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Proportionally Submodular Functions

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Submodular functions are well-studied in combinatorial optimization, game theory and economics. The natural diminishing returns property makes them suitable for many applications. We study an extension of monotone submodular functions, which we call *proportionally submodular functions*. Our extension includes some (mildly) supermodular functions. We show that several natural functions belong to this class and relate our class to some other recent submodular function extensions.

We consider the optimization problem of maximizing a proportionally submodular function subject to uniform and general matroid constraints. For a uniform matroid constraint, the "standard greedy algorithm" achieves a constant approximation ratio. More specifically, for any cardinality constraint p, the greedy algorithm has a constant approximation ratio bounded by a function $\alpha(p)$ that experimentally appears to be converging (from below) to 5.95 as p increases. For a general matroid constraint with rank s, we prove that the local search algorithm has constant approximation ratio bounded by a function $\rho(s)$ which analytically is converging (from above) to 10.22 as s increases.

Additional Key Words and Phrases: submodular functions, max-sum dispersion, greedy algorithms, local search

1. INTRODUCTION

There are many applications where the goal becomes a problem of maximizing a submodular function subject to some constraint. In many cases the submodular function f is also monotone, non-negative and normalized so that $f(\emptyset) = 0$. Such applications arise for example in the consideration of influence in a stochastic social network as formalized in [Kempe et al. 2003], diversified search ranking as in [Bansal et al. 2010] and document summarization as in [Lin and Bilmes 2011]. In another application, following the work of [Gollapudi and Sharma 2009], [Borodin et al. 2012] considered the linear combination of a monotone submodular function that measures the "quality" of a set of results combined with a diversity function given by the max-sum dispersion measure, a widely studied measure of diversity. Their analysis suggested that although the max-sum dispersion measure is a supermodular function, it possessed similar properties to monotone submodular functions. In this paper we develop this idea by introducing the class of *proportionally submodular* functions and show that greedy and local search algorithms can be used (respectively) to approximately maximize such functions subject to a cardinality (resp. general matroid) constraint.

The literature on the maximization of submodular functions is extensive. Here we only mention the most relevant work. Perhaps the most seminal paper concerning monotone submodular functions is the Nemhauser, Fisher and Wosley paper

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[Nemhauser et al. 1978] showing that natural greedy and local search algorithms for maximizing a monotone submodular function obtains approximation ratios $\frac{e}{e-1}$ (resp. 2) for maximizing any monotone submodular function subject to a cardinality (resp. arbitrary matroid) constraint. Our work shows that these algorithms still enjoy constant approximation ratios for the broader class of proportionally submodular functions. More recent work (see, [Feige et al. 2007], [Feldman et al. 2011], [Buchbinder et al. 2012], [Buchbinder et al. 2014]) provides constant approximation bounds for unconstrained and constrained non monotone submodular functions.

The remainder of the paper is as follows. In section 2, we provide the definition of proportionally submodular [1] functions. In section 3 we provide some basic observations about this class of functions along with a number of examples of monotone proportionally submodular function (that are not submodular). Section 4 contains a discussion of two other frameworks for extending submodular functions. Sections 5 and 6 contain analyses of the approximation ratios of the natural greedy (respectively local search) algorithms for maximizing monotone proportionally submodular functions subject to cardinality (respectively, matroid) constraints. We conclude in section 7 with some open problems.

2. PRELIMINARIES

Let $f : U \to \Re$ be a set function over a universe U. All of the specific set functions we consider are normalized and non-negative. That is, f satifies:

— $f(\emptyset) = 0$
— $f(S) \geq 0$ for all $S \subseteq U$

For the most part, we will focus attention on functions that are monotone. That is,

— $f(S) \leq f(T)$ for all $S \subseteq T \subseteq U$

A function $f(\cdot)$ is submodular if for any two sets S and T, we have

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

We define the following variant of submodular functions. We call a normalized, non-negative function $f(\cdot)$ *proportionally submodular* if for any two sets S and T, we have

$$|T|f(S) + |S|f(T) \geq |S \cap T|f(S \cup T) + |S \cup T|f(S \cap T).$$

Our extension of submodularity "normalizes" the submodularity definition in terms of the cardinality of the sets occuring in the definition. This allows for some supermodular functions since now large set unions with small intersections can possibly observe the required inequality. A similar idea can be found in the class of *meta-submodular functions* as introduced by [Kleinberg et al. 1998]. Such meta-submodular functions need not satisfy the submodular inequality when the sets are disjoint. We will see that monotone proportionally submoudular functions generalize monotone submodular (and monotone meta-submodular functions) and still retain the main algorithmic property of monotone submodular functions; namely that simple and efficient greedy and local search algorithms suffice to approximately maximize such functions subject to cardinality and general matroid constraints.

[1]In a previous version of this paper, we used "weakly submodular" as the name for this class. This name has been used before in the context of lattices by [Wild 2008]. It is difficult to find an appropriate name for the class of functions studied in this paper. For example, we might have preferred to have used the term *meta-submodular* but that term is already used in the computer science community [Kleinberg et al. 1998]. The name "proportionally submodular" was suggested by Sophie Laplante and we believe that it is more suggestive of the class we are defining.

3. EXAMPLES OF PROPORTIONALLY SUBMODULAR FUNCTIONS

In this section, we will first consider some natural proportionally submodular functions showing in particular that this class includes all monotone submodular functions as well as some supermodular functions. In Section 4, we will relate weak submodularity to the functions of supermodular degree defined by [Feige and Izsak 2013] and further studied in x‘[Feldman and Izsak 2014], and to the k-wise dependent functions of i [Conitzer et al. 2005], and the related MPH-k functions defined by [Feige et al. 2014].

3.1. Submodular Functions

From the proportionally submodular definition, it is not obvious that monotone submodular functions are a subclass of proportionally submodular functions. We will prove that this is indeed the case.

PROPOSITION 3.1. *Any monotone submodular function is proportionally submodular. This, of course, implies that every linear function (with non-negative weights) is proportionally submodular.*

PROOF.
Given a monotone submodular function $f(\cdot)$ and two subsets S and T, without loss of generality, we assume $|S| \leq |T|$, then

$$|T|f(S) + |S|f(T) = |S|[f(S) + f(T)] + (|T| - |S|)f(S).$$

By submodularity $f(S) + f(T) \geq f(T \cup S) + f(T \cap S)$ and monotonicity $f(S) \geq f(S \cap T)$, we have

$$\begin{aligned} |T|f(S) + |S|f(T) &= |S|[f(S) + f(T)] + (|T| - |S|)f(S) \\ &\geq |S|[f(S \cup T) + f(S \cap T)] + (|T| - |S|)f(S \cap T) \\ &= |S|f(S \cup T) + |T|f(S \cap T) \\ &= |S \cap T|f(S \cup T) + \mathrm{Big}[(|S| - |S \cap T|)f(S \cup T) + |T|f(S \cap T)\mathrm{Big}]. \end{aligned}$$

And again by monotonicity $f(S \cup T) \geq f(S \cap T)$, we have

$$(|S| - |S \cap T|)f(S \cup T) + |T|f(S \cap T) \geq (|S| + |T| - |S \cap T|)f(S \cap T) = |S \cup T|f(S \cap T).$$

Therefore

$$|T|f(S) + |S|f(T) \geq |S \cap T|f(S \cup T) + |S \cup T|f(S \cap T);$$

the proposition follows. □

We note that the proof of Proposition 3.1 did not require the function $f(\cdot)$ to be normalized or non-negative. But the proof did use the monotonicity of $f(\cdot)$. Non-monotone submodular functions (such as Max-Cut and Max-Di-Cut) are, of course, also widely studied. In contrast to Proposition 3.1, if we apply the proportionally submodular definition to non-monotone functions, then it is no longer the case that a non-monotone submodular function would necessarily be a non-monotone proportionally submodular function.

