CSC410
Data Flow Analyses

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First Structure

Partially Ordered Sets

$(S, \sqsubseteq)$: a set S and a (partial) order relation $\sqsubseteq$

- $\sqsubseteq$ is reflexive, transitive, and anti-symmetric

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Information Order

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$x \sqsubseteq x$ (reflexivity).

If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$ (transitivity).

If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$ (antisymmetry).

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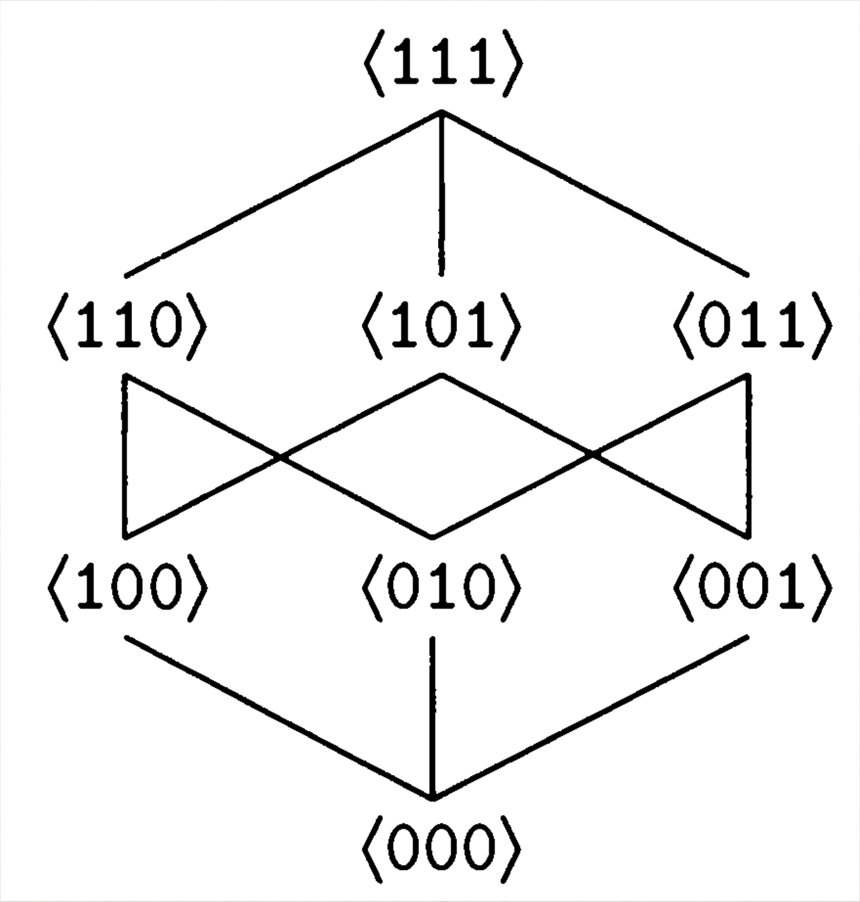

Second Structure

Semi-Lattices

A (meet) *semi-lattice* $\mathbf{L} = (S, \sqcap)$ is a set S with a binary operation, called *meet* ($\sqcap$), that has the following properties:

(1) For all $x, y \in S$, there exist a unique $z \in S$ such that $x \sqcap y = z$ (CLOSURE).

(2) For all $x, y, z \in S$, we have

$$x \sqcap x = x \quad \text{(idempotence)}$$

$$x \sqcap y = y \sqcap x \quad \text{(commutativity)}$$

$$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z \quad \text{(associativity)}$$

Complete Semi-Lattices

The unit for $\sqcap$ is $\top$:

$$\forall x : \; x \sqcap \top = \top \sqcap x = x$$

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The meet semi-lattice is called complete if $\top \in \mathbb{L}$

The Connection

The Connection Between The Structures

Given a semi-lattice and define a binary operation $\sqsubseteq$:

$$x \sqsubseteq y \text{ if and only if } x \sqcap y = x$$

$\sqsubseteq$ is provably a partial order relation.

$\sqcap$ is provably the greatest lower bound defined based on $\sqsubseteq$.

The Converse

Given a partially ordered set $(S, \sqsubseteq)$, where the greatest lower bound of every pair of elements is defined, let:

$$x \sqcap y = \text{the greatest lower bound according to } \sqsubseteq$$

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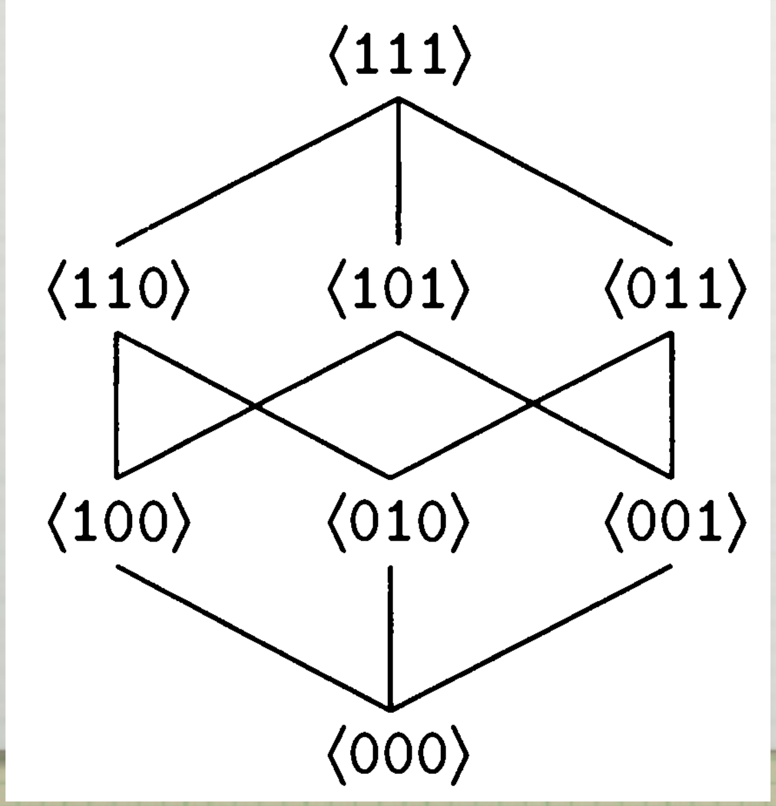

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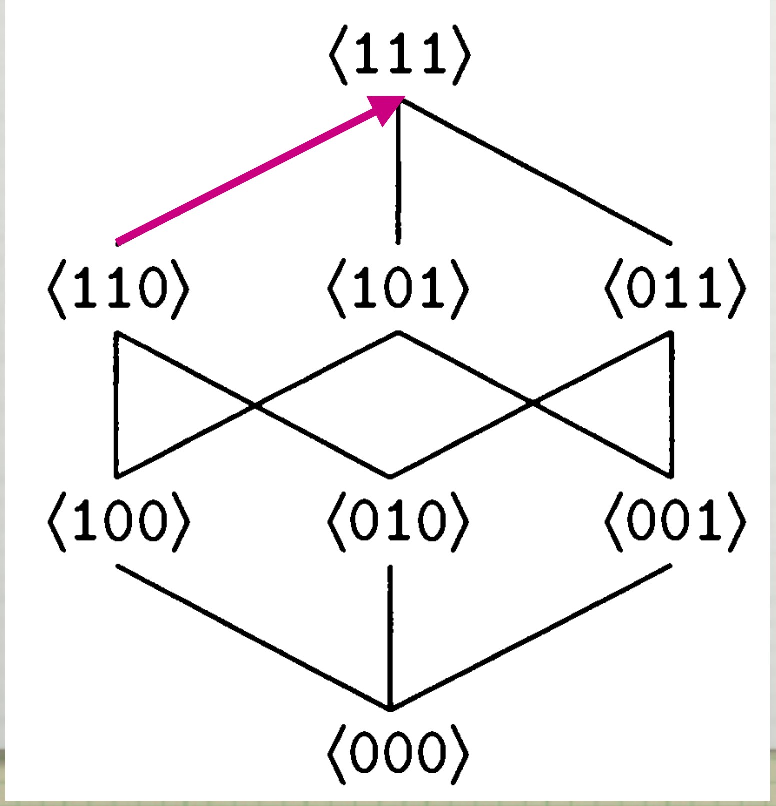

