# CSC 411: Lecture 05: Nearest Neighbors

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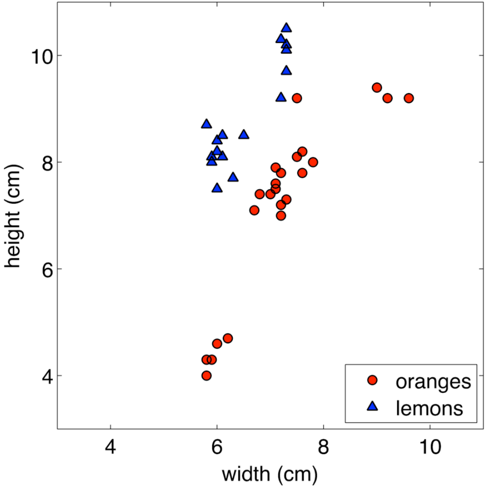
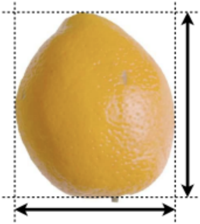
# Today

- Non-parametric models
  - distance
  - non-linear decision boundaries

Note: We will mainly use today's method for classification, but it can also be used for regression

# Classification: Oranges and Lemons



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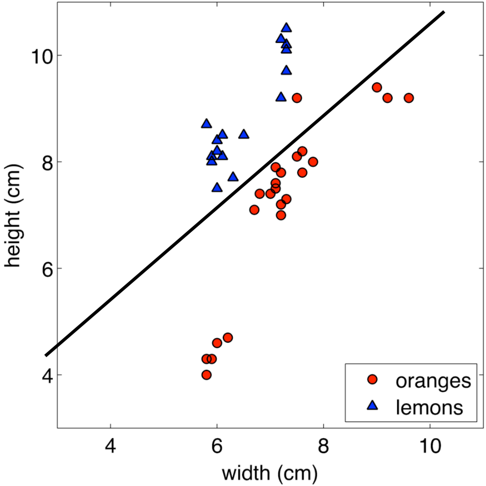
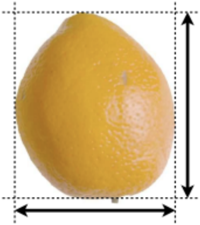



Can construct simple linear decision boundary:

$$y = \text{sign}(w_0 + w_1 x_1 + w_2 x_2)$$



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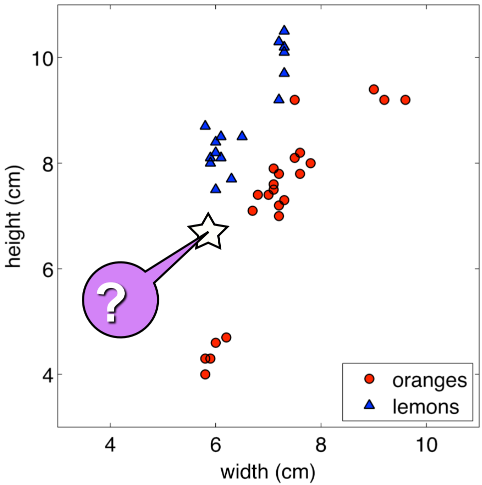
- What functions $f()$ have we seen so far in class?

# Classification as Induction



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- These are typically simple methods for approximating discrete-valued or real-valued target functions (they work for classification or regression problems)
- Learning amounts to simply storing training data
- Test instances classified using similar training instances
- Embodies often sensible underlying assumptions:
  - Output varies smoothly with input
  - Data occupies sub-space of high-dimensional input space

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- Note: we don't really need to compute the square root. Why?

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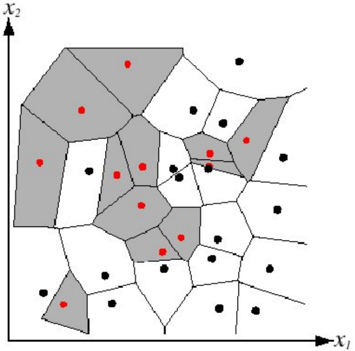
- Decision boundaries: Voronoi diagram visualization
  - show how input space divided into classes
  - each line segment is equidistant between two points of opposite classes



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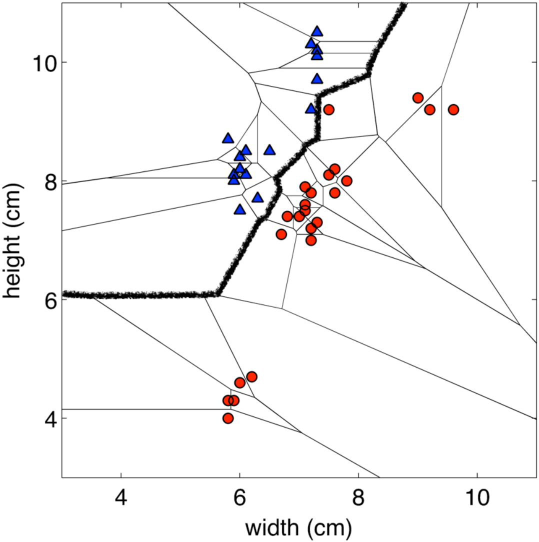


