it was observed previously that $F''_{x_1,x_2}(s) \leq 0$ on the entire interval $(0, 1)$. □

Proof of Proposition 3.14. The same approach used in the proof of Proposition 3.12 allows us to verify that a_2 satisfies the properties of Proposition 3.14. The proof is nearly identical to that of 3.14, so we omit it here. □