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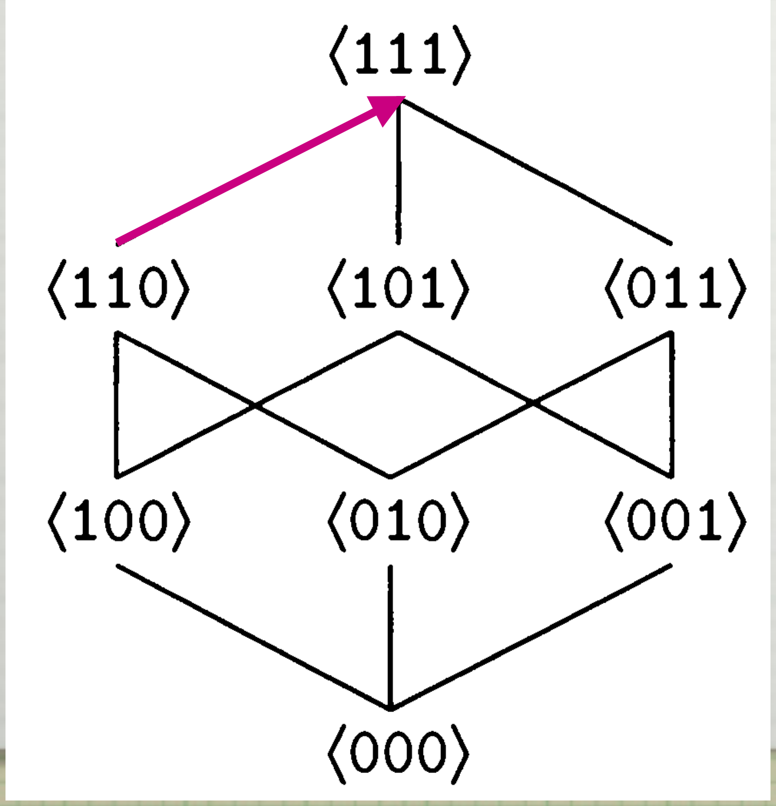

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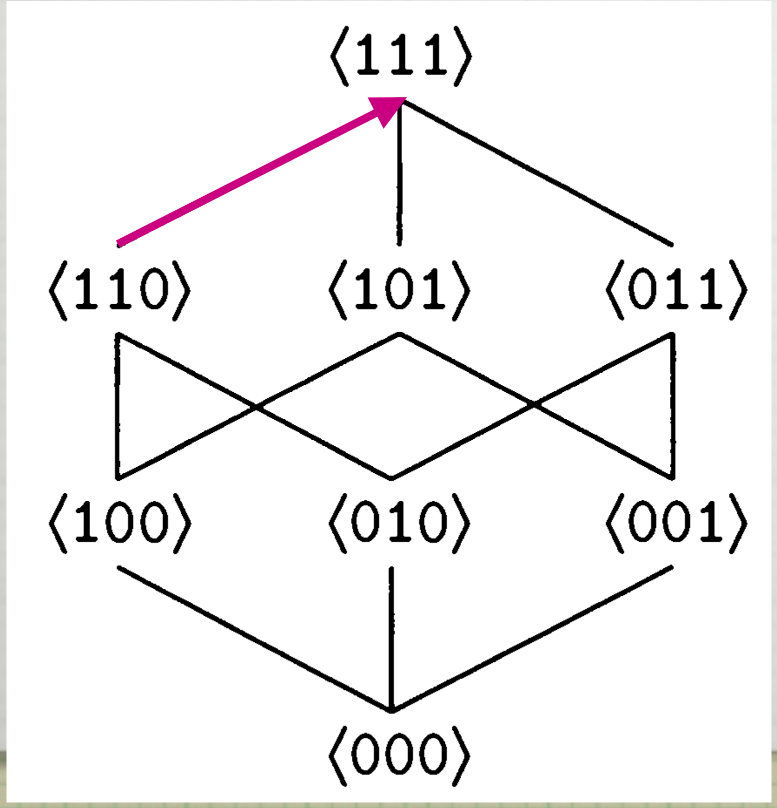

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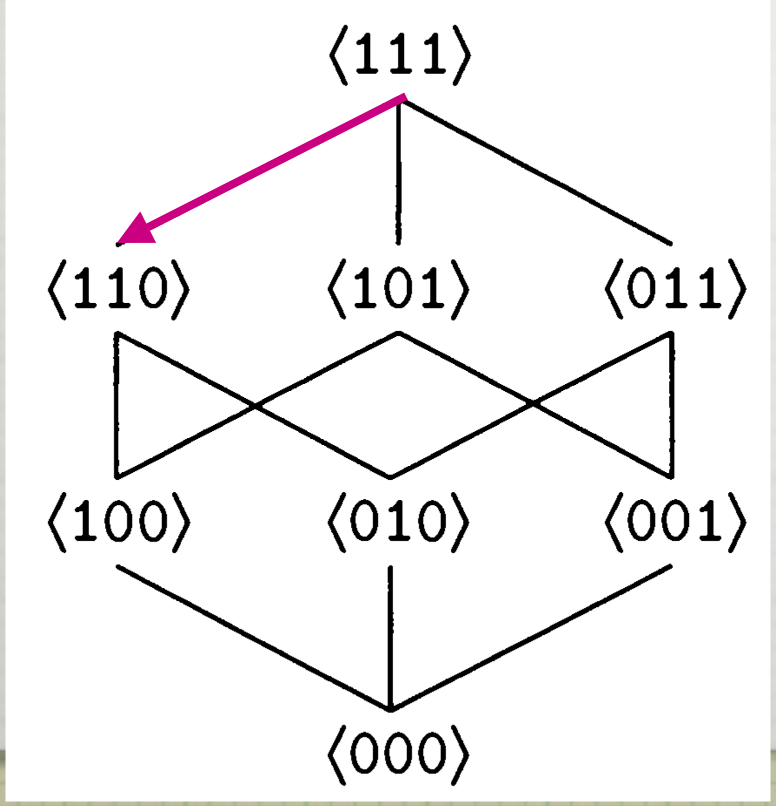

