(F)

Control Variates

~~Co Var~~

How can we estimate $\underset{s \sim p}{E}\, g(s)$

For some function g?

Naive Monte Carlo estimate:

$$\underset{s \sim p}{E}\, g(s) \approx \frac{1}{K} \sum_{i=1}^{K} g(s_i)$$

where $s_1 \ldots s_K \sim p$

This ~~is~~ estimate is unbiased but can have high variance if K is small.

$$Var = Var(g)/K$$

We can reduce the variance by introducing a control variate, a function $h(s)$ that is correlated with $g(s)$ & for which we can estimate $\underset{s \sim p}{E}\, h(s)$ with low var. ~~accurately~~. (even if biased)

In particular,

$$\underset{s}{E}\, g(s) \approx \frac{1}{K} \sum_i \left[g(s_i) - \eta \cdot h(s_i) \right] + \eta \underset{s}{E}\, h(s)$$

is an unbiased estimator of $\underset{s}{E}\, g(s)$

for any $\eta \in \mathbb{R}$.

Choose η to minimize the variance of the estimator. ~~thus~~

Exercise: show that $\eta = \dfrac{\mathrm{Cov}(g,h)}{\mathrm{Var}(h)}$

minimizes the variance of the estimator.

Hint: show that this value of η also minimizes the variance of

$$g(s) - \eta \cdot h(s) + \eta \underset{s}{E}\, h(s) \qquad (\text{ie, } K=1)$$

Hint: $\mathrm{Var}(\underset{g-h}{\cancel{g+h}}) = \mathrm{Var}(g) + \mathrm{Var}(h) - 2 \cdot \mathrm{Cov}(h,g)$

__Exercise:__ ~~for this This~~ for this value of n, the variance of the estimator is, $\leq \text{Var}(g)/K$ ~~for $K \geq 1$~~

$$\frac{1}{K}\left[\text{Var}(g) - \frac{[\text{Cov}(g,h)]^2}{\text{Var}(h)}\right]$$

$$= [1 - \text{Corr}(g,h)^2] \cdot \underbrace{\text{Var}(g)/K}_{\text{variance of naive estimator}} \quad \text{for } K \geq 1.$$

ie, the variance of $g(s)$ is reduced when g + h are correlated,

ie, $|\text{Corr}(g,h)| > 0$.

__def.__ $\text{corr}(g,h) = \cancel{\text{Var}(g} \dfrac{\text{Cov}(g,h)}{\sqrt{\text{Var}(g) \cdot \text{Var}(h)}}$

$-1 \leq \text{corr}(g,h) \leq 1$

$g \perp\!\!\!\perp h \Rightarrow \text{corr}(g,h) = 0$

$\text{corr}(g,h) = \pm 1$ iff $g = a \cdot h$ for some constant, a.

<u>Note.</u> Must be able to compute $\underset{s}{E}\, h(s)$ or have a low-variance estimate of it.

Special Case (our case)

Estimate $\nabla \underset{s\sim q}{E} f(s) = \underset{s\sim q}{E} f(s) \cdot \nabla \log q(s)$

use $g(s) = f(s) \cdot \nabla \log q(s)$
& $h(s) = \nabla \log q(s)$

(: g & h should be correlated if $f(s)$ has constant sign for all (most) s.)

Note. $\underset{s\sim q}{E} h(s) = \underset{s\sim q}{E} \nabla \log q(s)$

$$= \int q(s) \nabla \log q(s) \, ds$$

$$= \int \nabla q(s) \, ds$$

$$= \nabla \underbrace{\int q(s) \, ds}_{1}$$

$$= 0$$

$\therefore$ ~~Reduced Variance~~

$\therefore$ New estimator (using $K=1$) is

$$g(s) - \eta \cdot h(s) + \eta \cdot \underset{s \sim q}{E}\, h(s)$$

$$= g(s) - \eta \cdot h(s)$$

$$= f(s)\, \nabla \log q(s) - \eta\, \nabla \log q(s)$$

$$= \left[f(s) - \eta\right] \cdot \nabla \log q(s)$$

baseline, b, as before.

Also,

$\mathrm{cov}(g, h)$

$$= \underset{s \sim q}{E} [g(s) - E g(s)] \cdot [h(s) - E h(s)]$$

$$= \underset{s \sim q}{E} [g(s) - E g(s)] \cdot h(s)$$

$$= \underset{s \sim q}{E} [g(s) \cdot h(s)] - E g(s) \cdot E h(s)$$

$$= \underset{s \sim q}{E} g(s) \cdot h(s)$$

$$= \underset{s \sim q}{E} [f(s) \cdot \nabla \log q(s)] \cdot [\nabla \log q(s)]$$

$$= \underset{s \sim q}{E} f(s) \cdot [\nabla \log q(s)]^2$$

$\cancel{>>0}$

or $\cancel{<<0}$

$<< 0$ if $f(s) << 0$

i.e., if (s) is a typical likelihood function.

i.e., high (anti) covariance.