CSC 411: Lecture 02: Linear Regression

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(Most plots in this lecture are from Bishop's book)

Problems for Today

- What should I watch this Friday?

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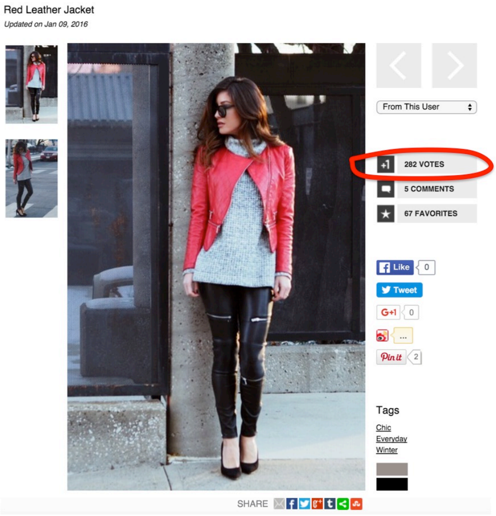
- **Goal**: Predict movie rating automatically!

Problems for Today

- **Goal:** How many followers will I get?



Problems for Today

- **Goal:** Predict the price of the house

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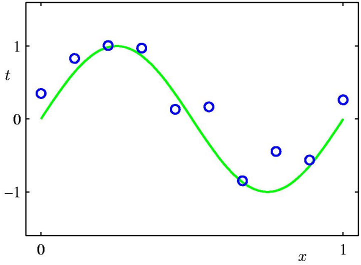
  - Optimization, a way of finding the parameters of our model that minimizes the loss function

Today: Linear Regression

- Linear regression
    - continuous outputs
    - simple model (linear)
- Introduce key concepts:
    - loss functions
    - generalization
    - optimization
    - model complexity
    - regularization

Simple 1-D regression



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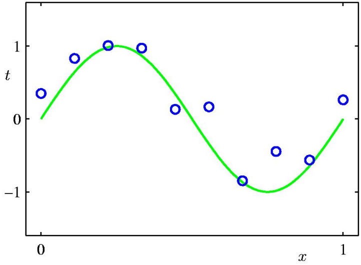

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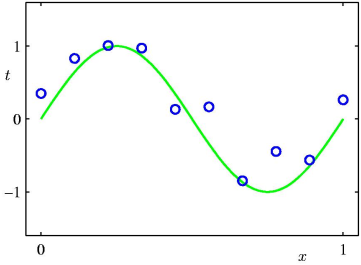

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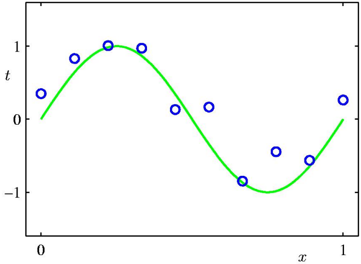

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- Goal: We want to fit a curve to these points

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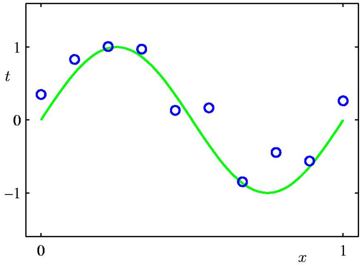

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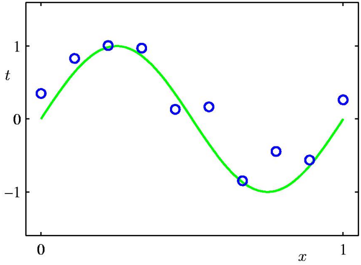

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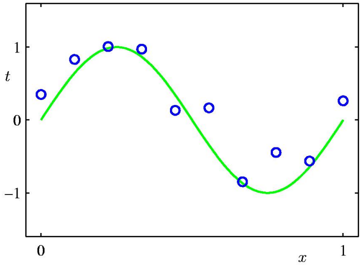

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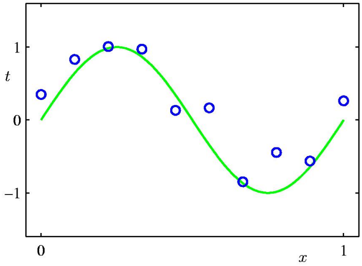

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  - How do we optimize fit to unseen test data (generalization)?