PROPOSITION 3.2. *There is a non-monotone submodular function $f(\cdot)$ that is not proportionally submodular. More specifically, the Max-Cut function (for a particular graph G) is not proportionally submodular.*

PROOF.
Consider a graph $G = (U, E)$ where $V = R \cup \{s\} \cup \{t\}$ and $E = \{(s, u), (u, t) | u \in R\}$. Letting $S = R \cup \{s\}$ and $T = R \cup \{t\}$, for $|R| = n$ we have the following:

— $f(S) = f(T) = n$

— $f(S \cup T) = f(U) = 0$
— $f(S \cap T) = f(R) = 2n$

Thus

(1) $|T|f(S) + |S|f(T) = (n+1)n + (n+1)n = 2n^2 + 2n$
(2) $|S \cap T|f(S \cup T) + |S \cup T|f(S \cap T) = n \cdot 0 + (n+2) \cdot 2n = 2n^2 + 4n$

This contradicts the proportionally submodular definition. □

PROPOSITION 3.3. *Let f be a proportionally submodular function. Then the complement function $\bar{f} = f(U/S)$ is proportionally submodular iff f is submodular.*

PROOF.
It is well known that submodular functions are closed under complememts so one direction of the proposition holds. We now show that when $\bar{f}$ is also proportionally submodular, then f is submodular.

For the other direction, let g be the complement function of a proportionally submodular function f and assume that g satisfies the proportionally submodular definition:
$|T|g(S) + |S|g(T) \geq |S \cap T|g(S \cup T) + |S \cup T|g(S \cap T)$
Simplifying the above inequality using the notation $\bar{S} = U \setminus S$, we have:
$(|U| - |T|)f(\bar{S}) + (|U| - |S|)f(\bar{T}) \geq (|U| - |\bar{S} \cup \bar{T}|f(\bar{S} \cap \bar{T}) + (|U| - |\bar{S} \cap \bar{T})f(\bar{S} \cup \bar{T})$
Rearranging the above expression:
$|U|[f(\bar{S}) + f(\bar{T}) - f(\bar{S} \cap \bar{T}) - f(\bar{S} \cup \bar{T})]$
$\geq |T|f(\bar{S}) + |S|f(\bar{T}) - |\bar{S} \cup \bar{T}|f(\bar{S} \cap \bar{T}) - |\bar{S} \cap \bar{T})f(\bar{S} \cup \bar{T}) \geq 0$
This shows that f is submodular since for all $\bar{S}, \bar{T} \subseteq U$, we have the desired condition for submodularity.

□

On the other hand, it is easy to construct non-monontone proportionally submodular functions from any monotone proportionally submodular function f having at least one positive valuation. Namely, let $f(S^*) > 0$ for some S with $\emptyset \subset S^* \subset U$. Then define the function g to be identical to f except that $g(U) = 0$. Clearly, g is non-monotone. We can verify that g is proportionally submodular by checking the cases where U appears in the inequality that defines weak submodularity, namely when either S or T is U, or when $S \cup T = U$. Furthermore, if f was say the metric dispersion function, g is then clearly not submodular.

3.2. Meta-Submodular and Average Non-Negative Segmentation Functions

Motivated by appliations in clustering and data mining, [Kleinberg et al. 1998] introduce the general class of segmentation functions. In their generality, segmentation functions need not be submodular nor monotone. They show that every segmentation function belongs to what they call the class of *meta-submodular functions* and consider the greedy algorithm for “proportionally montone” meta-submodular functions. A set function is a meta-submodular fuction if for any *non-disjoint* sets S and T, we have

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

Clearly every submodular function is meta-submodular and hence there are (non monotone) meta-submodular functions that are not proportionally submodular.

PROPOSITION 3.4. *Any monotone meta-submodular function is proportionally submodular.*

PROOF. If S and T are not disjoint then the proof of Proposition 3.1 applies immediately. If S and T are disjoint, then $|S \cap T| = 0$, and $|S \cup T| = |S| + |T|$. By monotonicity, we also have $\sigma(S) \geq \sigma(S \cap T)$ and $\sigma(T) \geq \sigma(S \cap T)$. Therefore,

$$|S \cap T|\sigma(S \cup T) + |S \cup T|\sigma(S \cap T) = |S|\sigma(S \cap T) + |T|\sigma(S \cap T) \leq |T|\sigma(S) + |S|\sigma(T)$$

□

We now consider a specific class of segmentation functions.

Given an $m \times n$ matrix M and any subset $S \subseteq [m]$, a *segmentation function* $\sigma(S)$ is the sum of the maximum elements of each column whose row indices appear in S; i.e.; $\sigma(S) = \sum_{j=1}^{n} \max_{i \in S} M_{ij}$. A segmentation function is *average non-negative* if for each row i, the sum of all entries of M is non-negative; i.e., $\sum_{j=1}^{n} M_{ij} \geq 0$. Letting columns index individuals, and rows index items, each entry of M_{ij} represents how much the individual j likes or dislikes the item i. The average non-negative property requires that for each item i, on average, people do not dislike it.

We can use columns to model individuals, and rows to model items, then each entry of M_{ij} represents how much the individual j likes the item i. The average non-negative property basically requires that for each item i, on average people do not dislike it. Next, we show that an average non-negative segmentation function is proportionally-submodular. We first prove the following two lemmas.

LEMMA 3.5. *An average non-negative segmentation function is monotone.*

PROOF. Let S be a proper subset of $[m]$, and e be an element in $[m]$ that is not in S. If S is empty, then by the average non-negative property, we have $\sigma(\{e\}) = \sum_{j=1}^{n} M_{ej} \geq 0$. Otherwise, by adding e to S we have $\max_{i \in S \cup \{e\}} M_{ij} \geq \max_{i \in S} M_{ij}$ for all $1 \leq j \leq n$. Therefore $\sigma(S \cup \{e\}) \geq \sigma(S)$. □

LEMMA 3.6. *For any non-disjoint set S and T and an average non-negative segmentation function $\sigma(\cdot)$, we have*

$$\sigma(S) + \sigma(T) \geq \sigma(S \cup T) + \sigma(S \cap T).$$

That is, σ is a meta-submodular function.

PROOF. For any non-disjoint set S and T and an average non-negative segmentation function $\sigma(\cdot)$, we let $\sigma_j(S) = \max_{i \in S} M_{ij}$. We show a stronger statement that for any $j \in [n]$, we have

$$\sigma_j(S) + \sigma_j(T) \geq \sigma_j(S \cup T) + \sigma_j(S \cap T).$$

Let e be an element in $S \cup T$ such that M_{ej} is maximum. Without loss of generality, assume $e \in S$, then $\sigma_j(S) = \sigma_j(S \cup T) = M_{ej}$. Since $S \cap T \subseteq T$, we have $\sigma_j(T) \geq \sigma_j(S \cap T)$. Therefore,

$$\sigma_j(S) + \sigma_j(T) \geq \sigma_j(S \cup T) + \sigma_j(S \cap T).$$

Summing over all $j \in [n]$, we have

$$\sigma(S) + \sigma(T) \geq \sigma(S \cup T) + \sigma(S \cap T)$$

as desired. □

The following proposition is immediate by the above two lemmas and Proposition 3.4.

PROPOSITION 3.7. *Any average non-negative segmentation function is proportionally submodular.*

We note that an average non-negative segmentation function need not be submodular. Consider the 2×2 matrix with rows $S = \{1, -1\}$ and $T = \{-1, 1\}$. Then it is easy to verify that the function defined by this matrix is an average non-negative segmentation function. However $\sigma(S) = \sigma(T) = 0$ while $\sigma(S \cup T) = 2$.

Hereafter, we will we restrict attention to monotone, non-negative and normalized functions. In the remaining subsections of section 3, we present a number of monotone proportionally submodular functions that are not submodular (and in fact are "mildly" supermodular).

3.3. Sum of Metric Distances of a Set

Let U be a metric space with a distance function $d(\cdot,\cdot)$. For any subset S, define $d(S)$ to be the sum of distances induced by S; i.e.,

$$d(S) = \sum_{\{u,v\} \subseteq S} d(u,v)$$

where $d(u,v)$ measures the distance between u and v. The problem of maximizing $d(S)$ subject to a cardinality constraint is called the *max-sum dispersion problem* and is one of many dispersion problems studied in location theory.

We extend the distance function to a pair of disjoint subsets X and Y and define $d(X,Y)$ to be the sum of pair-wise distances between X and Y; i.e.,

$$d(X,Y) = \sum_{u \in X, v \in Y} d(u,v).$$

We have the following proposition.

