CSC410
Data Flow Analyses

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Theorem 3

Let $(S, \sqsubseteq)$ be a partially ordered set such that for all $x, y \in S$ the greatest lower bound of x and y is always defined (and in S). Prove $(S, \sqcap)$ to be a semi-lattice if:

$$x \sqcap y = glb(x, y)$$

$$x \sqcap y = glb(x, y) \iff (x \sqcap y \sqsubseteq x \wedge x \sqcap y \sqsubseteq y \wedge \forall z : z \sqsubseteq x \wedge z \sqsubseteq y \implies z \sqsubseteq x \sqcap y)$$

(proof on the board)

Recall Live Variable Analysis

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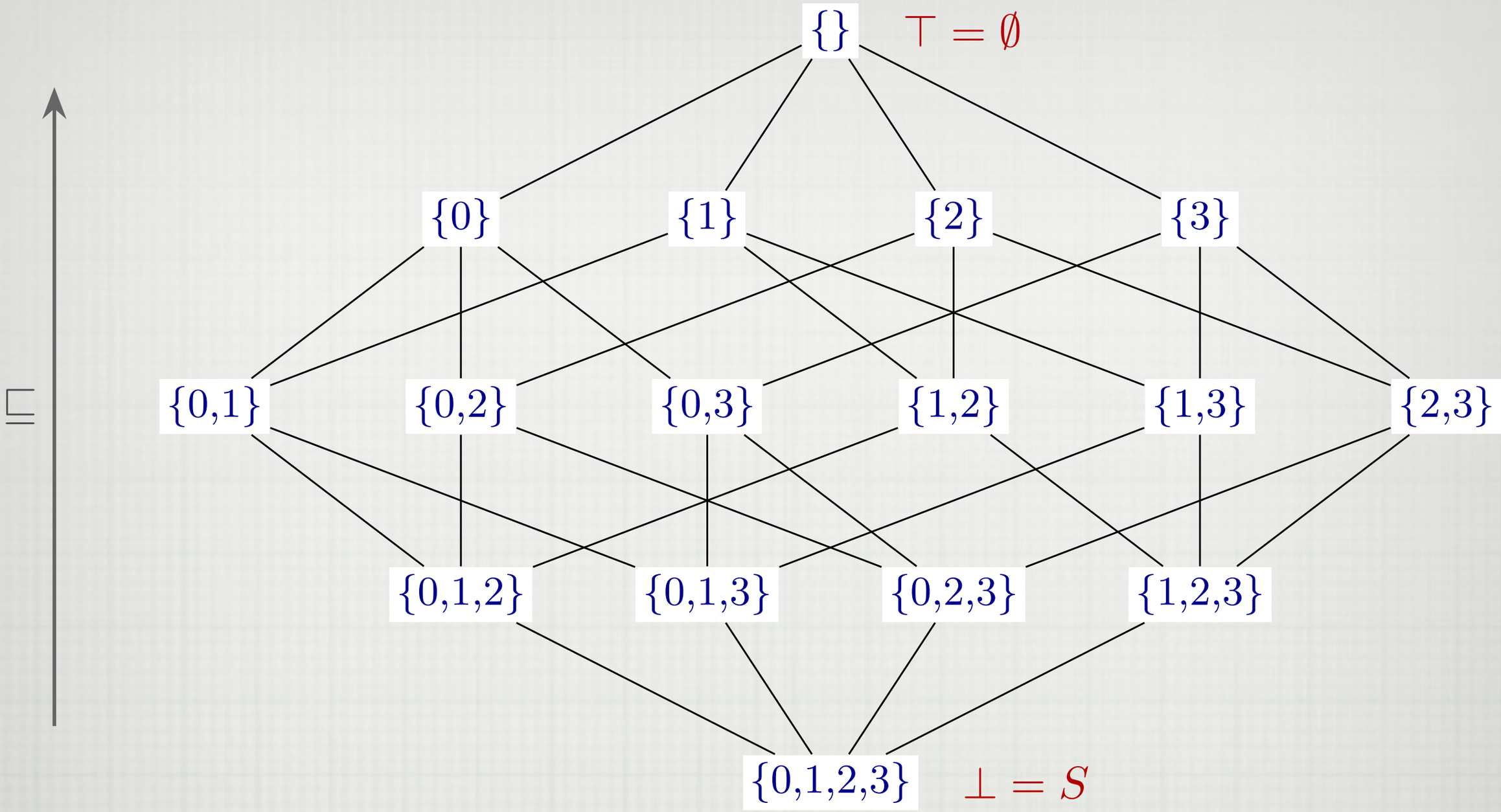
$$x \text{ is live at } l \Rightarrow x \in LVe_{exit}(l)$$