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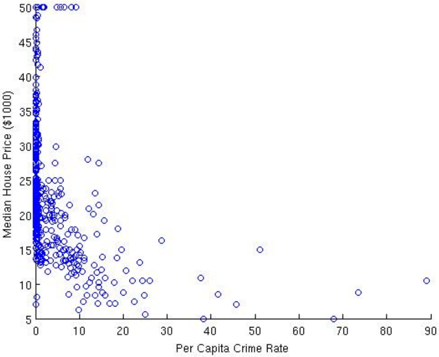

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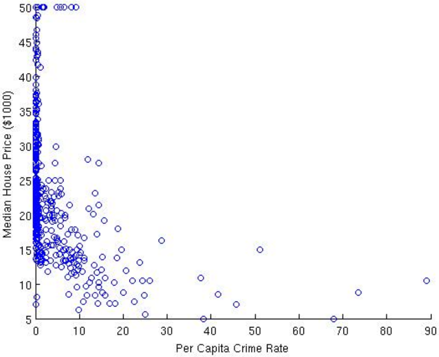

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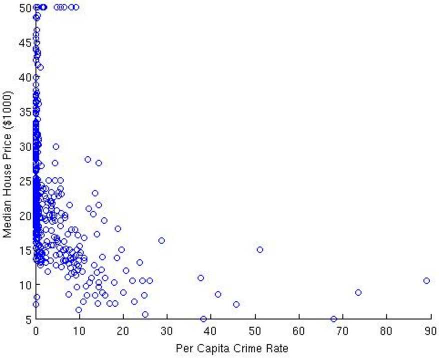

- Use this to predict house prices in other neighborhoods
- Is this a good input (attribute) to predict house prices?

Represent the Data

- Data is described as pairs $\mathcal{D} = \{(x^{(1)}, t^{(1)}), \cdots, (x^{(N)}, t^{(N)})\}$
  - $x \in \mathbb{R}$ is the input feature (per capita crime rate)
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  - Evaluate hypothesis on test set

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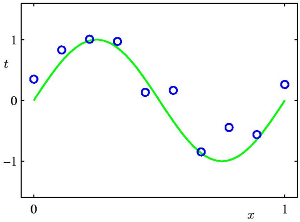
  - Model may be too simple to account for data targets

Least-Squares Regression



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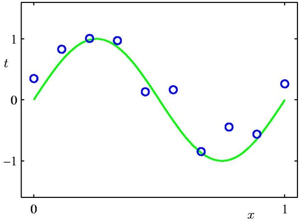

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$$y(x) = \text{function}(x, \mathbf{w})$$

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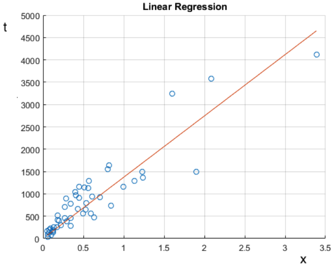

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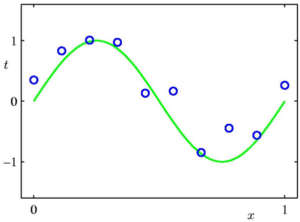

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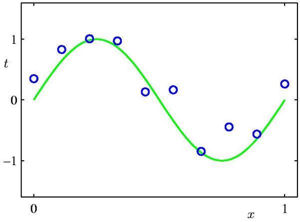

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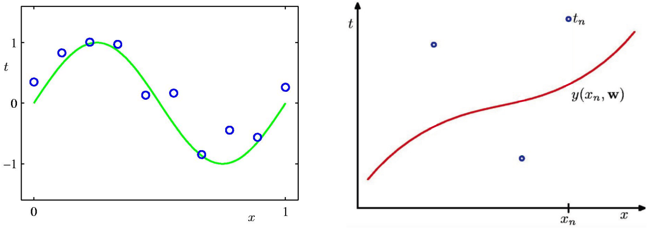

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- For a particular hypothesis ($y(x)$ defined by a choice of $\mathbf{w}$, drawn in red), what does the loss represent geometrically?

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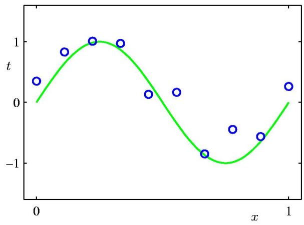

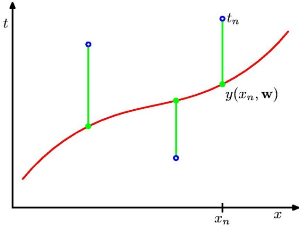

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- The loss for the red hypothesis is the **sum of the squared vertical errors** (squared lengths of green vertical lines)

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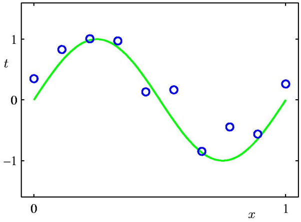