PROPOSITION 3.8. *The sum of metric distances $d(S)$ of a set is proportionally submodular (and clearly monotone).*

PROOF. Given two subsets S and T of U, let $A = S \setminus T$, $B = T \setminus S$ and $C = S \cap T$. Observe the fact that by the triangle inequality, we have

$$|B|d(A,C) + |A|d(B,C) \geq |C|d(A,B).$$

Therefore,

$$\begin{aligned}
& |T|d(S) + |S|d(T) \\
= \; & (|B| + |C|)[d(A) + d(C) + d(A,C)] + (|A| + |C|)[d(B) + d(C) + d(B,C)] \\
= \; & |C|[d(A) + d(B) + d(C) + d(A,C) + d(B,C)] + (|A| + |B| + |C|)d(C) \\
& + |B|d(A) + |A|d(B) + |B|d(A,C) + |A|d(B,C) \\
\geq \; & |C|[d(A) + d(B) + d(C) + d(A,C) + d(B,C)] + |S \cup T|d(S \cap T) + |C|d(A,B) \\
= \; & |C|[d(A) + d(B) + d(C) + d(A,C) + d(B,C) + d(A,B)] + |S \cup T|d(S \cap T) \\
= \; & |S \cap T|d(S \cup T) + |S \cup T|d(S \cap T).
\end{aligned}$$

□

3.4. Minimum Cardinality Functions

For any $k \geq 1$, let $f_k(S) = B > 0$ for $|S| \geq k$ and 0 otherwise.

PROPOSITION 3.9.

(1) *For $k = 1, 2$, f_k is proportionally submodular*
(2) *For $k \geq 3$, f_k is not proportionally submodular on any universe of size at least k*

PROOF.

In all cases, we need only restrict attention to non empty sets S and T in the weak submodularity definition since we are assuming $f(\emptyset) = 0$.

(1) For $k = 1$, weak submodularity follows from the fact that $|S| + |T| = |S \cap T| + |S \cup T|$ given that $f_1(Z) = B$ for all non empty sets Z.
(2) For $k = 2$, we can verify that f is proportionally submodular by considering the possible cardinalities of the sets in the proportionally submodular definition; that is, when say $|S| \leq |T|$ we consider the cases $|S| < 2$ and $|S| \geq 2$. For $|S| < 2$, either $S \subseteq T$ or $|S \cap T| = \emptyset$ and we can easily verify that f satisfies the weak submodularity definition in either case. If $|S|$ and $|T|$ are both ≥ 2, then weak submodularity follows as in the proof for $k = 1$ since $f_2(Z) = B$ for all sets Z with cardinality at least 2.
(3) If $k \geq 3$, let $S = \{a_1, \ldots a_{k-1}\}$ and $T = \{a_{k-1}, a_k\}$ for distinct elements $a_1 \ldots a_k$. Then
— $|T|f_k(S) + |S|f_k(T) = 0$
— $|S \cap T|f_k(S \cup T) + |S \cup T|f_k(S \cap T)$ = B ¿ 0

which contradicts the proportionally-submodular definition.

□

3.5. Powers of the Cardinality of a Set

Clearly, for any positive integer k, the functions $f(S) = |S|^k$ can be computed in time $O(\log k)$. However, given Lemma 3.13 below, it is still useful to know what simple functions can be used in conjuction with other submodular and proportionally submodular functions.

It is immediate to see that the functions $f(S) = |S|^0$ and $f(S) = |S|^1$ are linear and hence submodular. We now show that the square and the cube of the cardinality of a set are also proportionally submodular.

PROPOSITION 3.10. *The square of cardinality of a set is proportionally submodular.*

PROOF. Given two subsets S and T of U, let $a = |S \setminus T|$, $b = |T \setminus S|$ and $c = |S \cap T|$.

$$
\begin{aligned}
& |T|f(S) + |S|f(T) \\
= \ & (b+c)(a+c)^2 + (a+c)(b+c)^2 \\
= \ & (a+b+2c)(b+c)(a+c) \\
= \ & (a+b+2c)(ab+ac+bc+c^2) \\
\geq \ & (a+b+2c)(ac+bc+c^2) \\
= \ & (a+b+2c)c(a+b+c) \\
= \ & c(a+b+c)^2 + (a+b+c)c^2 \\
= \ & |S \cap T|f(S \cup T) + |S \cup T|f(S \cap T).
\end{aligned}
$$

□

PROPOSITION 3.11. *The cube of cardinality of a set is proportionally submodular.*

PROOF. Given two subsets S and T of U, let $a = |S \setminus T|$, $b = |T \setminus S|$ and $c = |S \cap T|$.

$$\begin{aligned}
& |T|f(S) + |S|f(T) \\
= \; & (b+c)(a+c)^3 + (a+c)(b+c)^3 \\
= \; & (a^2+b^2+2c^2+2ac+2bc)(b+c)(a+c) \\
= \; & [(a+b+c)^2+c^2-2ab][ab+c(a+b+c)] \\
= \; & [(a+b+c)^2+c^2][c(a+b+c)] + ab[(a+b+c)^2+c^2] - 2a^2b^2 - 2abc(a+b+c) \\
= \; & c(a+b+c)^3 + c^3(a+b+c) + ab[(a+b+c)^2+c^2-2ab-2c(a+b+c)] \\
= \; & |S \cap T|f(S \cup T) + |S \cup T|f(S \cap T) + ab(a^2+b^2+c^2+2ab+2ac+2bc+c^2-2ab-2ac-2bc-2c^2) \\
= \; & |S \cap T|f(S \cup T) + |S \cup T|f(S \cap T) + ab(a^2+b^2) \\
\geq \; & |S \cap T|f(S \cup T) + |S \cup T|f(S \cap T).
\end{aligned}$$

□

We now give an example that shows $f(S) = |S|^4$ is not proportionally submodular.

PROPOSITION 3.12. *$f(S) = |S|^4$ is not proportionally submodular.*

PROOF. Given two subsets S and T of U, let $a = |S \setminus T|$, $b = |T \setminus S|$ and $c = |S \cap T|$. Suppose $a = 4, b = 4, c = 1$.

$$|T|f(S) + |S|f(T) = (b+c)(a+c)^4 + (a+c)(b+c)^4 = 6250$$

On the other hand, we have

$$|S \cap T|f(S \cup T) + |S \cup T|f(S \cap T) = c(a+b+c)^4 + (a+b+c)c^4 = 9^4 + 9 = 6570$$

Therefore, the function is not proportionally submodular.

□

Similarly, one can see that $f(S| = |S|^k$ is not proportionally submodular for all intergers $k \geq 4$.

3.6. Linear combinations of proportionally submodular functions

Next we show a basic but important property of proportionally submodular functions.

LEMMA 3.13. *Non-negative linear combinations of proportionally submodular functions are weakly submodular.*

PROOF. Consider proportionally submodular functions $f_1, f_2, \ldots, f_n$ and non-negative numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let $g(S) = \sum_{i=1}^{n} \alpha_i f_i(S)$, then for any two set S and

T, we have

$$\begin{aligned}
& |T|g(S) + |S|g(T) \\
= \; & |T|\sum_{i=1}^{n}\alpha_i f_i(S) + |S|\sum_{i=1}^{n}\alpha_i f_i(T) \\
= \; & \sum_{i=1}^{n}\alpha_i[|T|f_i(S) + |S|f_i(T)] \\
\geq \; & \sum_{i=1}^{n}\alpha_i[|S\cap T|f_i(S\cup T) + |S\cup T|f_i(S\cap T)] \\
= \; & |S\cap T|\sum_{i=1}^{n}\alpha_i f_i(S\cup T) + |S\cup T|\sum_{i=1}^{n}\alpha_i f_i(S\cap T) \\
= \; & |S\cap T|g(S\cup T) + |S\cup T|g(S\cap T).
\end{aligned}$$

Therefore, $g(S)$ is proportionally submodular. □

COROLLARY 3.14. *The welfare maximization problem (also known as the maximization problem for combinatorial auctions) for agents with proportionally submodular valuations is a special case of the maximization of a proportionally submodular function subject to a partition matroid.*

PROOF.
In the maximum welfare problem, n agents $A = \{1, \ldots, n\}$ have valuation functions $v_i := U \to \Re$. A feasible allocation is a disjoint allocation of subsets S_i to each agent $(1 \leq i \leq n)$ so as to maximize the social welfare function $f(S) = \sum_{i=1}^{n} v_i(S_i)$. It is well known then how to view the disjointness constraint as a partition matroid constraint. Namely, we consider a universe $U' = A \times U$ where the elements of U' are partitioned into blocks $B_u = \{i, u)|i \in A\}$ for each $u \in U$. For $S' = \cup B'_u$, we let the partition matroid be defined by the independence condition that a subset $S' \subseteq U'$ is independent iff $|B'_u| \leq 1$; that is, it does not contain any two elements (i, u) and (i', u) for some $u \in U$ and $i \neq i'$. Letting $\pi_i(S') = \{u : (i, u) \in S'\}$, define $f'_i(S) = v_i(\pi_i(S')$ and $f'(S') = \sum_{i=1}^{n} v_i(\pi_i(S'))$ for any subset $S' \subseteq U'$. Given that each v_i is proportionally submodular on the universe $\pi_i(U')$ and that the class of proportionally submodular functions is closed under linear combinations, $f'(S')$ is a proportionally submodular function when all the valuations v_i are proportionally submodular.