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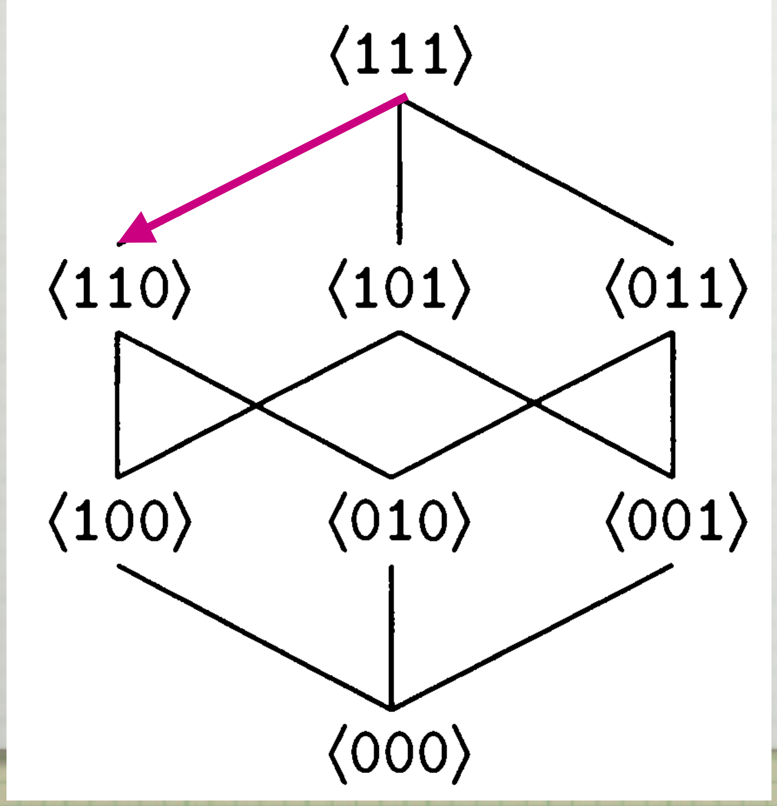

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Theorem 1

Given a semi-lattice and define a binary operation $\sqsubseteq$:

$$x \sqsubseteq y \text{ if and only if } x \sqcap y = x$$

$\sqsubseteq$ is provably a partial order relation.

(proof on the board)

Theorem 2

$\sqcap$ is provably the greatest lower bound defined based on $\sqsubseteq$.

(proof on the board)

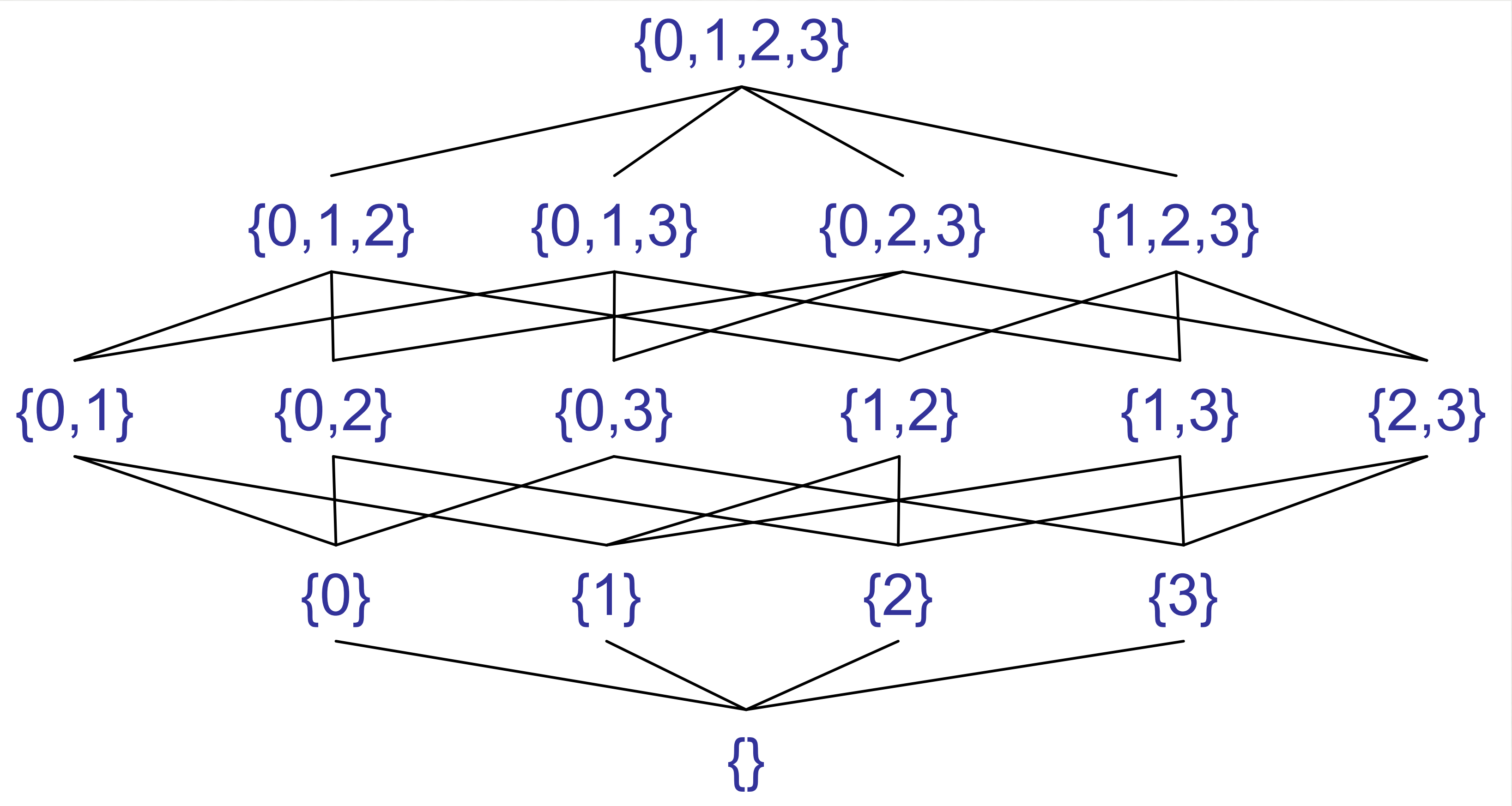
Theorem 3

Let $(S, \sqsubseteq)$ be a partially ordered set such that for all $x, y \in S$ the greatest lower bound of x and y is always defined (and in S). Prove $(S, \sqcap)$ to be a semi-lattice if:

$$x \sqcap y = glb(x, y)$$

(proof on the board)