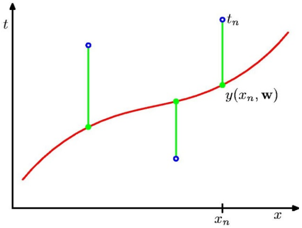

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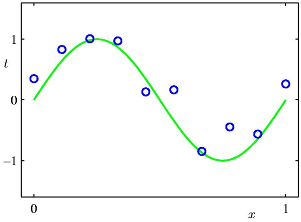

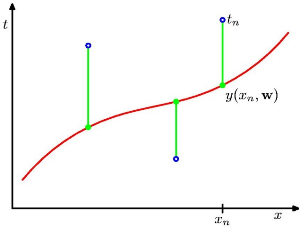

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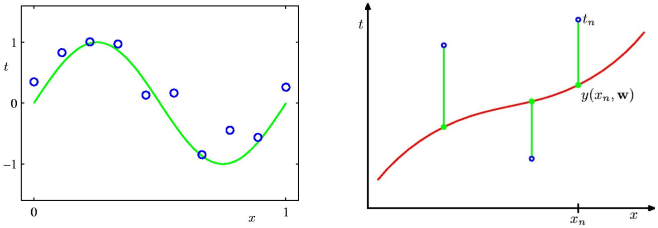