Example: 2D decision boundary

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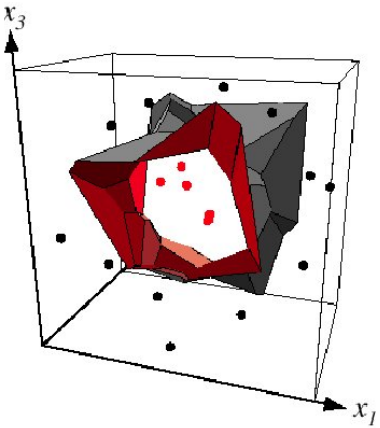
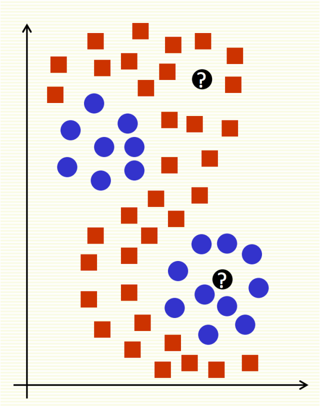



Example: 3D decision boundary

# Nearest Neighbors: Multi-modal Data

- Nearest Neighbor approaches can work with multi-modal data




[Slide credit: O. Veksler]

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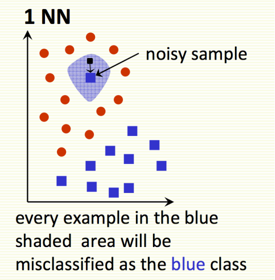


- Nearest neighbors sensitive to mis-labeled data ("class noise"). Solution?

# k-Nearest Neighbors

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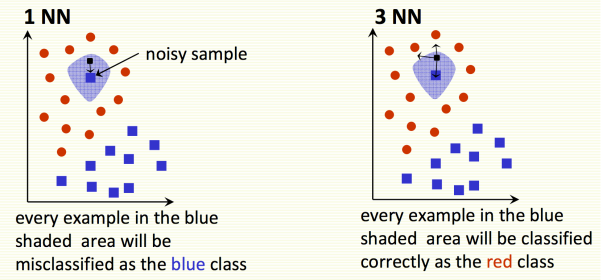


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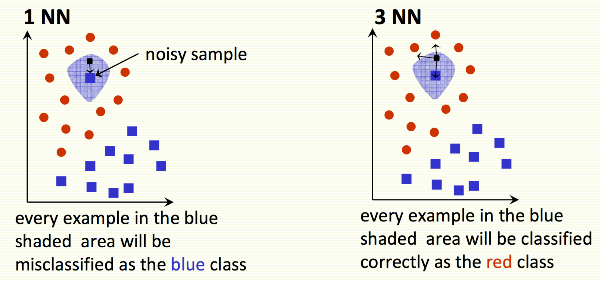


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**Algorithm (kNN)**:

1. Find k examples $\{\mathbf{x}^{(i)}, t^{(i)}\}$ closest to the test instance $\mathbf{x}$
2. Classification output is majority class

$$y = arg \max_{t^{(z)}} \sum_{r=1}^{k} \delta(t^{(z)}, t^{(r)})$$

# k-Nearest Neighbors

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- But if we set $k$ too large we may end up looking at samples that are not neighbors (are far away from the query)
- We can use cross-validation to find $k$
- Rule of thumb is $k < sqrt(n)$, where $n$ is the number of training examples

[Slide credit: O. Veksler]

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- Non-metric attributes (symbols)
  - Hamming distance

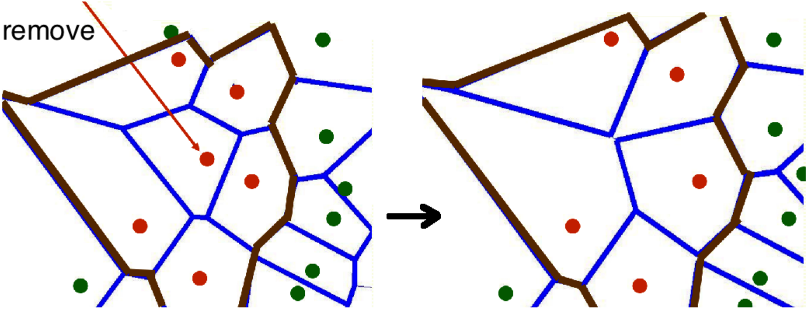
# k-Nearest Neighbors: Issues (Complexity) & Remedies