□

We now show two more examples of proportionally submodular function using Lemma 3.13.

3.7. The Objective Function of Max-Sum Diversification

COROLLARY 3.15. *The objective function of the max-sum diversification problem, $f(S) = g(S) + \sum_{\{u,v\}\subseteq S} d(u, v)$, is proportionally submodular when g is monotone submodular (or proportionally submodular) and d is a metric.*

PROOF. This follows immediate from Proposition 3.1 and 3.8 and Lemma 3.13. □

3.8. Restricted Polynomial Function on the Cardinality of a Set

COROLLARY 3.16. *For polynomial functions on the cardinality of a set, if the degree is less than four and coefficients are all non-negative, then the function is proportionally submodular.*

PROOF. This follows immediate from Proposition 3.10 and 3.11 and Lemma 3.13. □

4. RELATED WORK

It is well known that submodular functions are a strictly smaller class than that of subadditive functions. In their study of subadditive valuations for combinatorial auctions, [Lehmann et al. 2001] introduce XOS functions as a subadditive monotone extension of monotone submodular functions. A function f is an XOS function is there exists a (possibly exponentially large) collection $\mathcal{L}$ of linear functions such that $f(S) = \max_{L \in \mathcal{L}} L(S)$. To date this class has been mainly used to facilitate analysis as in [Feldman et al. 2015].

PROPOSITION 4.1. *There is an XOS (and hence subadditive function) f that is not proportionally submodular.*

PROOF. As shown in [Lehmann et al. 2001], for the universe $U = \{a, b, c\}$, the following function is XOS but not submodular:
$f(S) = 0$ for $|S| = 0; 2; 3$ (respectively) for $|S| = 0; 1, 2; 3$ (respectively).
The same function is easily seen to violate the proportionally submodular definition by considering $S = \{a, b\}$ and $T = \{b, c\}$. □

Recently, there have been other generalizations of monotone submodular functions[2]. In particular with regard to combinatorial auctions, [Feige and Izsak 2013] defined the concept of the *supermodularity degree* of a set function as a measure of the degree of complementarity. Intuitvely, for each item u, its supermodular degree is the number of other items v that increase the marginal value of u with respect to some subset not containing u. This induces a supermodular dependency graph and the supermodular degree of an item is its degree in this dependency graph. The supermodular degree of a set function is the maximum of the item supermodular degrees. Set functions with supermodular degree 0 are precisely the submodular functions and every set function on a universe U has supermodular degree at most $|U| - 1$. Feige and Izsak consider the welfare maximization problem when each agent has a valuation function with supermodular degree at most d. Amongst their results, they show that given the supermodular dependency graph, and a value oracle to access the valuation function of each agent, a greedy algorithm approximates the welfare maximization problem to within a factor [3] of $d + 2$. [Feldman and Izsak 2014] consider the maximization of set functions with supermodular degree d degree subject to independence in a matroid and more generally to independence in a k-extendible system as defined by Mestre [Mestre 2006]. They show that a natural greedy algorithm achieves approximation ratio $k(d + 1) + 1$ assuming a value oracle (for accessing the set function) and an independence oracle (for determining if a set is indepedendent in $\mathcal{I}$).

It is easy to see that the class of proportionally submodular functions does not correspond to functions having bounded supermodular degree. For example, the function f_2

[2]We note that the class of proportionally submodular functions (named "weakly submodular functions") was introduced in the PHD thesis of [Ye 2013] and followed from observations made with regard to the diversifcation problem in [Borodin et al. 2012]. As such this class was studied independently from the work relating to supermodular degree and the MPH-k hierarchy that will now be discussed.
[3]We are stating all of our approximation ratios to be greater than or equal to 1 whereas Feige and Izsak use fractional approximation ratios.

in Proposition 3.9 is proportionally submodular and has supermodular degree $|U| - 1$ for any universe U with at least 3 elements. Furthermore, [Feige et al. 2014] show that there are instances of the metric sum dispersion problem (even with unit distance on the complete graph $G = (U, U \times U)$) that do not have bounded supermodular degree. In fact, Feige et al show that for this instance of the dispersion function, a function of supermodular degree d cannot provide an approximation better than $\frac{|U|}{d+1} - 1$. On the other hand, we have the following observation:

PROPOSITION 4.2.
There are simple functions having supermodular degree 1 that are not proportionally submodular.

PROOF. For the universe $U = \{a_1, a_2, b\}$, let $f(S) = B > 0$ if $\{a_1, a_2\} \subseteq S$ and 0 otherwise. Letting $S = \{a_1, b_1\}$ and $T = \{a_2, b_1\}$, we have

— $|T|f(S) + |S|f(T) = 0$
— $|S \cap T|f(S \cup T) + |S \cup T|f(S \cap T)$ = B

which violates the definition of weak submodularity. □

Another generalization of submodular functions was introduced in [Conitzer et al. 2005] and further developed in the expessive MPH-k hierarchy of [Feige et al. 2014]. They consider the representation of a set function $f(S)$ by its unique hypergraph $h(S)$ (called hypercube in [Conitzer et al. 2005]) representation. Functions in which the only non zero elements $h(S)$ in the hypergraph representation are positive and further satisfy $|S| \leq k$ are called PH-k functions. A monotone function is in the class MPH-k if it can be expressed as maximum over a finite collection of PH-k functions. Feige et al establish a number of significant results amongst which (most relevant to our results) are the facts that all monotone functions of supermodular degree $k - 1$ are in MPH-k for $k \geq 1$ and that using demand oracles and given the hypergraph representation of agent set functions, the welfare maximization problem for agents with MPH-k valuations can be solved by an LP-rounding algorithm with approximation ratio $k + 1$. As a special case, we note that the sum dispersion problem is a MPH-2 function (even for non metric distances). As they show (in their appendix L), the expressiveness of the MPH-k framework may require some simple functions (even in MPH-1) to require exponentially many hypergraphs to be so represented. While functions in any MPH-k are closed under linear combinations, maximizing such functions to a cardinality constraint (and hence to matroid constraints) would require a breakthrough for the densest subgraph problem since the densest subgraph problem subject to a cardinality constraint can be reduced to the MPH-2 non metric dispersion problem (see [Feige et al. 2001], [Andersen and Chellapilla 2009] and [Khuller and Saha 2009]).

Finally, we mention the related classes of weakly submodular and quasi submodular functions as (respectively) defined and studied by [Wild 2008] and [Mei et al. 2015]). A function f is said to be weakly submodular if it satisfies the following condition:

$$f(S \cap T) = f(S) \Rightarrow f(S \cup T) = f(T)$$

.

A function is quasi submodular if the following two conditions are satifisfied:

$$f(S \cap T) \geq f(S) \Rightarrow f(T) \geq f(S \cup T)$$

$$f(S \cap T) > f(S) \Rightarrow f(T) > f(S \cup T)$$

.

Both classes generalize the concept of submodular functions. It is easy to see that for monotone functions these classes are equivalent, and moroever, *every* strictly increasing monotone function is weakly submodular (and hence quasi submodular). We note that the monotone function f_2 as defined in Proposition 3.9 is not quasi submodular. From an algorithmic point of view, these classes seem to be mainly of interest in the non-monotone case and Mei et al study approximations for the unconstrained maximization problem for such functions.