Example: Subset SemiLattice



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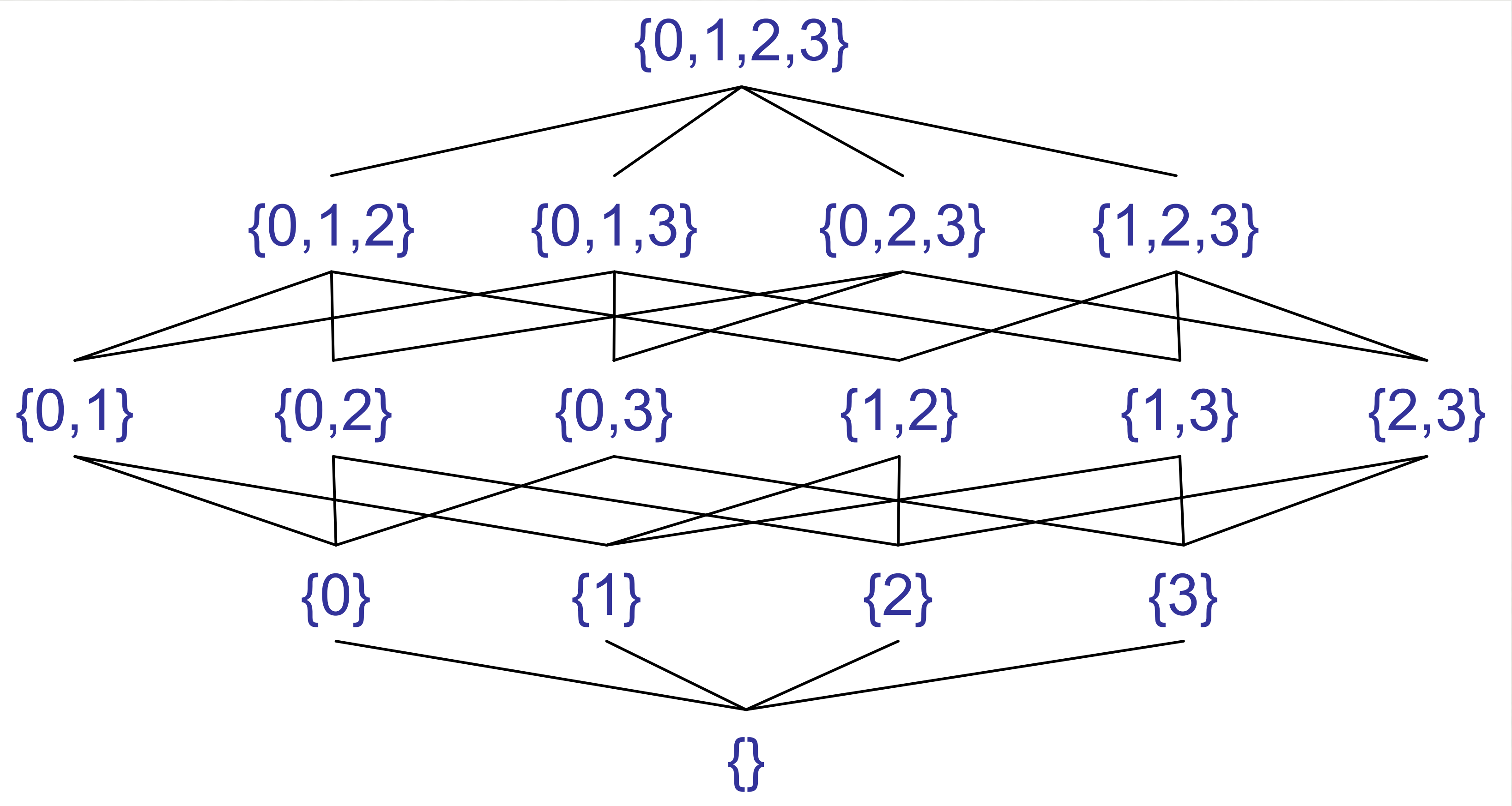

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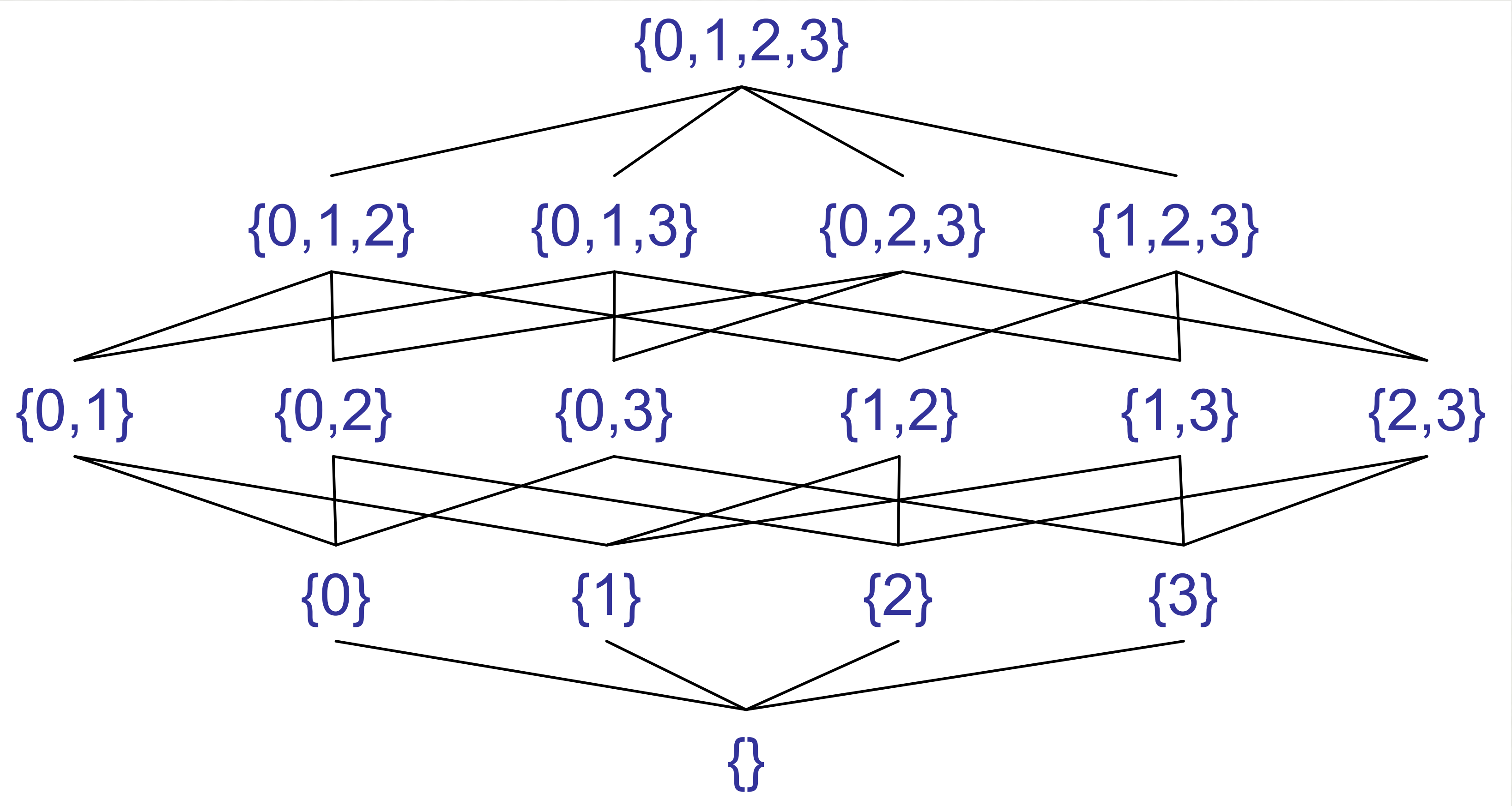

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$\sqsubseteq = \subseteq \qquad \top = S$

Recall Live Variable Analysis

Any solution $Live_{entry}$, $Live_{exit}$ to this system of equations overapproximates the true set of live variables.

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Our domain is a subset lattice where S is the set of all variables!

Descending Chains

A descending chain is a sequence of elements related by the order:

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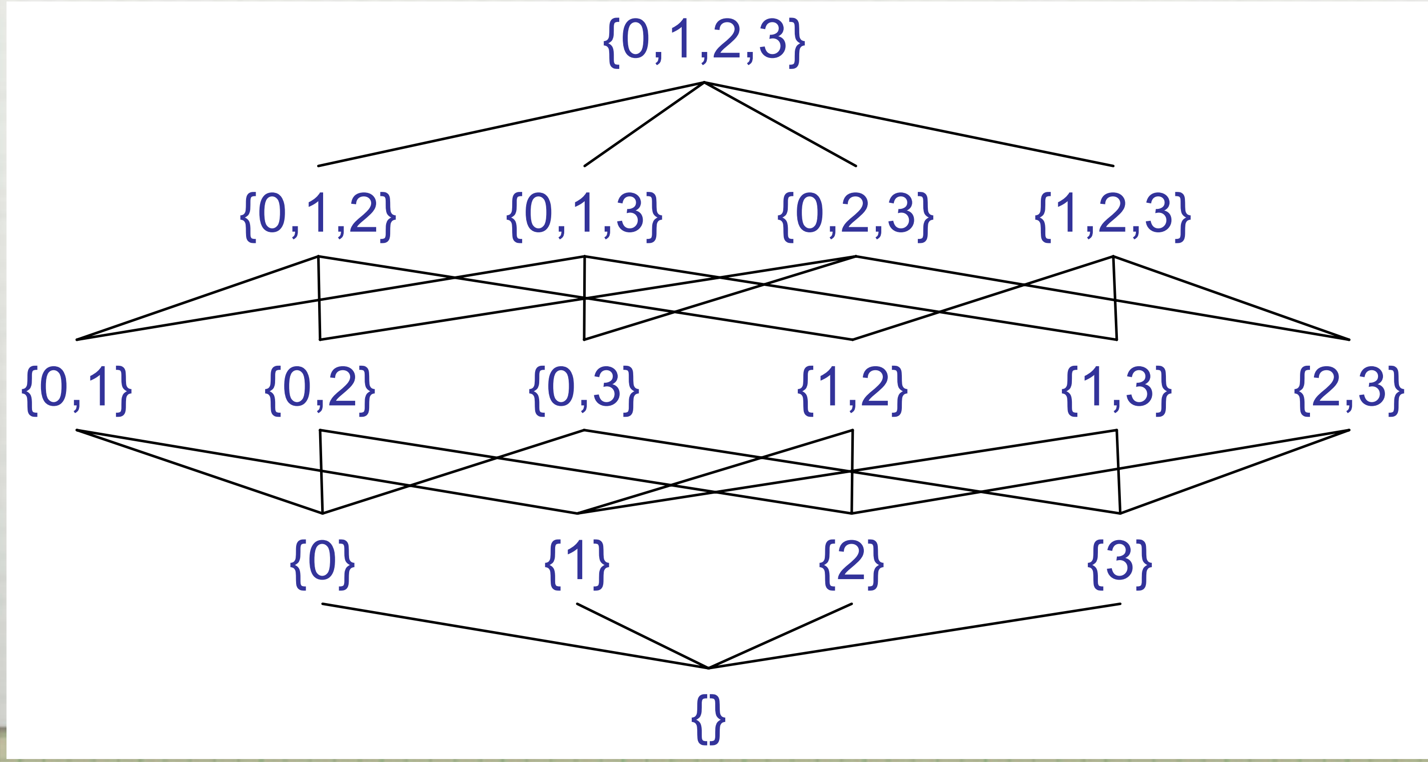

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Useful for Algorithmic convergence: a finite height!

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What is the height of this lattice?

The worklist algorithm terminates because this lattice has a finite height!

An Infinite Lattice with Finite Height

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> For each program point, whether or not a variable has a constant value whenever execution reaches that point.