Recall Live Variable Analysis

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$$x \text{ is live at } l \Rightarrow x \in LVe_{exit}(l)$$

Our domain is a subset lattice where S is the set of all variables!


{}
⊤ = ∅
{0}
{1}
{2}
{3}
{0,1}
{0,2}
{0,3}
{1,2}
{1,3}
{2,3}
{0,1,2}
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{1,2,3}
{0,1,2,3}
⊥ = S
⊑
⊑=⊒
⊓ = ∪
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Descending Chains

A descending chain is a sequence of elements related by the order:

$$x_1 \sqsupseteq x_2 \sqsupseteq \cdots \sqsupseteq x_n$$

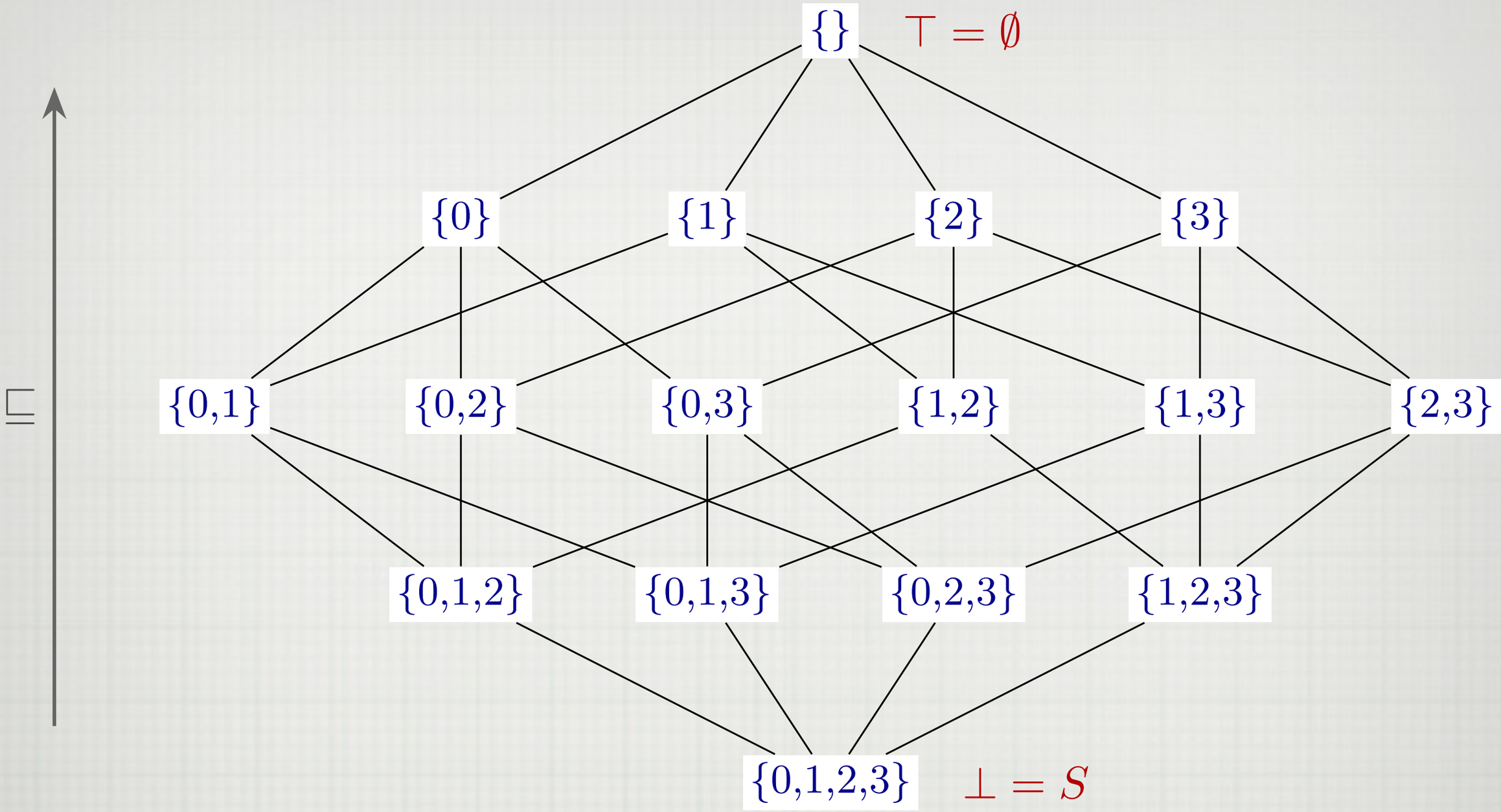
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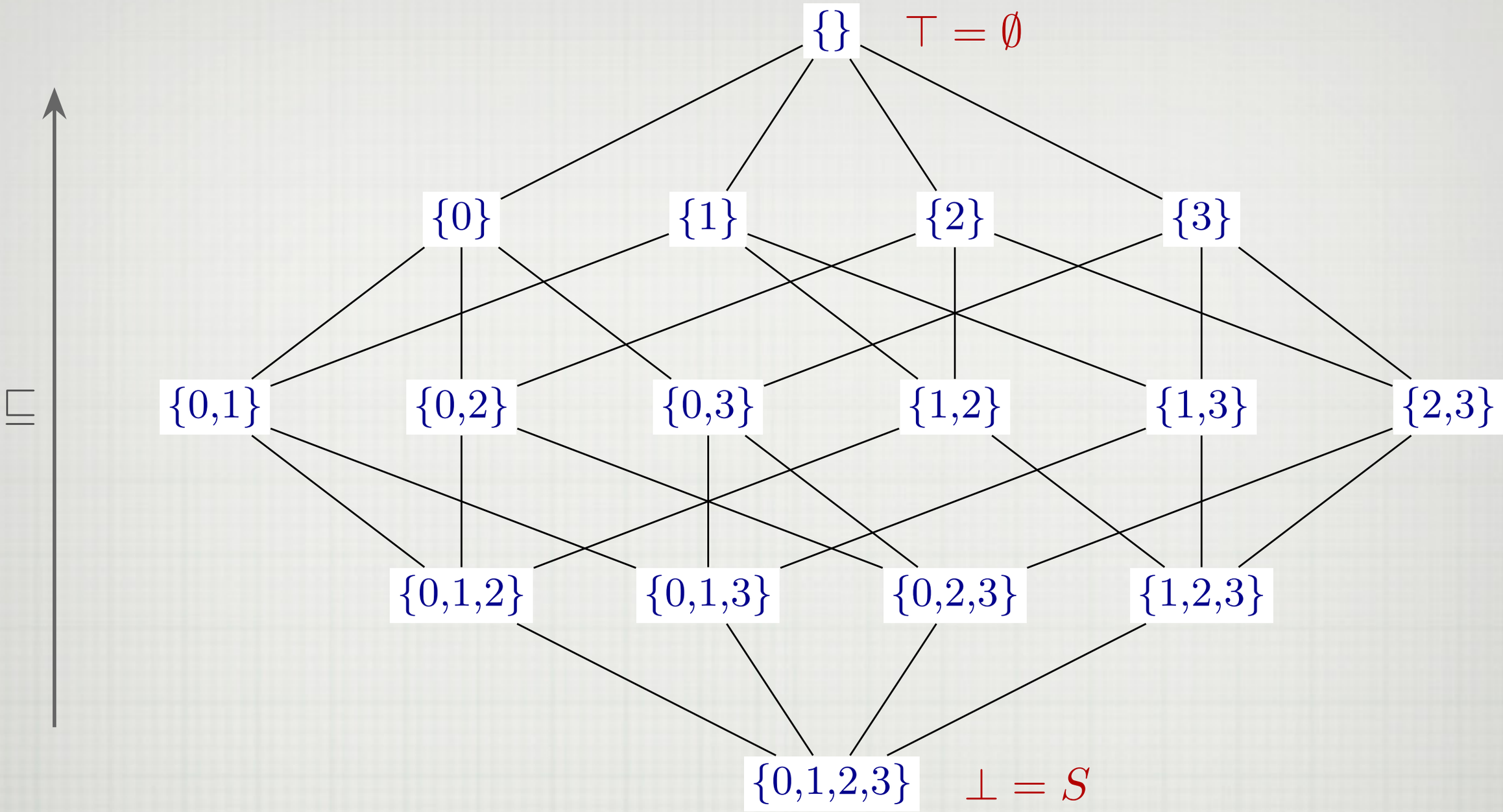
A descending chain is a sequence of elements related by the order:

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The height of a lattice (or semi-lattice) is the length of its longest descending chain.

Useful for Algorithmic convergence: a finite height!


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The worklist algorithm terminates because this lattice has a finite height!

Finite height means finite lattice?

An Infinite Lattice with Finite Height

The aim of the *Constant Propagation Analysis* is to determine

For each program point, whether or not a variable has a constant value whenever execution reaches that point.

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```
x := 2
y := 5
x := 1
z := 0
if (x <= 0) {
    z := x + 2
} else {
    z := y * y
}
x := z
```

An Infinite Lattice

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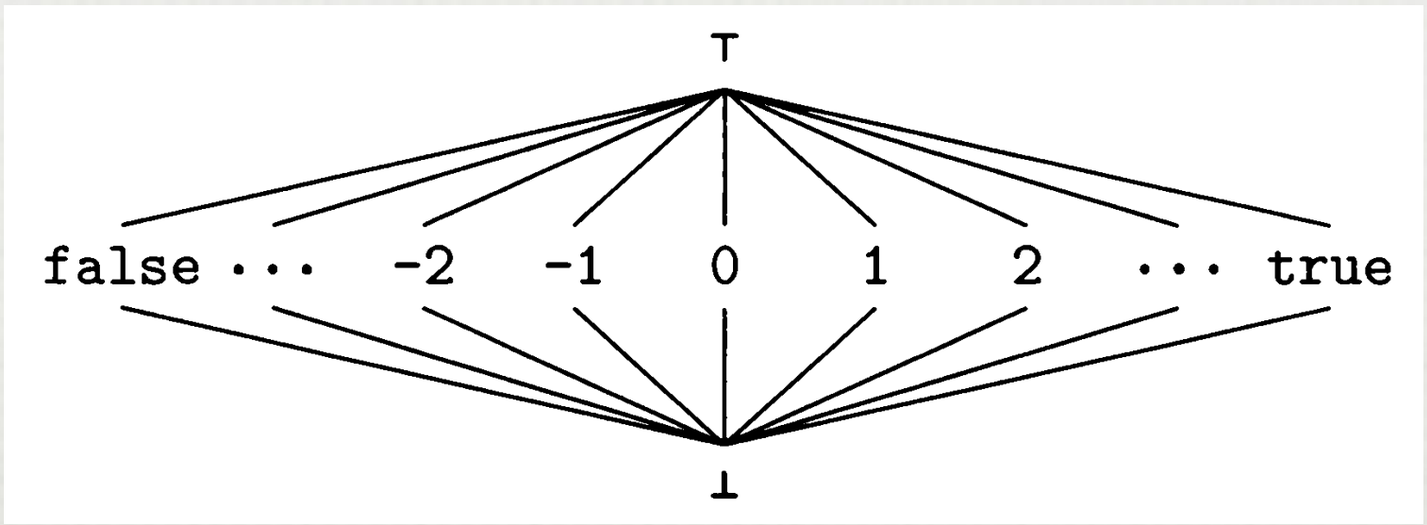
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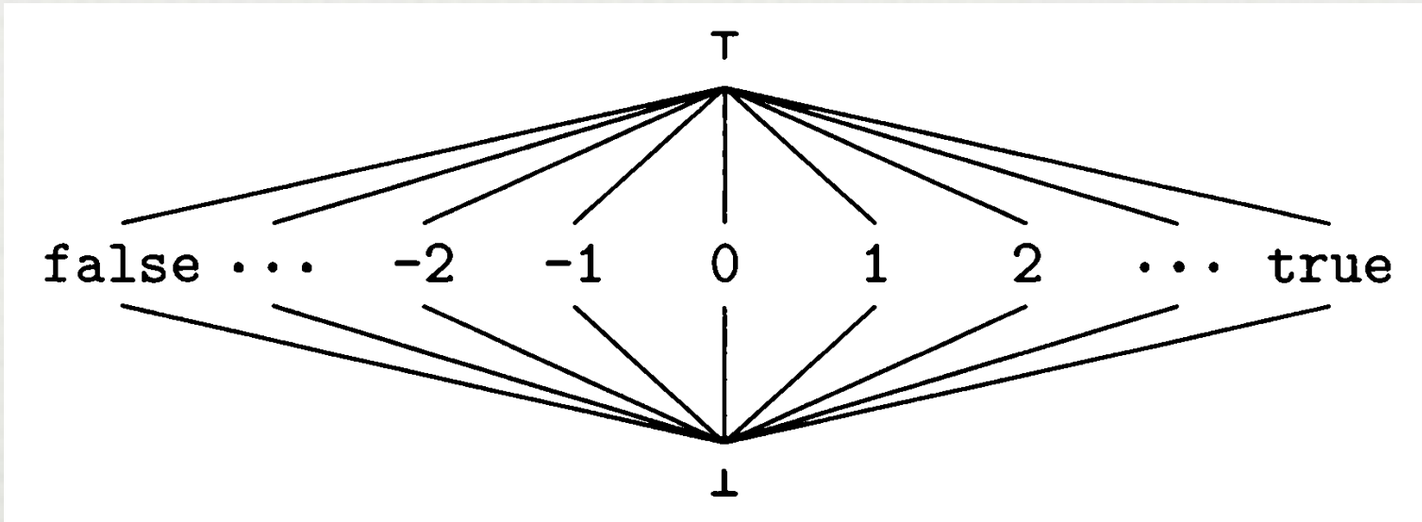


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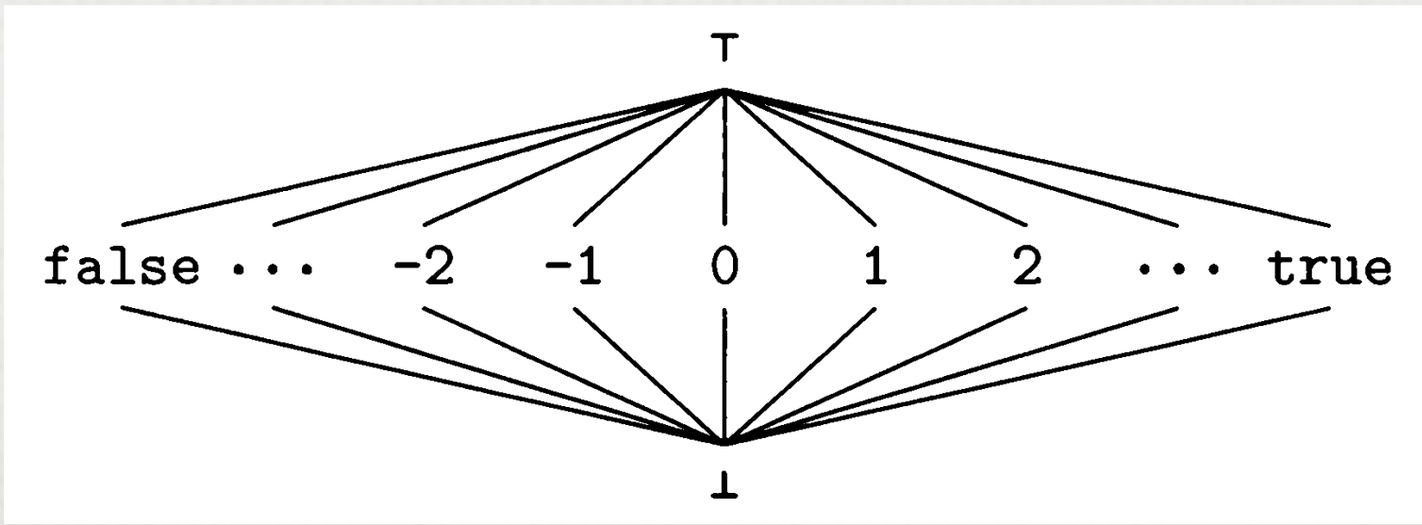
$\top$: non-constant

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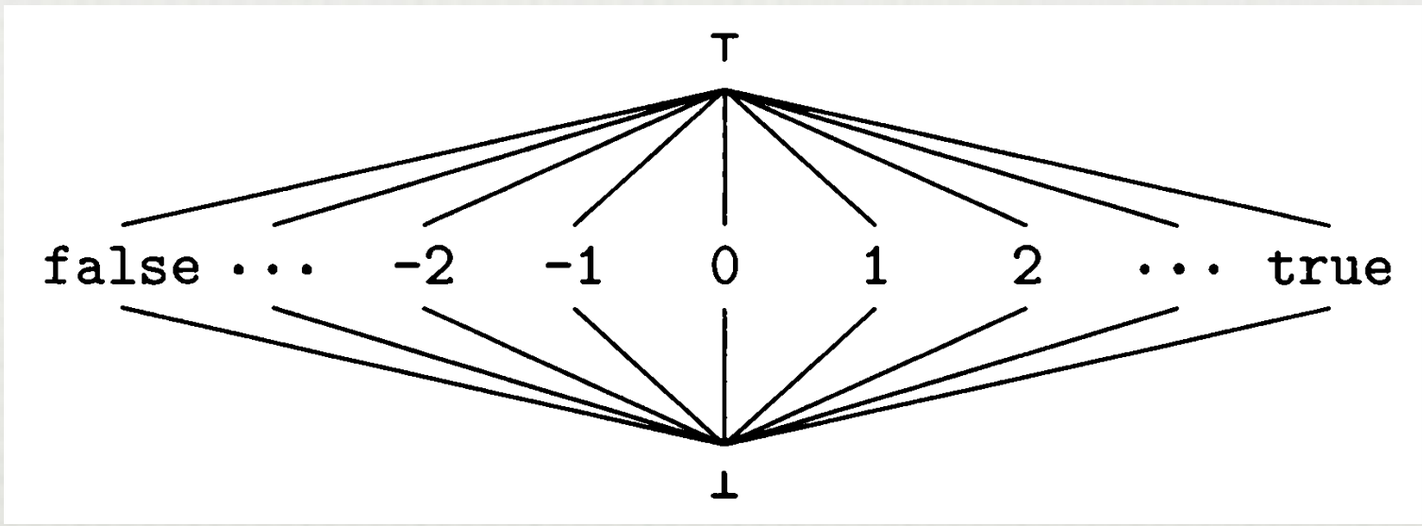
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$\top$: non-constant

$\bot$: no-information

$$v_1 \sqcap v_2 = \begin{cases} v_1 & \text{if } v_1 = v_2 \vee v_2 = \bot \\ \top & \text{otherwise} \end{cases}$$

A Dual Lattice Over The Same Structure

Semi-Lattices