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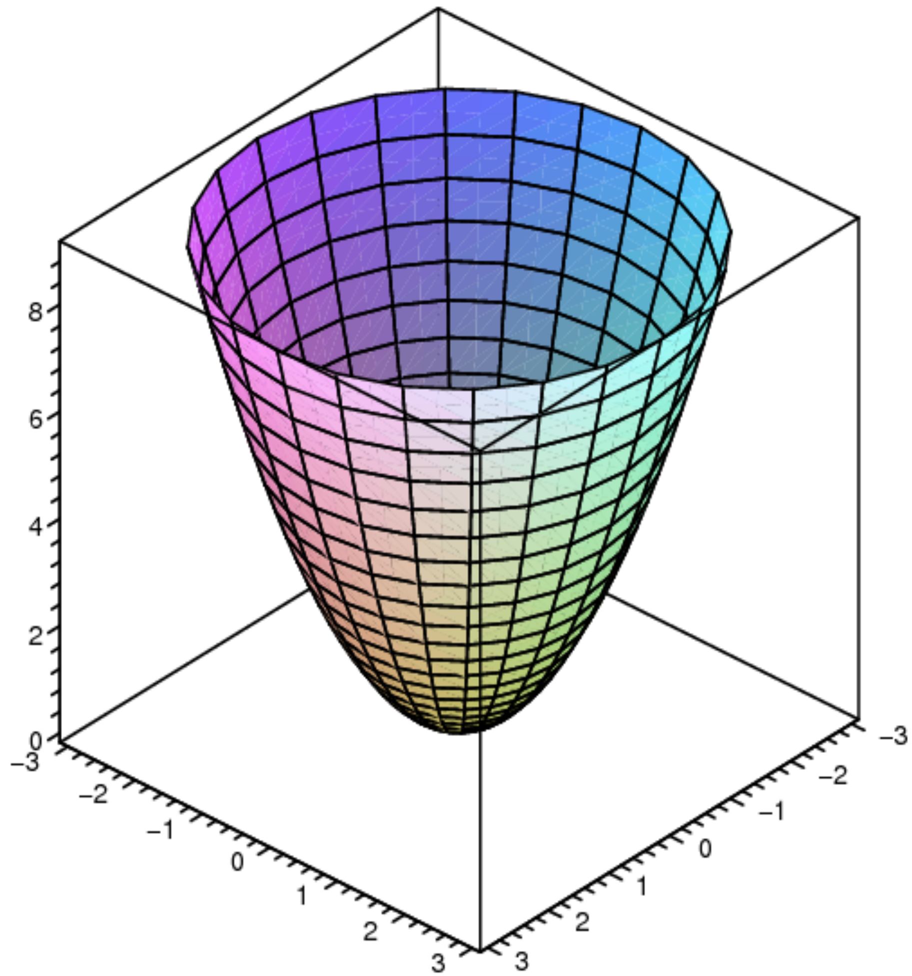
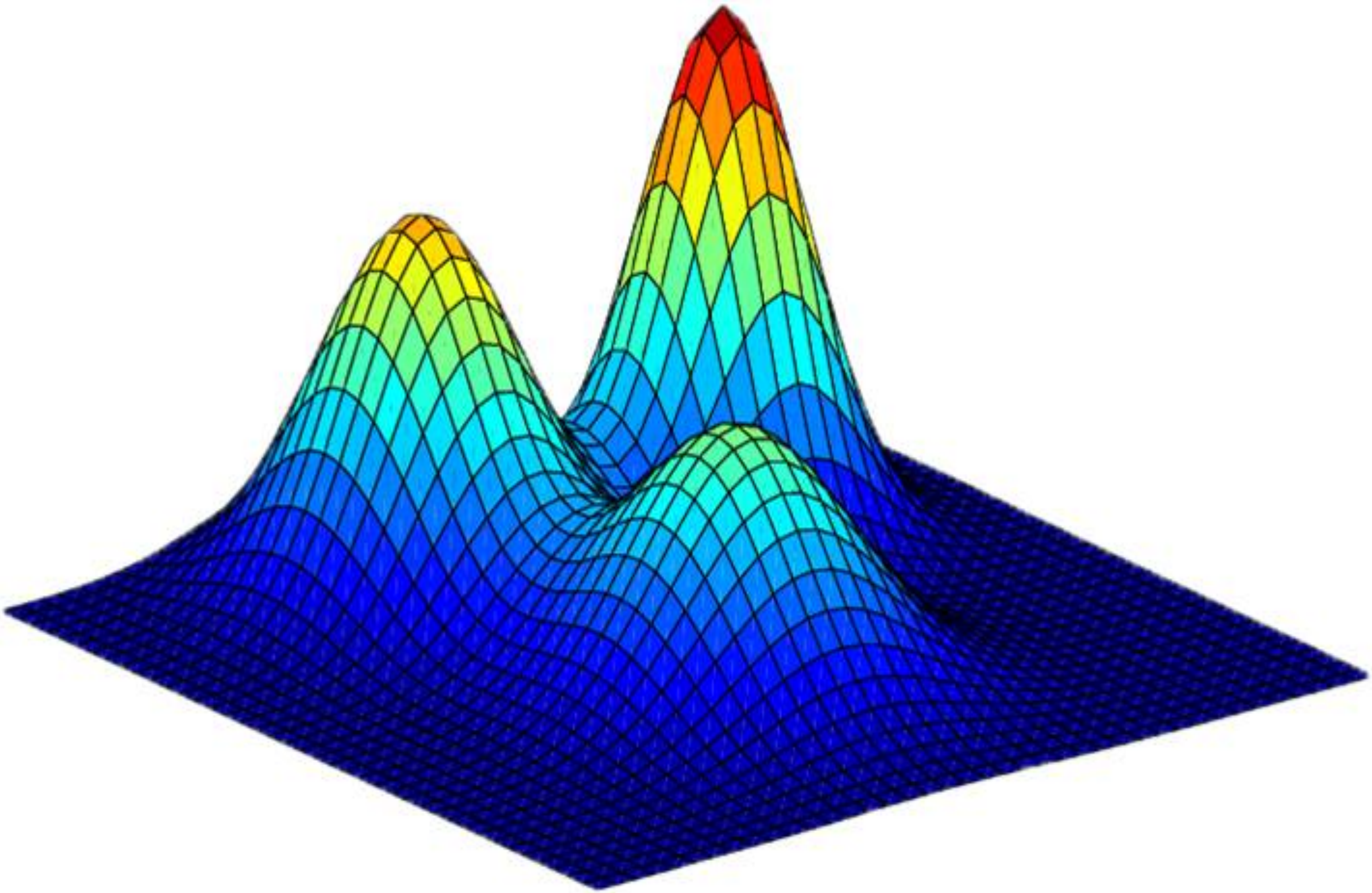
- How do we obtain weights $\mathbf{w} = (w_0, w_1)$?
- For the linear model, what kind of a function is $\ell(\mathbf{w})$?