- Expensive at test time: To find one nearest neighbor of a query point **x**, we must compute the distance to all N training examples. Complexity: $O(kdN)$ for kNN
  - Use subset of dimensions
  - Pre-sort training examples into fast data structures (e.g., kd-trees)
  - Compute only an approximate distance (e.g., LSH)
  - Remove redundant data (e.g., condensing)
- Storage Requirements: Must store all training data
  - Remove redundant data (e.g., condensing)
  - Pre-sorting often increases the storage requirements
- High Dimensional Data: "Curse of Dimensionality"
  - Required amount of training data increases exponentially with dimension
  - Computational cost also increases

[Slide credit: David Claus]

# k-Nearest Neighbors Remedies: Remove Redundancy

- If all Voronoi neighbors have the same class, a sample is useless, remove it

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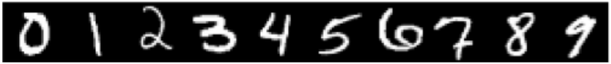
# Example: Digit Classification

- Decent performance when lots of data



- Yann LeCunn – MNIST Digit Recognition
  - Handwritten digits
  - 28x28 pixel images: $d = 784$
  - 60,000 training samples
  - 10,000 test samples
- Nearest neighbour is competitive

| | Test Error Rate (%) |
|---|---|
| Linear classifier (1-layer NN) | 12.0 |
| K-nearest-neighbors, Euclidean | 5.0 |
| K-nearest-neighbors, Euclidean, deskewed | 2.4 |
| K-NN, Tangent Distance, 16x16 | 1.1 |
| K-NN, shape context matching | 0.67 |
| 1000 RBF + linear classifier | 3.6 |
| SVM deg 4 polynomial | 1.1 |
| 2-layer NN, 300 hidden units | 4.7 |
| 2-layer NN, 300 HU, [deskewing] | 1.6 |
| LeNet-5, [distortions] | 0.8 |
| Boosted LeNet-4, [distortions] | 0.7 |

# Fun Example: Where on Earth is this Photo From?

- Problem: Where (e.g., which country or GPS location) was this picture taken?

[Paper: James Hays, Alexei A. Efros. im2gps: estimating geographic information from a single image. CVPR'08. Project page: http://graphics.cs.cmu.edu/projects/im2gps/]

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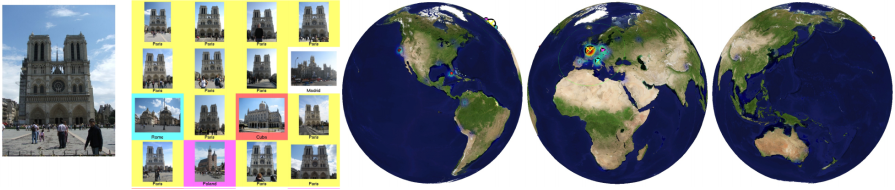
  - Represent each image with meaningful features
  - Do kNN!

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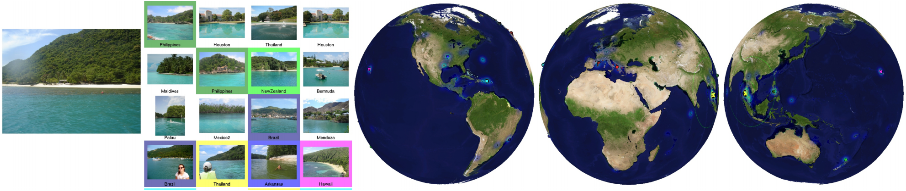
  - Do kNN (large $k$ better, they use $k = 120$)!

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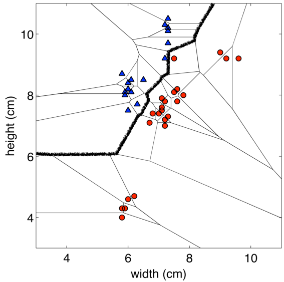


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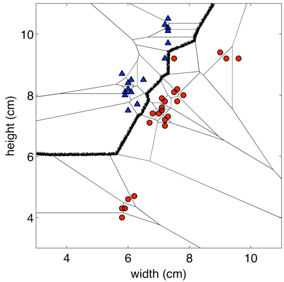


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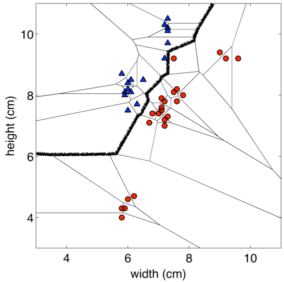


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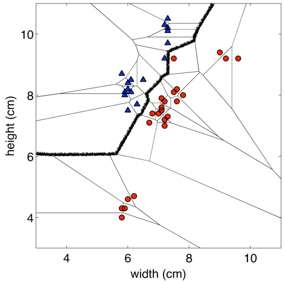


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- Inductive Bias: What kind of decision boundaries do we expect to find?