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```
x := 2
y := 5
x := 1
z := 0
if (x <= 0) {
   z := x + 2
} else {
   z := y * y
}
x := z
```

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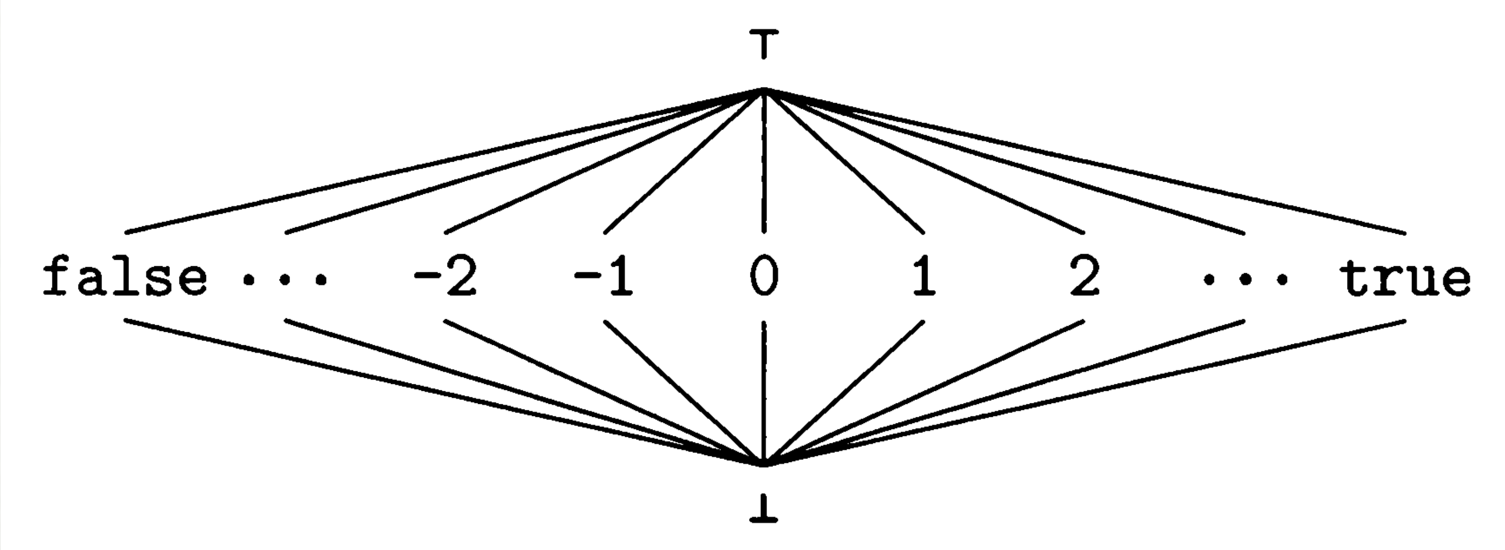

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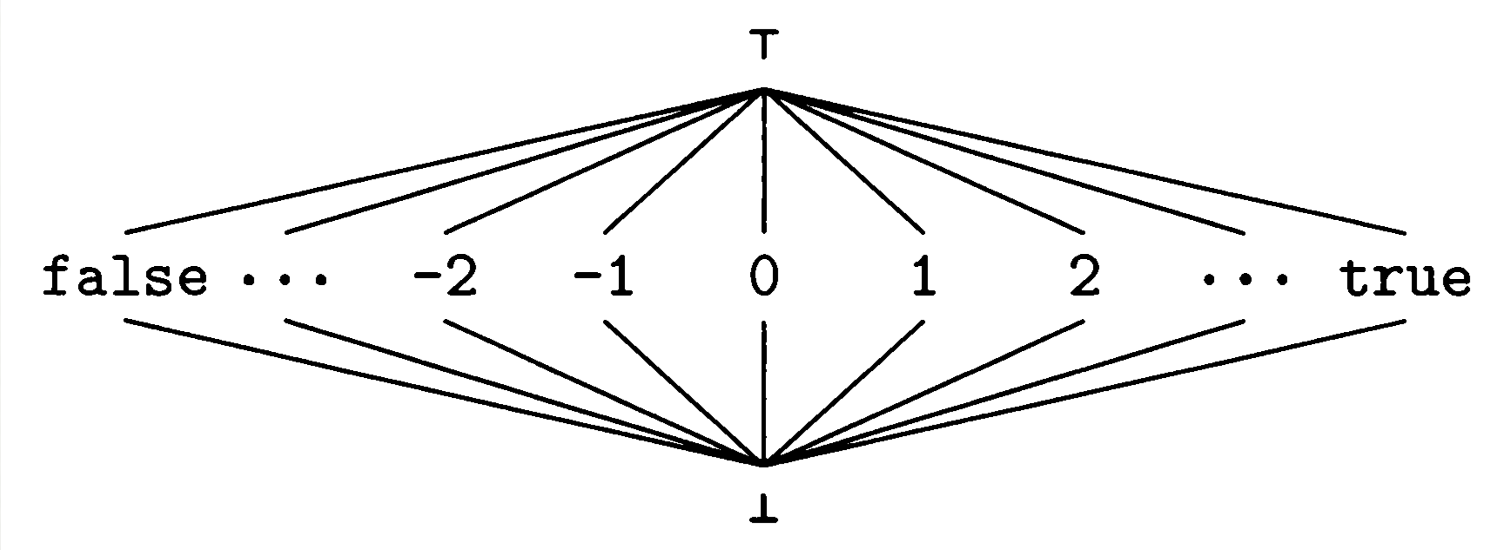

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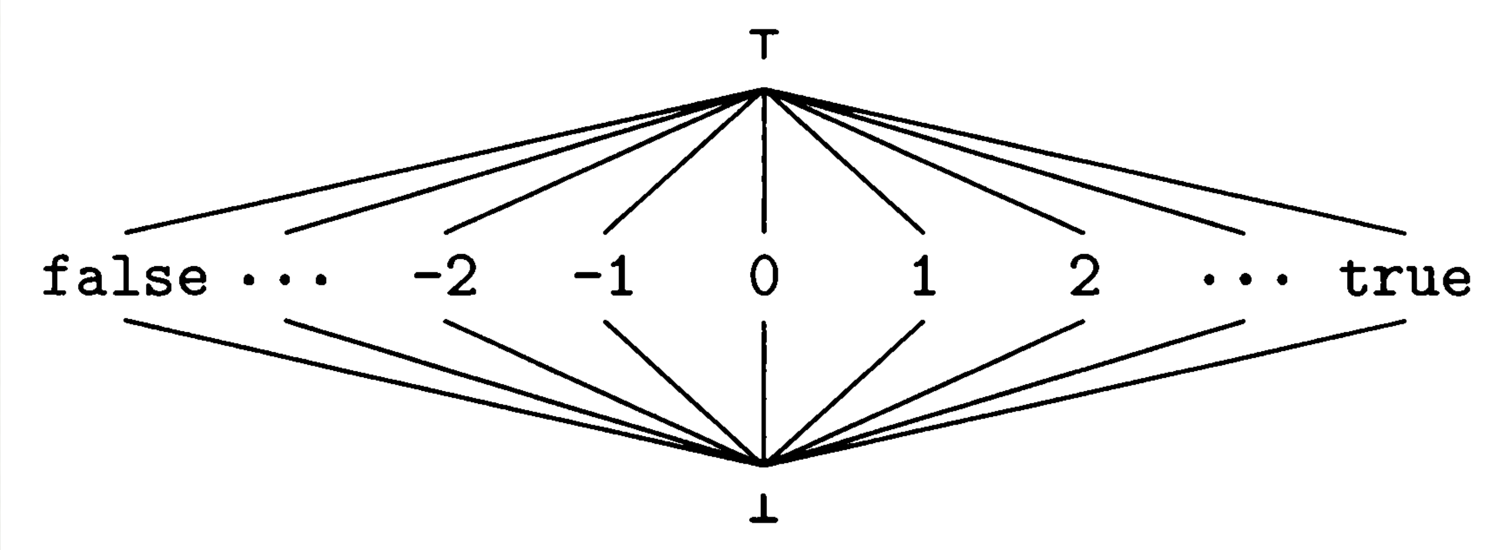

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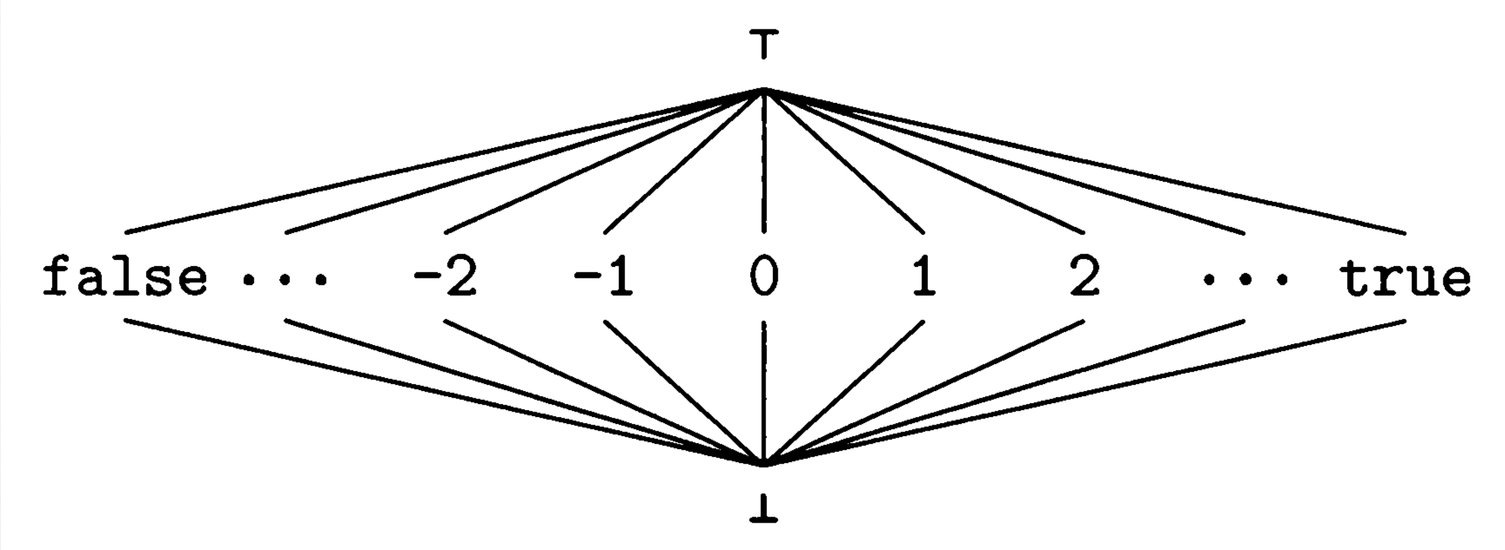

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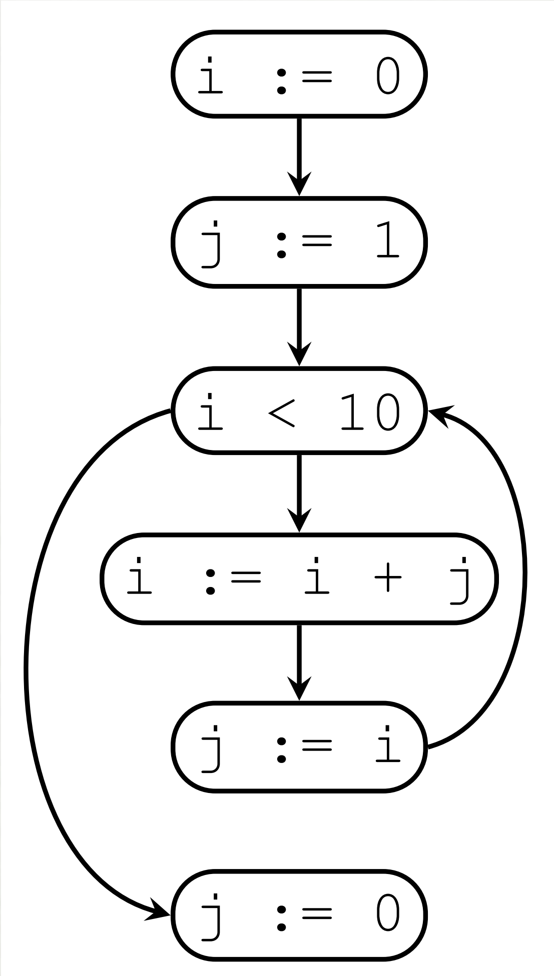
$$v_1 \sqcap v_2 = \begin{cases} v_1 & \text{if } v_1 = v_2 \\ \top & \text{if } v_1 \neq v_2 \end{cases}$$

A General Framework for Dataflow Analyses based on Basic Lattice Theory

Component 1:
Domains are (semi)-lattices of finite height!

Example: Live Variable Analysis

The domain is a (complete) subset lattice $(\mathcal{P}(S), \cup, \emptyset, S)$ where S is the set of all program variables.



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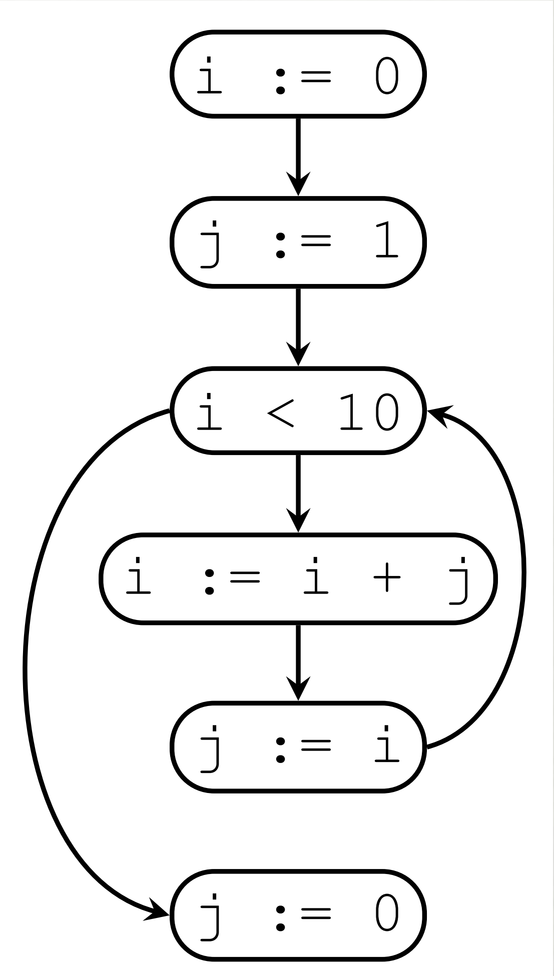

Why $\cup$? Because a variable is live if it is live on some path from the node.

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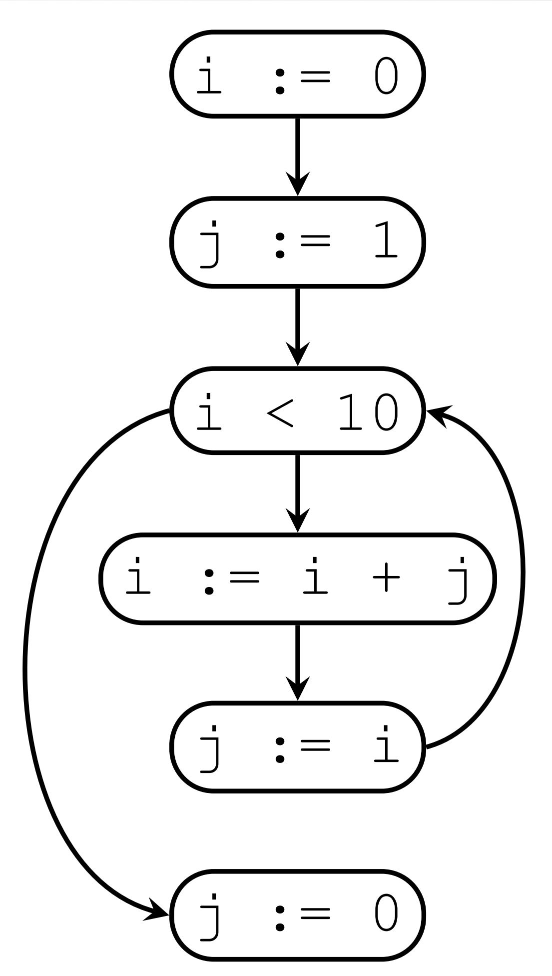

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By choosing to call variables live at exit, we also decide that this is a backward dataflow problem.

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- Define the (semi) lattice: dataflow facts and how to combine them!

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To fully define the domain:

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- Decide on the direction of the analysis: forward or backward!

Component 2:
Transfer Functions

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A *transfer function* models, for a particular data flow analysis problem, the effect of the programming language constructs as a mapping from the lattice (used in the analysis) to itself).

$$\forall st \in \text{Statements},\ f_{st} : \mathbf{L} \mapsto \mathbf{L}$$

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Example:

$$\begin{aligned} LV_{entry}(\ell) &= LV_{exit}(\ell) \setminus write(\ell) \cup read(\ell) \\ LV_{exit}(\ell) &= \bigcup_{\ell \to \ell' \in E} LV_{entry}(\ell') \end{aligned}$$

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Example:

backward: would reverse for forward!

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Properties of Transfer Functions

Monotonicity: $\forall x, y \in \mathbf{L} : x \sqsubseteq y \implies f(x) \sqsubseteq f(y).$

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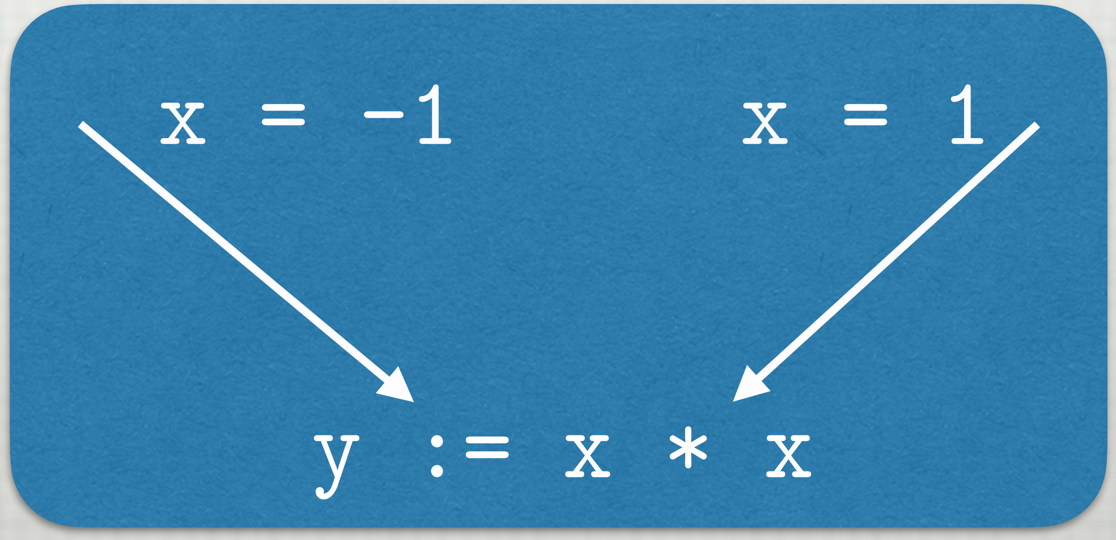

Component 3:
The Computation

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meet-over-all-paths (MOP) solutions.

- Start from the beginning (entry node, or exist note for backward flow problems) with some initial information.

- Walk down a path and apply transfer functions along these paths to each node in the flow graph.

- For each node, compute the *meet* of all paths to this point.

Formally

For a path $\pi = init \ldots l$

$$f_\pi = f_{init} \circ \cdots \circ f_l$$

$$MOP_\circ(l) = \textstyle\bigsqcap_{\pi \in Path(l)} f_\pi(\iota).$$