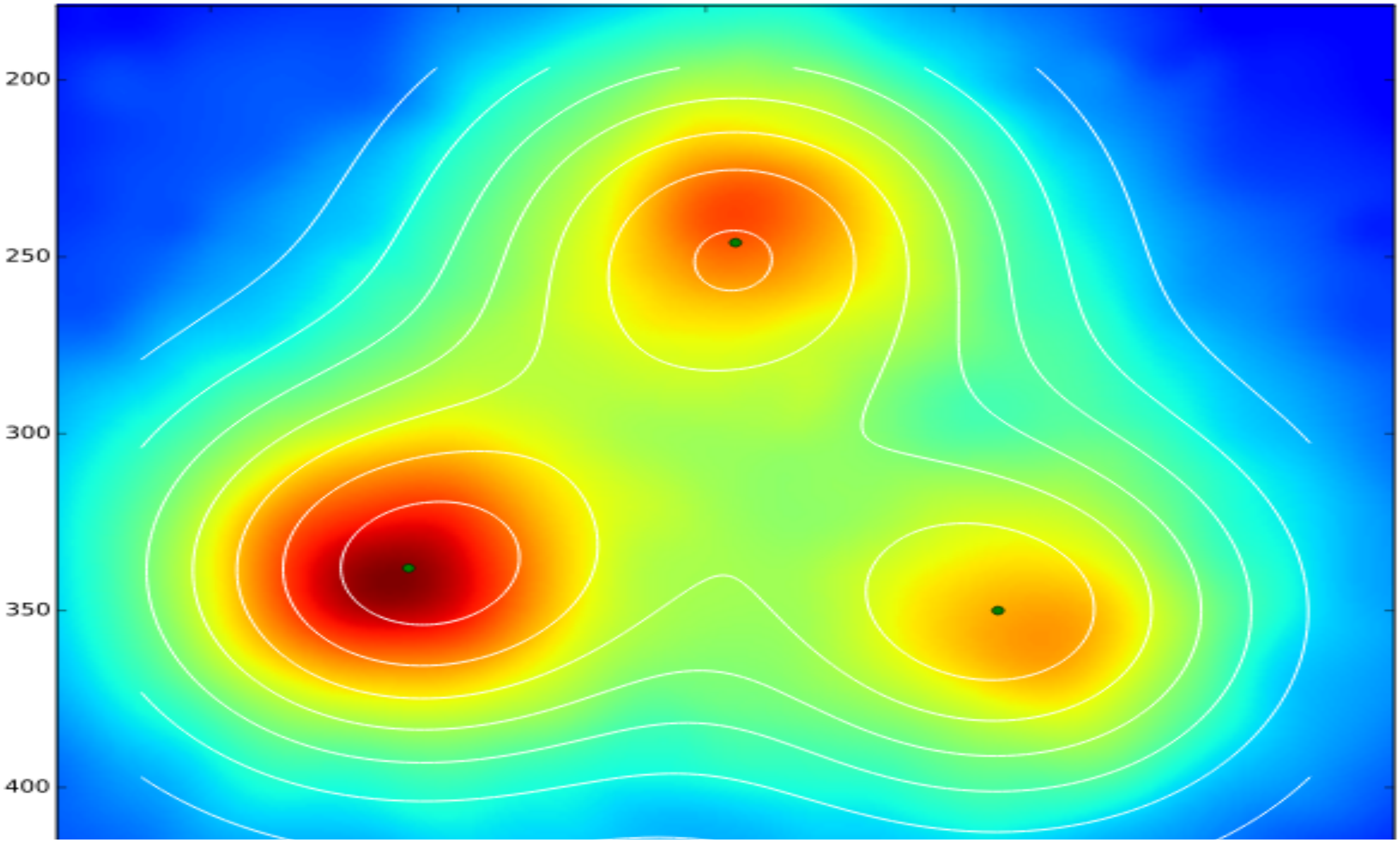
$$\ell(\mathbf{w}) = {w_0}^2 + {w_1}^2$$