5. PROPORTIONALLY SUBMODULAR FUNCTION MAXIMIZATION SUBJECT TO A CARDINALITY CONSTRAINT

We emphasize again that we restrict attention to monotone, non-negative and normalized functions. In this section, we discuss a greedy approximation algorithm for maximizing proportionally submodular functions subject to a uniform matroid (i.e. a cardinality constraint). In section 6 we consider an arbitrary matroid constraint.

Given an underlying set U and a proportionally submodular function $f(\cdot)$ defined on every subset of U, the goal is to select a subset S maximizing $f(S)$ subject to a cardinality constraint $|S| \leq p$. We consider the following standard greedy algorithm that achieves an approximation ratio of $\frac{e}{e-1}$ for monotone submodular maximization by a classic result of Nemhauser, Fisher and Wolsey [Nemhauser et al. 1978]. [Feige 1998] showe the By a result of [Birnbaum and Goldman 2009], it is known that the same greedy algorithm [4] is a 2-approximation for the metric dispersion problem subject to a cardinality constraint.

GREEDY ALGORITHM FOR PROPORTIONALLY SUBMODULAR FUNCTION MAXIMIZATION
WHILE $|S| < p$
  Find $u \in U \setminus S$ maximizing $f(S \cup \{u\}) - f(S)$
  $S = S \cup \{u\}$
ENDWHILE

THEOREM 5.1. *For all p, the standard greedy algorithm achieves a constant approximation ratio.*
In particular, the approximation ratio is 3.74 (resp. 5.62) when $p = 10$ (resp. when $p = 100$). Computer evaluations suggest that the approximation ratio converges to 5.95 as p tends to ∞.

Before getting into the proof of Theoren 5.1, we first prove two algebraic identities.

LEMMA 5.2.

$$\sum_{j=1}^{n} (\frac{i+1}{i})^{j-1} = i(\frac{i+1}{i})^n - i.$$

[4]While greedy algorithms are conceptually simple to state and understand operationally, it can be the case that the analysis of an approximation ratio is not at all simple. For example, the Birnbaum and Goldman proof that the greedy algorithm is a 2-approximation for the cardinality constrained metric sum dispersion problem is such a proof. Their proof answered an explicit 12 year old conjecture by [Hassin et al. 1997] following the 4-approximation by [Ravi et al. 1994]. In fact, one can view the Ravi et al paper as an implicit conjecture given their example showing that the greedy algorithm was no better than a 2-approximation for the dispersion problem.

PROOF. Note that the expression on the left-hand side is a geometric sum. Therefore, we have

$$\sum_{j=1}^{n}(\frac{i+1}{i})^{j-1} = \frac{(\frac{i+1}{i})^n - 1}{\frac{i+1}{i} - 1} = i(\frac{i+1}{i})^n - i.$$

□

LEMMA 5.3.

$$\sum_{j=1}^{n} j(\frac{i+1}{i})^{j-1} = ni^2(\frac{i+1}{i})^{n+1} - (n+1)i^2(\frac{i+1}{i})^n + i^2.$$

PROOF. Consider the function $f(x) = \sum_{j=1}^{n} x^j$ with $x \neq 1$, its derivative $f'(x) = \sum_{j=1}^{n} jx^{j-1}$. Since $f(x)$ is a geometric sum and $x \neq 1$, we have

$$f(x) = \frac{x^{n+1} - 1}{x - 1} - 1.$$

Taking derivatives on both sides we have

$$f'(x) = \frac{(n+1)x^n(x-1) - x^{n+1} + 1}{(x-1)^2} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$$

Therefore, we have

$$\sum_{j=1}^{n} jx^{j-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$$

Substituting x with $\frac{i+1}{i}$, we have

$$\sum_{j=1}^{n} j(\frac{i+1}{i})^{j-1} = \frac{n(\frac{i+1}{i})^{n+1} - (n+1)(\frac{i+1}{i})^n + 1}{(\frac{i+1}{i} - 1)^2} = ni^2(\frac{i+1}{i})^{n+1} - (n+1)i^2(\frac{i+1}{i})^n + i^2.$$

□

Now we proceed to the proof to Theorem 5.1.

PROOF. Let S_i be the greedy solution after the i^{th} iteration; i.e., $|S_i| = i$. Let O be an optimal solution, and let $C_i = O \setminus S_i$. Let $m_i = |C_i|$, and $C_i = \{c_1, c_2, \dots, c_{m_i}\}$. By the proportionally submodularity definition, we get the following m_i inequalities for each $0 < i < p$:

$$(i + m_i - 1)f(S_i \cup \{c_1\}) + (i+1)f(S_i \cup \{c_2, \dots, c_{m_i}\}) \geq (i)f(S_i \cup \{c_1 \dots, c_{m_i}\}) + (i + m_i)f(S_i)$$
$$(i + m_i - 2)f(S_i \cup \{c_2\}) + (i+1)f(S_i \cup \{c_3, \dots, c_{m_i}\}) \geq (i)f(S_i \cup \{c_2 \dots, c_{m_i}\}) + (i + m_i - 1)f(S_i)$$
$$\vdots$$
$$(i+1)f(S_i \cup \{c_{m_i-1}\}) + (i+1)f(S_i \cup \{c_{m_i}\}) \geq (i)f(S_i \cup \{c_{m_i-1}, c_{m_i}\}) + (i+2)f(S_i)$$
$$(i)f(S_i \cup \{c_{m_i}\}) + (i+1)f(S_i) \geq (i)f(S_i \cup \{c_{m_i}\}) + (i+1)f(S_i).$$

Multiplying the j^{th} inequality by $(\frac{i+1}{i})^{j-1}$, and summing all of them up (noting that the second term of the left hand side of the j^{th} inequality then cancels the first term of the $j+1^{st}$ inequality), we have

$$\sum_{j=1}^{m_i}(i+m_i-j)(\frac{i+1}{i})^{j-1}f(S_i\cup\{c_j\})+(i+1)(\frac{i+1}{i})^{m_i-1}f(S_i)$$
$$\geq(i)f(S_i\cup\{c_1,\ldots,c_{m_i}\})+\sum_{j=1}^{m_i}(i+m_i-j+1)(\frac{i+1}{i})^{j-1}f(S_i).$$

By monotonicity, we have $f(S_i\cup\{c_1,\ldots,c_{m_i}\})\geq f(O)$. Rearranging the inequality,

$$\sum_{j=1}^{m_i}(i+m_i-j)(\frac{i+1}{i})^{j-1}f(S_i\cup\{c_j\})\geq(i)f(O)+\sum_{j=1}^{m_i-1}(i+m_i-j+1)(\frac{i+1}{i})^{j-1}f(S_i).$$

By the greedy selection rule, we know that $f(S_{i+1})\geq f(S_i\cup\{c_j\})$ for any $1\leq j\leq m_i$, therefore we have

$$\sum_{j=1}^{m_i}(i+m_i-j)(\frac{i+1}{i})^{j-1}f(S_{i+1})\geq(i)f(O)+\sum_{j=1}^{m_i-1}(i+m_i-j+1)(\frac{i+1}{i})^{j-1}f(S_i).$$

For the ease of notation, we let

$$a_i=\sum_{j=1}^{m_i}(i+m_i-j)(\frac{i+1}{i})^{j-1}\qquad b_i=\sum_{j=1}^{m_i-1}(i+m_i-j+1)(\frac{i+1}{i})^{j-1}$$

so that we have $a_if(S_{i+1})-b_if(S_i)\geq(i)f(O)$

We first simplify a_i and b_i.

$$\begin{aligned}a_i&=\sum_{j=1}^{m_i}(i+m_i-j)(\frac{i+1}{i})^{j-1}\\&=\sum_{j=1}^{m_i}(i+m_i)(\frac{i+1}{i})^{j-1}-\sum_{j=1}^{m_i}j(\frac{i+1}{i})^{j-1}.\end{aligned}$$

