Partial Derivatives Review

CSC311

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Single-Variable Differentiation

Rule	$f(x)$	Scalar derivative notation with respect to x	Example
Constant	c	0	$\frac{d}{dx}99 = 0$
Multiplication by constant	cf	$c\frac{df}{dx}$	$\frac{d}{dx}3x = 3$
Power Rule	x^n	nx^{n-1}	$\frac{d}{dx}x^3 = 3x^2$
Sum Rule	$f + g$	$\frac{df}{dx} + \frac{dg}{dx}$	$\frac{d}{dx}(x^2 + 3x) = 2x + 3$
Difference Rule	$f - g$	$\frac{df}{dx} - \frac{dg}{dx}$	$\frac{d}{dx}(x^2 - 3x) = 2x - 3$
Product Rule	fg	$f\frac{dg}{dx} + \frac{df}{dx}g$	$\frac{d}{dx}x^2x = x^2 + x2x = 3x^2$
Chain Rule	$f(g(x))$	$\frac{df(u)}{du}\frac{du}{dx}$, let $u = g(x)$	$\frac{d}{dx}ln(x^2) = \frac{1}{x^2}2x = \frac{2}{x}$

Partial Derivatives

- **Multivariate functions** have more than one variable:

e.g. $f(x_1, x_2, x_3) = a_1 x_1 + a_2 x_2 + a_3 x_3 + b$

- **Partial derivatives:** derivative of one variable with all others fixed:

$$\frac{\partial}{\partial x_1}[f(x_1, x_2, x_3)] = a_1$$
$$\frac{\partial}{\partial x_2}[f(x_1, x_2, x_3)] = a_2$$
$$\frac{\partial}{\partial x_3}[f(x_1, x_2, x_3)] = a_3$$

Partial Derivatives

- **More Examples:**

$$\mathrm{g}(x_1, x_2) = x_1 x_2^2$$

$$\frac{\partial}{\partial x_1}[g(x_1, x_2)] = x_2^2$$

$$\frac{\partial}{\partial x_2}[g(x_1, x_2)] = 2x_1 x_2$$

Gradient

- **Gradient:** is a vector containing partial derivative i in position i.

- i.e. $$[\nabla f(x_1, x_2, \dots, x_n)]_i = \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_i}$$

- Equivalently, $$\nabla f(x_1, x_2, \dots, x_n) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Gradient

- **Examples:**

$$f(x_1, x_2, x_3) = a_1x_1 + a_2x_2 + a_3x_3 + b$$

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\mathrm{g}(x_1, x_2) = x_1x_2^2$$

$$\nabla g(x_1, x_2) = \begin{bmatrix} x_2^2 \\ 2x_1x_2 \end{bmatrix}$$

Hessian Matrix

- We can define the second derivative of a function f, which is generally referred to as the **Hessian** of f. It is a matrix and its i-th, j-th entry is given by:

$$[\nabla f(x_1, x_2, \dots, x_n)]_{ij} = \frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_i \partial x_j}$$

- Equivalently, $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$

Hessian Matrix

- **Example: find the gradient and hessian of:**

$$f(x_1, x_2) = x_1^2 + 2x_2^2 - x_1x_2 - 3x_1 - 9x_2 + 3$$

- **Answer:**

$$\nabla f(x_1, x_2) = \begin{bmatrix} 2x_1 - x_2 - 3 \\ 4x_2 - x_1 - 9 \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$