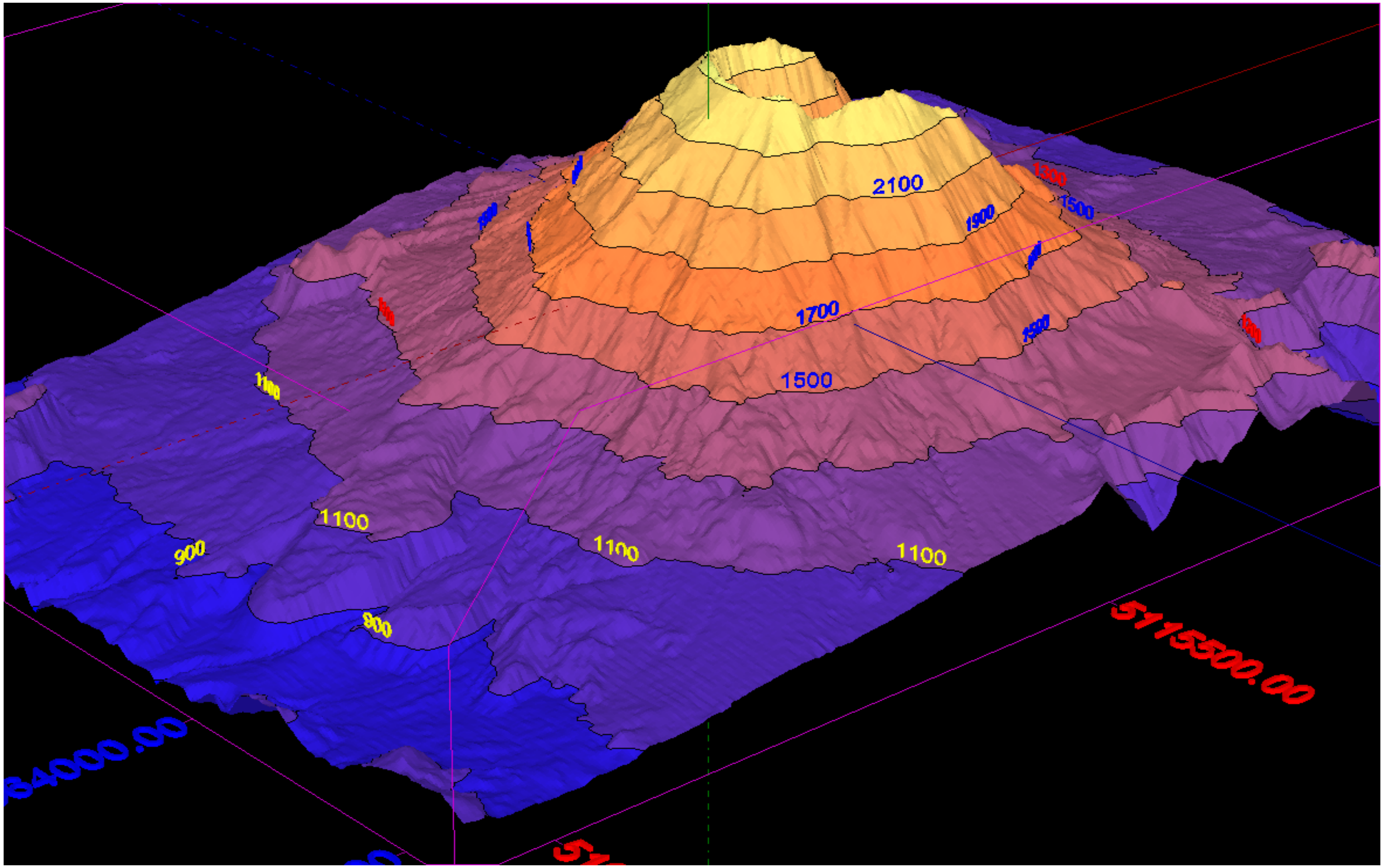
Mixture of three Gaussians



MOG contours



Contour Maps



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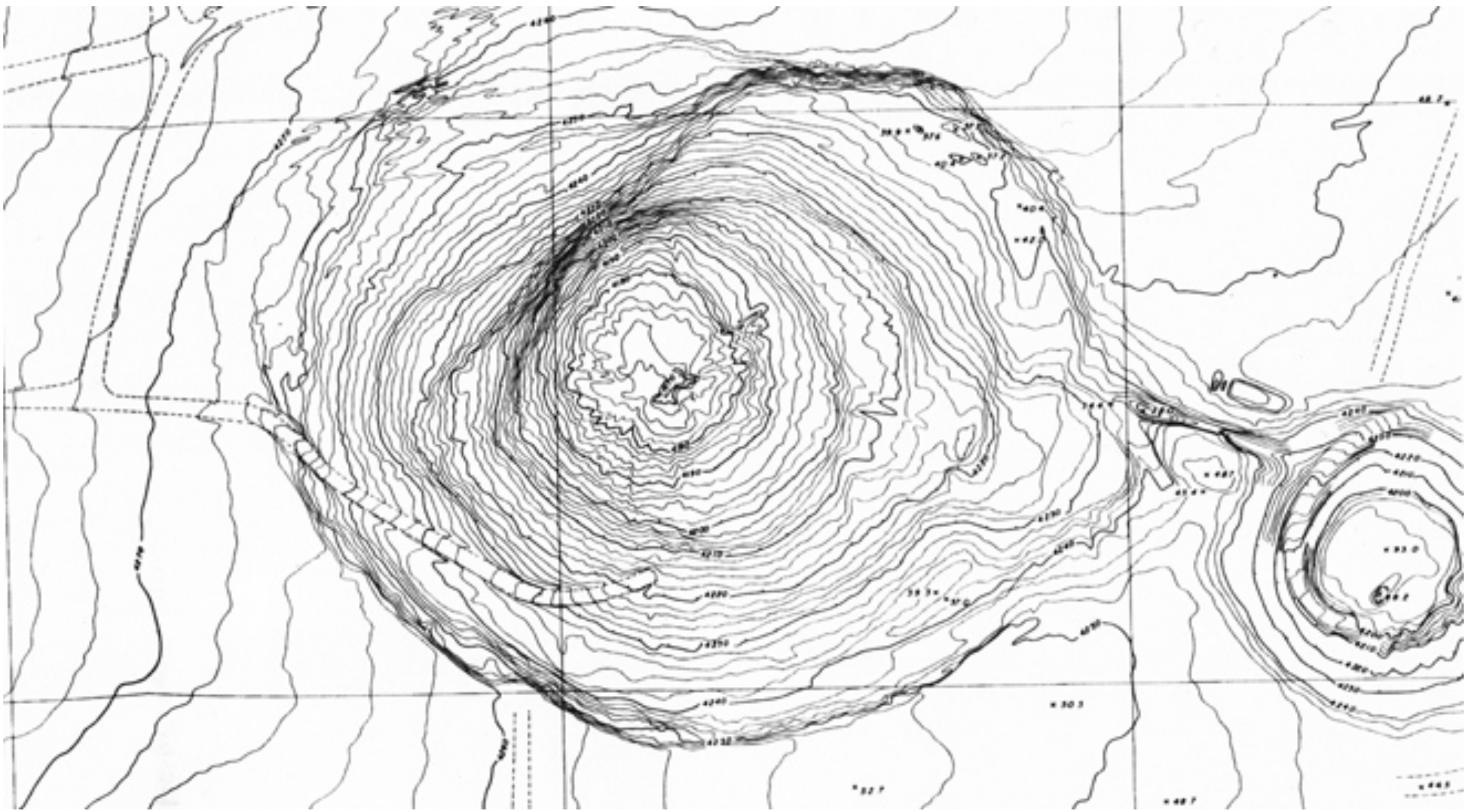