By Lemma 5.2 and 5.3, we have

$$\begin{aligned}a_i&=(i+m_i)[i(\frac{i+1}{i})^{m_i}-i]-m_ii^2(\frac{i+1}{i})^{m_i+1}+(m_i+1)i^2(\frac{i+1}{i})^{m_i}-i^2\\&=[i^2+im_i-m_i(i^2+i)+(m_i+1)i^2](\frac{i+1}{i})^{m_i}-2i^2-im_i\\&=2i^2(\frac{i+1}{i})^{m_i}-2i^2-im_i.\end{aligned}$$

Similarly, we have

$$\begin{aligned}
b_i &= \sum_{j=1}^{m_i-1}(i+m_i-j+1)(\frac{i+1}{i})^{j-1} \\
&= \sum_{j=1}^{m_i-1}(i+m_i+1)(\frac{i+1}{i})^{j-1} - \sum_{j=1}^{m_i-1} j(\frac{i+1}{i})^{j-1} \\
&= (i+m_i+1)[i(\frac{i+1}{i})^{m_i-1} - i] - (m_i-1)i^2(\frac{i+1}{i})^{m_i} + m_i i^2(\frac{i+1}{i})^{m_i-1} - i^2 \\
&= [i^2 + im_i + i - (m_i-1)(i^2+i) + m_i i^2](\frac{i+1}{i})^{m_i-1} - 2i^2 - im_i - i \\
&= 2i(i+1)(\frac{i+1}{i})^{m_i-1} - 2i^2 - im_i - i \\
&= 2i^2(\frac{i+1}{i})^{m_i} - 2i^2 - im_i - i.
\end{aligned}$$

Now let

$$a_i^* = \sum_{j=1}^{p}(i+p-j)(\frac{i+1}{i})^{j-1} \qquad b_i^* = \sum_{j=1}^{p-1}(i+p-j+1)(\frac{i+1}{i})^{j-1}$$

The simplication of a_i and b_i makes it clear that $a_i - b_i = i$ for any value of m_i. Since a_i^* (resp. b_i^*) can be thought of as a_i (resp. b_i) with $m_i = p$, we have

$$a_i^* - a_i = b_i^* - b_i \geq 0$$

Therefore,

$$a_i^* f(S_{i+1}) - b_i^* f(S_i) = a_i f(S_{i+1}) - b_i f(S_i) + (a_i^* - a_i)[f(S_{i+1}) - f(S_i)].$$

Since $f(\cdot)$ is monotone, we have $f(S_{i+1}) - f(S_i) \geq 0$. Therefore,

$$a_i^* f(S_{i+1}) - b_i^* f(S_i) \geq a_i f(S_{i+1}) - b_i f(S_i) \geq i f(O).$$

Then we have the following set of inequalities:

$$\begin{aligned}
a_1^* f(S_2) &\geq 1 f(O) + b_1^* f(S_1) \\
a_2^* f(S_3) &\geq 2 f(O) + b_2^* f(S_2) \\
&\vdots \\
a_{p-2}^* f(S_{p-1}) &\geq (p-2) f(O) + b_{p-2}^* f(S_{p-2}) \\
a_{p-1}^* f(S_p) &\geq (p-1) f(O) + b_{p-1}^* f(S_{p-1}).
\end{aligned}$$

Multiplying the i^{th} inequality by $\frac{\prod_{j=1}^{i-1} a_j^*}{\prod_{j=2}^{i} b_j^*}$, summing all of them up and ignoring the term $b_1^* f(S_1)$,

$$\frac{\prod_{j=1}^{p-1} a_j^*}{\prod_{j=2}^{p-1} b_j^*} f(S_p) \geq \sum_{i=1}^{p-1} \frac{i \prod_{j=1}^{i-1} a_j^*}{\prod_{j=2}^{i} b_j^*} f(O).$$

Therefore the approximation ratio

$$\frac{f(O)}{f(S_p)} \leq \frac{\frac{\prod_{j=1}^{p-1} a_j^*}{\prod_{j=2}^{p-1} b_j^*}}{\sum_{i=1}^{p-1} \frac{i \prod_{j=1}^{i-1} a_j^*}{\prod_{j=2}^{i} b_j^*}} = \left(\sum_{i=1}^{p-1} \frac{i \prod_{j=i+1}^{p-1} b_j^*}{\prod_{j=i}^{p-1} a_j^*}\right)^{-1} = \left(\sum_{i=1}^{p-1}\left[\frac{i}{a_i^*} \cdot \prod_{j=i+1}^{p-1} \frac{b_j^*}{a_j^*}\right]\right)^{-1}.$$

Note that the approximation ratio is simply a function of p. As stated in the theorem statement, this ratio converges[5] to 5.95 as p tends to ∞. $\square$

In terms of hardness of approximation, [Nemhauser et al. 1978] showed that in the value oracle model, $\frac{e}{e-1}$ is the best approximation possible for monotone the value oracle model; that is, to achieve a better ratio would require exponentially many oracle calls. Feige [Feige 1998] showed the same inapproximation holds for the explictly defined max coverage problem (an example of monotone submodular maximization subject to a cardinality constraint) subject to the conjecture that $P \neq NP$. The max-sum dispersion problem is known to be NP-hard by an easy reduction from Max-Clique, and as noted by Alon [Alon 2014], there is evidence that the problem is hard to compute in polynomial time with approximation $2-\epsilon$ for any $\epsilon > 0$ when $p = n^r$ for $1/3 \leq r < 1$. (See the discussion in Section 3 of [Borodin et al. 2014].)

6. PROPORTIONALLY SUBMODULAR FUNCTION MAXIMIZATION SUBJECT TO AN ARBITRARY MATROID CONSTRAINT

It is natural to consider a general matroid constraint for the problem of proportionally submodular function maximization. For this more general problem, the greedy algorithm in the previous section no longer achieves any constant approximation ratio. See the example presented in the Appendix of [Borodin et al. 2014]. Following the result for max-sum diversification subject to a matroid constraint in [Borodin et al. 2012], we will analyze the following oblivious local search algorithm:

LOCAL SEARCH ALGORITHM FOR PROPORTIONALLY SUBMODULAR FUNCTION MAXIMIZATION
Let S be a basis of matroid $\mathcal{M} = (U, \mathcal{F})$ where $\mathcal{F}$ denotes the independent sets of the matroid.
WHILE $\exists u \in U \setminus S$ and $v \in S$ such that $S \cup \{u\} \setminus \{v\} \in \mathcal{F}$ and $f(S \cup \{u\} \setminus \{v\}) > f(S)$
  $S = S \cup \{u\} \setminus \{v\}$
ENDWHILE

We first state and prove a purely technical lemma:

LEMMA 6.1. *Given three non-increasing non-negative sequences:*

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0,$$
$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0,$$
$$x_1 \geq x_2 \geq \cdots \geq x_n \geq 0.$$

Then we have

$$\sum_{i=1}^{n} \alpha_i x_i \sum_{i=1}^{n} \beta_i \geq \sum_{i=1}^{n} \beta_i x_{n+1-i} \sum_{i=1}^{n} \alpha_i.$$

[5]This number is obtained by a computer program.

PROOF. Consider the following:

$$\begin{aligned}
n\sum_{i=1}^{n}\alpha_i x_i &= n\alpha_1 x_1 + n\alpha_2 x_2 + \cdots + n\alpha_n x_n \\
&= \sum_{i=1}^{n}\alpha_i x_1 + (n\alpha_1 - \sum_{i=1}^{n}\alpha_i)x_1 + n\alpha_2 x_2 + \cdots + n\alpha_n x_n \\
&\geq \sum_{i=1}^{n}\alpha_i x_1 + (n\alpha_1 + n\alpha_2 - \sum_{i=1}^{n}\alpha_i)x_2 + \cdots + n\alpha_n x_n \\
&= \sum_{i=1}^{n}\alpha_i x_1 + \sum_{i=1}^{n}\alpha_i x_2 + (n\alpha_1 + n\alpha_2 - 2\sum_{i=1}^{n}\alpha_i)x_2 + \cdots + n\alpha_n x_n \\
&\vdots \\
&\geq \sum_{i=1}^{n}\alpha_i x_1 + \sum_{i=1}^{n}\alpha_i x_2 + \cdots + \sum_{i=1}^{n}\alpha_i x_n + (n\alpha_1 + n\alpha_2 + \cdots + n\alpha_n - n\sum_{i=1}^{n}\alpha_i)x_n \\
&= \sum_{i=1}^{n}\alpha_i \sum_{i=1}^{n} x_i
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
n\sum_{i=1}^{n}\beta_i x_{n+1-i} &= n\beta_1 x_n + n\beta_2 x_{n-1} + \cdots + n\beta_n x_1 \\
&= \sum_{i=1}^{n}\beta_i x_n + (n\beta_1 - \sum_{i=1}^{n}\beta_i)x_n + n\beta_2 x_{n-1} + \cdots + n\beta_n x_1 \\
&\leq \sum_{i=1}^{n}\beta_i x_n + (n\beta_1 + n\beta_2 - \sum_{i=1}^{n}\beta_i)x_{n-1} + \cdots + n\beta_n x_1 \\
&= \sum_{i=1}^{n}\beta_i x_n + \sum_{i=1}^{n}\beta_i x_{n-1} + (n\beta_1 + n\beta_2 - 2\sum_{i=1}^{n}\beta_i)x_{n-1} + \cdots + n\beta_n x_1 \\
&\vdots \\
&\leq \sum_{i=1}^{n}\beta_i x_n + \sum_{i=1}^{n}\beta_i x_{n-1} + \cdots + \sum_{i=1}^{n}\beta_i x_1 + (n\alpha_1 + n\beta_2 + \cdots + n\beta_n - n\sum_{i=1}^{n}\beta_i)x_1 \\
&= \sum_{i=1}^{n}\beta_i \sum_{i=1}^{n} x_i
\end{aligned}$$

Therefore the lemma follows. □

The following lemma on the exchange property of matroid bases was first stated in [Brualdi 1969].

LEMMA 6.2 (BRUALDI). *For any two sets $X, Y \in \mathcal{F}$ with $|X| = |Y|$, there is a bijective mapping $g : X \to Y$ such that $X \cup \{g(x)\} \setminus \{x\} \in \mathcal{F}$ for any $x \in X$.*

Before we prove the theorem, we need to establish several lemmas related to this bijective mapping. Let O be the optimal solution, and S, the solution at the end of the local search algorithm. Let s be the size of a basis; let $A = O \cap S$, $B = S \setminus A$ and $C = O \setminus A$. By Lemma 6.2, there is a bijective mapping $g : B \to C$ such that $S \cup \{b\} \setminus \{g(b)\} \in \mathcal{F}$ for any $b \in B$. Let $B = \{b_1, b_2, \ldots, b_t\}$, and let $c_i = g(b_i)$ for all $i = 1, \ldots, t$. We reorder $b_1, b_2, \ldots, b_t$ in different ways. Let $b'_1, b'_2, \ldots, b'_t$ be an ordering such that the corresponding $c'_1, c'_2, \ldots, c'_t$ maximizes the sum $\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c'_i\})$; and let $b''_1, b''_2, \ldots, b''_t$ be an ordering such that the corresponding $c''_1, c''_2, \ldots, c''_t$ minimizes the sum

$$\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c''_i\}).$$

LEMMA 6.3.

$$\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c'_i\}) \leq sf(S) + \sum_{i=1}^{t}(s+1-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c'_i\} \setminus \{b'_i\}) - (s+1)(\frac{s+1}{s})^{t-1} f(S \setminus \{b'_1, \ldots, b'_t\}).$$

PROOF. By the definition of proportionally submodular, we have

$$sf(S) + sf(S \cup \{c'_1\} \setminus \{b'_1\}) \geq (s-1)f(S \cup \{c'_1\}) + (s+1)f(S \setminus \{b'_1\})$$
$$sf(S \setminus \{b'_1\}) + (s-1)f(S \cup \{c'_2\} \setminus \{b'_2\}) \geq (s-2)f(S \cup \{c'_2\}) + (s+1)f(S \setminus \{b'_1, b'_2\})$$
$$\vdots$$
$$sf(S \setminus \{b'_1, \ldots, b'_{t-1}\}) + (s-t+1)f(S \cup \{c'_t\} \setminus \{b'_t\}) \geq (s-t)f(S \cup \{c'_t\}) + (s+1)f(S \setminus \{b'_1, \ldots, b'_t\})$$

Multiplying the i^{th} inequality by $(\frac{s+1}{s})^{i-1}$, and summing all of them up to get

$$sf(S) + \sum_{i=1}^{t}(s+1-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c'_i\} \setminus \{b'_i\}) \geq \sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c'_i\}) + (s+1)(\frac{s+1}{s})^{t-1} f(S \setminus \{b'_1, \ldots, b'_t\}).$$

After rearranging the inequality, we get

$$\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c'_i\}) \leq sf(S) + \sum_{i=1}^{t}(s+1-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c'_i\} \setminus \{b'_i\}) - (s+1)(\frac{s+1}{s})^{t-1} f(S \setminus \{b'_1, \ldots, b'_t\}).$$

□

LEMMA 6.4.

$$\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c''_i\}) - \sum_{i=1}^{t}(s+t+1-i)(\frac{s+1}{s})^{i-1} f(S) \geq sf(S \cup \{c''_1, \ldots, c''_t\}) - (s+1)(\frac{s+1}{s})^{t-1} f(S)$$

PROOF. By the definition of proportionally submodular, we have

$$(s+t-1)f(S\cup\{c''_1\})+(s+1)f(S\cup\{c''_2,\dots,c''_t\})\geq sf(S\cup\{c''_1,\dots,c''_t\})+(s+t)f(S)$$
$$\vdots$$
$$(s+1)f(S\cup\{c''_{t-1}\})+(s+1)f(S\cup\{c''_t\})\geq sf(S\cup\{c''_{t-1},c''_t\})+(s+2)f(S)$$
$$sf(S\cup\{c''_t\})+(s+1)f(S)\geq sf(S\cup\{c''_t\})+(s+1)f(S).$$

Multiplying the i^{th} inequality by $(\frac{s+1}{s})^{i-1}$, and summing all of them up, we have

$$\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1}f(S\cup\{c''_i\})+(s+1)(\frac{s+1}{s})^{t-1}f(S)$$
$$\geq sf(S\cup\{c''_1,\dots,c''_t\})+\sum_{i=1}^{t}(s+t+1-i)(\frac{s+1}{s})^{i-1}f(S).$$

Therefore, we have

$$\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1}f(S\cup\{c''_i\})$$
$$\geq sf(S\cup\{c''_1,\dots,c''_t\})+\sum_{i=1}^{t}(s+t+1-i)(\frac{s+1}{s})^{i-1}f(S)-(s+1)(\frac{s+1}{s})^{t-1}f(S).$$

□

Let

$$W=\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1},\qquad X=\sum_{i=1}^{t}(s+1-i)(\frac{s+1}{s})^{i-1},$$
$$Y=\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1},\qquad Z=\sum_{i=1}^{t}(s+t+1-i)(\frac{s+1}{s})^{i-1}.$$

LEMMA 6.5.

$$Y\text{Big}[\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1}f(S\cup\{c'_i\})\text{Big}]\geq W\text{Big}[\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1}f(S\cup\{c''_i\})\text{Big}].$$

PROOF. Let $\{c^*_i\}$ be an ordering of the $\{c_i\}$ such that $f(S\cup\{c^*_1\})\geq f(S\cup\{c^*_2\})\dots\geq f(S\cup\{c^*_1\})$. We then have:

$$Y\text{Big}[\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1}f(S\cup\{c'_i\})\text{Big}]\geq Y\text{Big}[\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1}f(S\cup\{c^*_i\})\text{Big}]$$

by definition of the $\{c'_i\}$

$$\geq W\text{Big}[\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1}f(S\cup\{c^*_{t+1-i}\})\text{Big}]$$

by applying Lemma 6.1 with $\alpha_i = (s-i)(\frac{s+1}{s})^{i-1}, \beta_i = (s+t-i)(\frac{s+1}{s})^{i-1}$, and $x_i = f(S \cup \{c_i^*\}$

$$\geq W[\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c_i''\})\text{Big}].$$

by definition of the $\{c_i''\}$. □

THEOREM 6.6. *Let s be the size of a basis, the local search algorithm achieves an approximation ratio bounded by a function $\rho(s)$. For all s, $\rho(s) \leq 14.5$ and $\rho(s)$ converges to 10.22 as s tends to ∞.*