$$MOP_\bullet(l) = f_l(MOP_\circ(l)).$$

Formally

For a path $\pi = init \ldots l$

Transfer Function for location l

$$f_\pi = f_{init} \circ \cdots \circ f_l$$

Set of all paths to l

Initial information at "init"

$$MOP_\circ(l) = \bigsqcap_{\pi \in Path(l)} f_\pi(\iota).$$

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Can this solution be computed effectively?

Bad News!

For an arbitrary data flow analysis problem where transfer functions are only monotone, one can show that there may be no algorithm to compute the MOP solution.

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For an arbitrary data flow analysis problem where transfer functions are only monotone, one can show that there may be no algorithm to compute the MOP solution.

Lemma

The MOP solution for Constant Propagation is undecidable.

Proof: Let $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ be strings over the alphabet $\{1, \cdots, 9\}$; let $| u |$ denote the length of u; let $[\![u]\!]$ be the natural number denoted.

The Modified Post Correspondence Problem is to determine whether or not $u_{i_1} \cdots u_{i_m} = v_{i_1} \cdots v_{i_n}$ for some sequence $i_1, \cdots, i_m$ with $i_1 = 1$.

x:=$[\![u_1]\!]$; y:=$[\![v_1]\!]$;
while [$\cdots$] do
  (if [$\cdots$] then x:=x * $10^{|u_1|}$ + $[\![u_1]\!]$; y:=y * $10^{|v_1|}$ + $[\![v_1]\!]$ else
   ⋮
   if [$\cdots$] then x:=x * $10^{|u_n|}$ + $[\![u_n]\!]$; y:=y * $10^{|v_n|}$ + $[\![v_n]\!]$ else skip)
[z:=abs((x-y)*(x-y))]$^\ell$

Then $MOP_\bullet(\ell)$ will map z to 1 if and only if the Modified Post Correspondence Problem has no solution. This is undecidable.

So, what do we do?

Good News

Instead, compute the *maximal fixed point solution* (MFP).

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Consider the set of constraints below:

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What is a solution to these set of constraints?

A solution is a fixed point!

Is it unique?

Algebra brings it all together!

Fixed Point Solutions

Theorem: (Knaster-Tarski Fixpoint Theorem)

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Let $\mathbf{L}$ be a complete lattice and $F : \mathbf{L} \to \mathbf{L}$ be a monotone function. Then, the set of fixpoints of F in $\mathbf{L}$ is also a complete lattice.

Fixed Point Solutions

Theorem: (Knaster-Tarski Fixpoint Theorem)

Let $\mathbf{L}$ be a complete lattice and $F : \mathbf{L} \to \mathbf{L}$ be a monotone function. Then, the set of fixpoints of F in $\mathbf{L}$ is also a complete lattice.

Corollary: We have a set of solutions (fixed points), with a guarantee for the existence of a maximal (also minimal) solution.

How do we compute it?

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DFA Algorithm

$\forall k \in N \,.\, \text{IN}_k = \text{OUT}_k = \top$
repeat
  foreach $k \in N$ **do** {
    $\text{IN}_k = \sqcap\{\text{OUT}_p \mid p \in pred(k)\}$
    $\text{OUT}_k = F_k(\text{IN}_k)$
  }
while solution changes

The Coincidence

If transfer functions are monotone:

$$MOP_\circ(l) \sqsupseteq MFP_\circ(l) \qquad MOP_\bullet(l) \sqsupseteq MFP_\bullet(l)$$

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The two solutions coincide!

Let’s make another instance of our framework!

Very Busy Expressions