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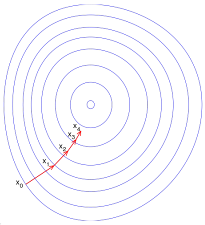

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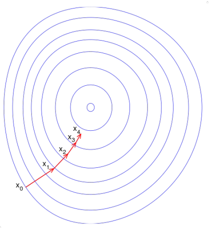

- λ is the learning rate

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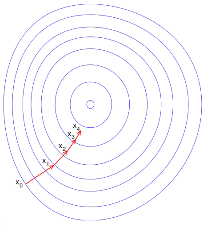

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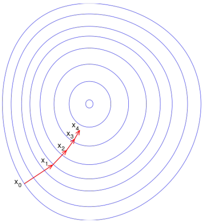

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$$\mathbf{w} \leftarrow \mathbf{w} + 2\lambda \underbrace{\left(t^{(n)} - y(x^{(n)})\right)}_{\text{error}} x^{(n)}$$

- Note: As error approaches zero, so does the update ($\mathbf{w}$ stops changing)

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---

Algorithm 1 Stochastic gradient descent

1: Randomly shuffle examples in the training set
2: **for** $i = 1$ to N **do**
3: Update:

$$\mathbf{w} \leftarrow \mathbf{w} + 2\lambda(t^{(i)} - y(x^{(i)}))x^{(i)} \qquad \text{(update for a linear model)}$$

4: **end for**

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    - Underlying assumption: sample is independent and identically distributed (i.i.d.)

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- Then:

$$\mathbf{w} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{t}$$

(work it out!)

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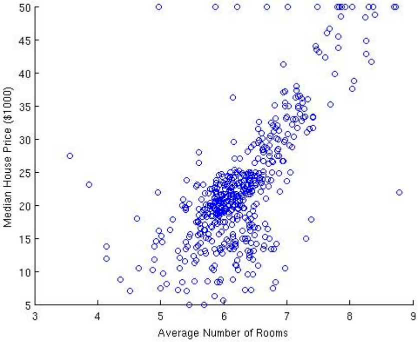
- In the Boston housing example, we can look at the number of rooms



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- We can use gradient descent to solve for each coefficient, or compute $\mathbf{w}$ analytically (how does the solution change?)

More Powerful Models?

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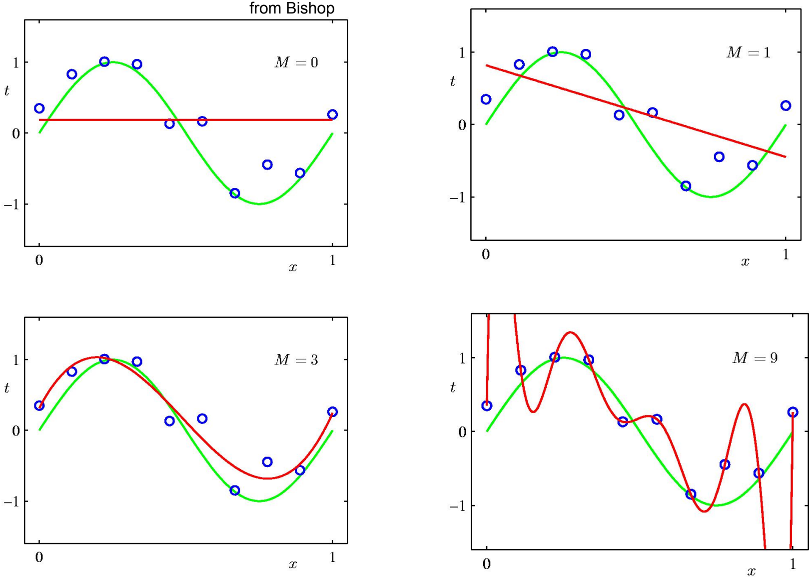
Which Fit is Best?