PROOF. Since S is a locally optimal solution, we have

$$f(S) \geq f(S \cup \{c_i'\} \setminus \{b_i'\}).$$

Since $f(S \setminus \{b_1', \dots, b_t'\}) \geq 0$, by Lemma 6.3, we have

$$\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c_i'\}) \leq sf(S) + \sum_{i=1}^{t}(s+1-i)(\frac{s+1}{s})^{i-1} f(S).$$

Therefore,

$$\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c_i'\}) \leq (s+X)f(S).$$

On the other hand, we have $O \subseteq S \cup \{c_1'', \dots, c_t''\}$, by monotonicity, we have $f(O) \leq f(S \cup \{c_1'', \dots, c_t''\})$. By Lemma 6.4, we have

$$\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c_i''\}) \geq sf(O) + [Z - (s+1)(\frac{s+1}{s})^{t-1}]f(S).$$

By Lemma 6.5, we have

$$Y\sum_{i=1}^{t}(s-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c_i'\}) \geq W\sum_{i=1}^{t}(s+t-i)(\frac{s+1}{s})^{i-1} f(S \cup \{c_i''\}).$$

Therefore

$$Y(s+X)f(S) \geq Wsf(O) + X[Z - (s+1)(\frac{s+1}{s})^{t-1}]f(S)$$

Hence the approximation ratio:

$$\frac{f(O)}{f(S)} \leq \frac{YX - WZ + Ys + W(s+1)(\frac{s+1}{s})^{t-1}}{Ws} = \frac{YX - WZ + Ys}{Ws} + (\frac{s+1}{s})^t.$$

Simplifying the notation, we have

$$\frac{f(O)}{f(S)} \leq \frac{\sum_{i=1}^{t}(s^2 + st + ti - si)(\frac{s+1}{s})^{i-1} + \sum_{i=t+1}^{2t-1} t(2t-i)(\frac{s+1}{s})^{i-1}}{\sum_{i=1}^{t} s(s-i)(\frac{s+1}{s})^{i-1}} + (\frac{s+1}{s})^t.$$

Using Lemma 5.2 and 5.3 to simply it further, we have

$$\frac{f(O)}{f(S)} \leq \frac{2s(\frac{s+1}{s})^{2t} - 2t(\frac{s+1}{s})^t - 2s}{(2s-t)(\frac{s+1}{s})^t - 2s}.$$

Let $x = (\frac{s+1}{s})^s$ and $r = \frac{t}{s}$, we study the continuous version of the above function

$$g(x,r) = \frac{2x^{2r} - 2rx^r - 2}{(2-r)x^r - 2}.$$

Since S is a local optimum with respect to the swapping of any single element and by the definition of x, s and t, we have $2 \le t \le s$ and hence $2.25 \le x \le e$ and $0 < r \le 1$. Our goal then is to establish an upper bound on $g(x,r)$ for $2.25 \le x \le e$ and $0 < r \le 1$. We will think of $g(x,r)$ as implictly defining x as a function of r at points where $g(x,r)$ can possibly take on a maximum value, namely when when $\frac{\partial g(x,r)}{\partial x} = 0$ and at the boundary points for x.

Note that since $x \ge 2.25$,

$$x > \left(\frac{2}{2-r}\right)^{\frac{1}{r}},$$

for all $0 < r \le 1$. Therefore, we have $(2-r)x^r - 2 > 0$ for given x and r. It is easy to verify that function $g(x,r)$ is continuous and differentiable. For any fixed r, the function has two boundary points at $x = 2.25$ and $x = e$, and taking partial derivative with respect to x, we have

$$\frac{\partial g(x,r)}{\partial x} = \frac{2rx^{r-1}(x^r-1)[(2-r)x^r - (2+r)]}{[(2-r)x^r - 2]^2}.$$

Therefore the only point where the partial derivative equals to zero is

$$x^* = (\frac{2+r}{2-r})^{\frac{1}{r}}.$$

Plugging this into the original expression for $g(x,r)$, we have

$$g(x^*,r) = \frac{2r^2+8}{(r-2)^2}.$$

The function $g(x^*,r)$ is monotonically increasing with respect to $r \in (0,1]$ and it has a maximum value of 10 when $r = 1$.

Now it only remains to check the two boundary points $x = 2.25$ and $x = e$. Note that these are fixed values. We now fix x, and take partial derivative with respect to r:

$$\frac{\partial g(x,r)}{\partial r} = \frac{2x^r(x^r-1)[(2\ln x - r\ln x + 1)x^r - (2\ln x + r\ln x + 1)]}{[(2-r)x^r - 2]^2}.$$

Since $x^r > 0$, $x^r - 1 > 0$ and $[(2-r)x^r - 2]^2 > 0$. If we can show that

$$(2\ln x - r\ln x + 1)x^r - (2\ln x + r\ln x + 1) > 0$$

then the function after fixing x is monotonically increasing with respect to r. We use the Taylor expansion of x^r at $x = 0$.

$$x^r > 1 + r\ln x + \frac{1}{2}r^2\ln^2 x.$$

Therefore,

$$(2\ln x - r\ln x + 1)x^r - (2\ln x + r\ln x + 1) > r\ln x(2\ln x + r\ln^2 x - \frac{1}{2}r^2\ln^2 x - \frac{1}{2}r\ln x - 1).$$

Note that we only need to check for the case when $x = e$ and $x = 2.25$.

(1) Case $x = e$:

$$2\ln x + r\ln^2 x - \frac{1}{2}r^2\ln^2 x - \frac{1}{2}r\ln x - 1 = 1 + \frac{1}{2}r - \frac{1}{2}r^2 > 0.$$

(2) Case $x = 2.25$:

$$2\ln x + r\ln^2 x - \frac{1}{2}r^2\ln^2 x - \frac{1}{2}r\ln x - 1 > 0.6 + 0.6r - 0.5r - 0.4r^2 > 0.$$

Therefore $(2\ln x - r\ln x + 1)x^r - (2\ln x + r\ln x + 1) > 0$, and hence $\frac{\partial g(x,r)}{\partial r} > 0$ for $x = 2.25$ and $x = e$. Therefore the maximum is obtained when $r = 1$. Plug $r = 1$ into the original formula, we have

$$g(x,1) = \frac{2x^2 - 2x - 2}{x - 2}.$$

Evaluating at $x = e$ and $x = 2.25$, we have $g(e,1) = 10.22$ and $g(2.25,1) = 14.5$. We define the function $\rho(s)$ (as in the theorem statement) to be $\max_x\{g(x,1)\}$. This completes the proof. □

While the function $\rho(s)$ is decreasing in s, we are not claiming that the approximation ratio of the algorithm is decreasing in s; we are only providing an analysis that yields $\rho(s)$ as a bound on the approximation ratio.

□

7. CONCLUSION AND OPEN PROBLEM

Motivated by the max-sum diversification problem we are led to study a generalization of monotone submodular functions that we call proportionally-submodular functions. This class includes the supermodular max-sum dispersion problem.

There are several open problems regarding the class of proportionally submodular functions. First we would like to find other natural functions that are monotone and non-monotone proportionally submodular. As we have shown, our class does for example contain some but not all functions with small supermodular degree as well as some functions that do not have small submodular degree. Indeed, proportionally submodular functions are incomparable with functions having small supermodular degree. Another obvious question is whether there is an analogue of the marginal decreasing property that characterizes submodular functions or at least analogues that would be a consequence of weak submodularity and would be useful in analyzing algorithms.

In terms of computational problems regarding the optimization of monotone proportionally submodular functions many interesting questions remain. Similar to the maximization for an arbitrary matroid constraint using local search, we would like to have a proof of the convergence of the greedy approximation bound for the cardinality constraint. Another immediate open problem is to close the gap between the upper and lower bounds we know for approximating an arbitrary monotone proportionally submodular function subject to cardinality or matroid constraints. We note that although all of our individual examples in section 3 can either be computed optimally or have better approximation ratios than we can prove for the class of monotone proportionally submodular functions, it does not follow that a sum of such functions can be computed with such good polynomial time approximations. It would also be of interest to consider an approximation for maximizing a proportionally submodular function subject to a knapsack constraint. Finally, are the efficient constant appropximation algorithms for maximizing non monotone proportionally submodular functions.

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