```
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An expression is very busy at the exit from a label if, no matter what path is taken from the label, the expression is always used before any of the variables occurring in it are redefined.

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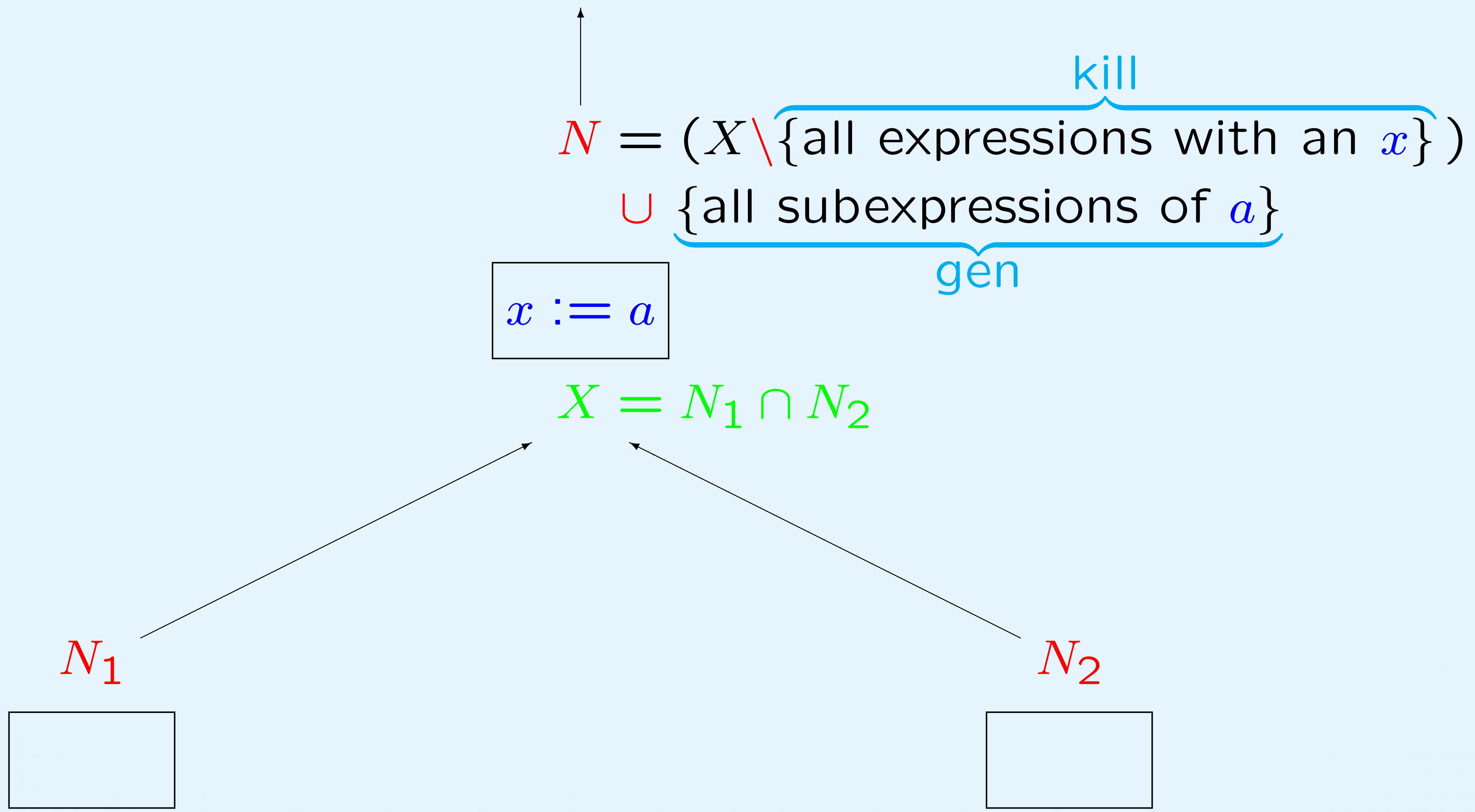
    - Sanity check: Monotonicity!

The Design Process



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unlike live variables: here we want the greatest fixed point!

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