from Bishop



Generalization

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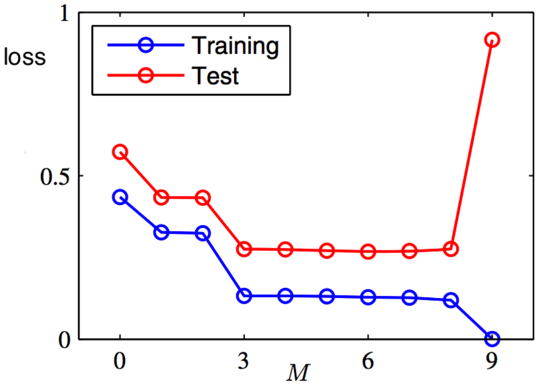

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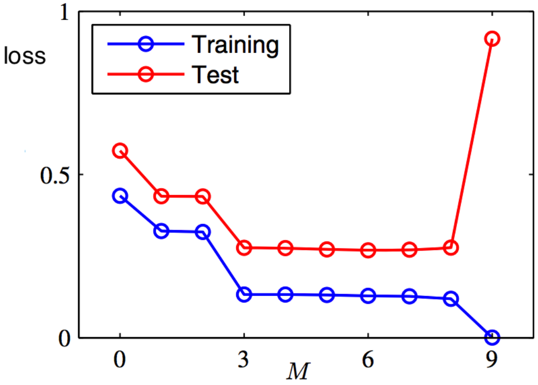

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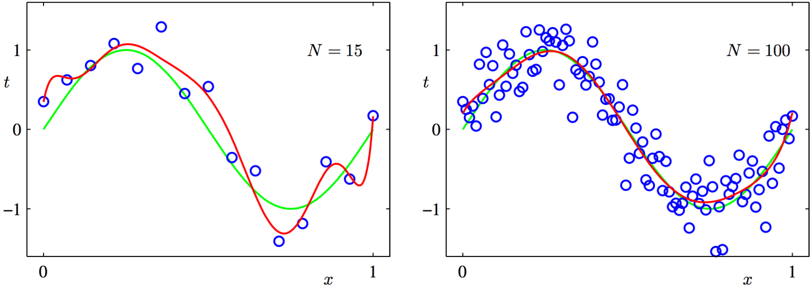

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	$M=0$	$M=1$	$M=6$	$M=9$
$w_0^\star$	0.19	0.82	0.31	0.35
$w_1^\star$		-1.27	7.99	232.37
$w_2^\star$			-25.43	-5321.83
$w_3^\star$			17.37	48568.31
$w_4^\star$				-231639.30
$w_5^\star$				640042.26
$w_6^\star$				-1061800.52
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- The weights are becoming huge to compensate for the noise
- One way of dealing with this is to encourage the weights to be small (this way no input dimension will have too much influence on prediction). This is called regularization

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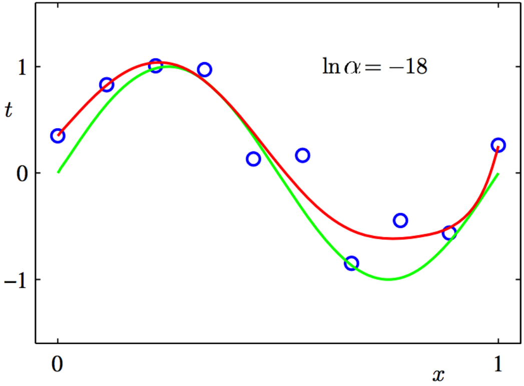
- Also has an analytical solution: $\mathbf{w} = (\mathbf{X}^T\mathbf{X} + \alpha\, \mathbf{I})^{-1}\mathbf{X}^T\mathbf{t}$ (verify!)

Regularized least squares

- Better generalization
- Choose α carefully



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- One method of assessing fit: test generalization = model's ability to predict the held out data
- Optimization is essential: stochastic and batch iterative approaches; analytic when available

So...

- Which movie will you watch?