A (join) *semi-lattice* $\mathbf{L} = (S, \sqcup)$ is a set S with a binary operation, called *join* ($\sqcup$), that has the following properties:

(1) For all $x, y \in S$, there exist a unique $z \in S$ such that $x \sqcup y = z$ (CLOSURE).

(2) For all $x, y, z \in S$, we have

$$x \sqcup x = x \quad \text{(idempotence)}$$

$$x \sqcup y = y \sqcup x \quad \text{(commutativity)}$$

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z \quad \text{(associativity)}$$

Complete Join Semi-Lattice

The unit for $\sqcup$ is $\bot$:

$$\forall x: \; x \sqcup \bot = \bot \sqcup x = x$$

The join semi-lattice is called complete if $\bot \in \mathbb{L}$

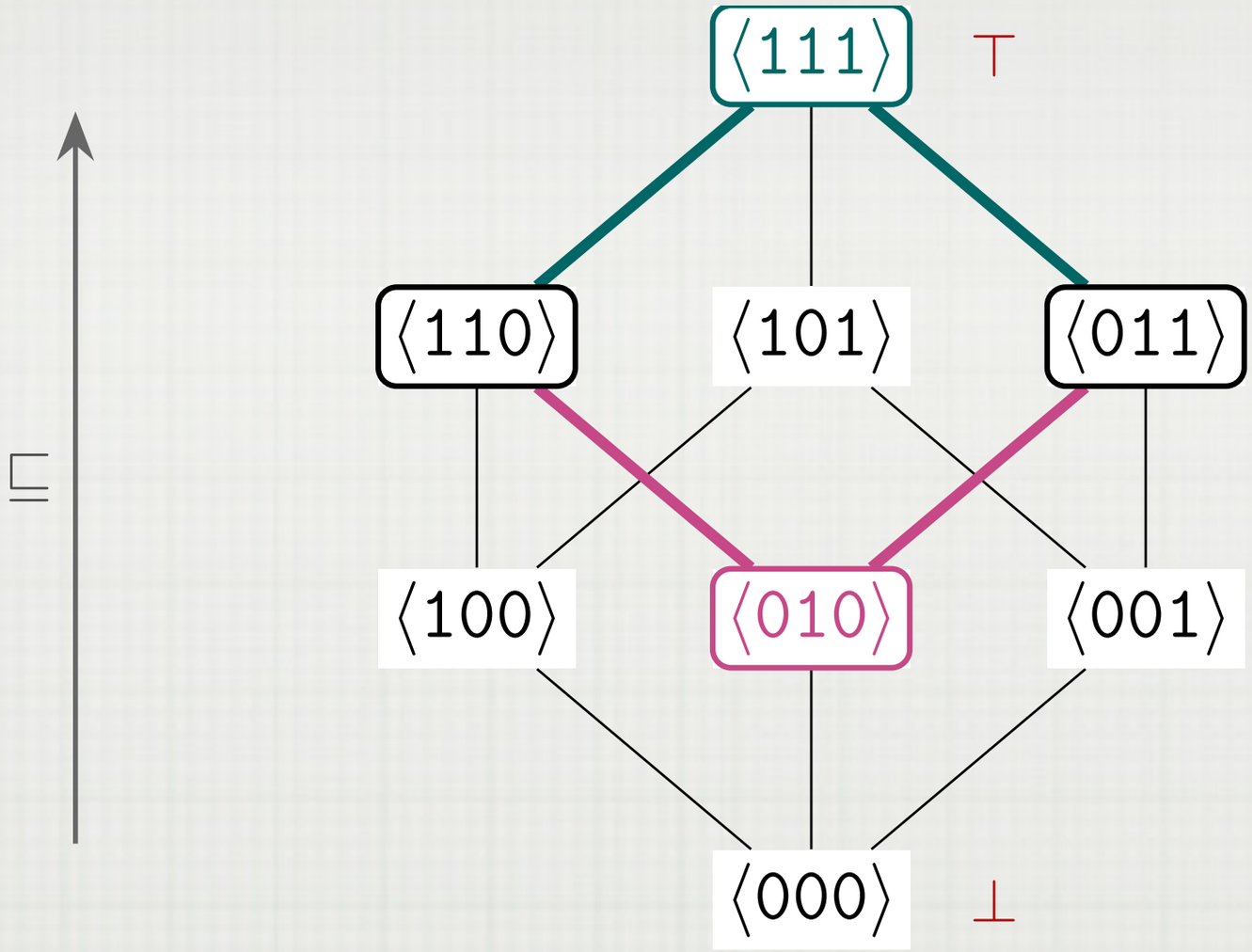
Defining The Order Based on Join

Given a semi-lattice and define a binary operation $\sqsubseteq$:

$$x \sqsubseteq y \text{ if and only if } x \sqcup y = y$$

$\sqsubseteq$ is provably a partial order relation.

$\sqcup$ is provably the least upper bound defined based on $\sqsubseteq$.



$$\sqcap = \wedge \qquad \sqcup = \vee$$

$$\langle 110 \rangle \sqcap \langle 011 \rangle = \langle 110 \rangle \wedge \langle 011 \rangle = \langle 010 \rangle$$

$$\langle 110 \rangle \sqcup \langle 011 \rangle = \langle 110 \rangle \vee \langle 011 \rangle = \langle 111 \rangle$$

Lattices

A *lattice* $\mathbf{L} = (S, \sqcap, \sqcup)$ is a set S with two binary operation, a *meet* ($\sqcap$) and a *join* ($\sqcup$), such that:

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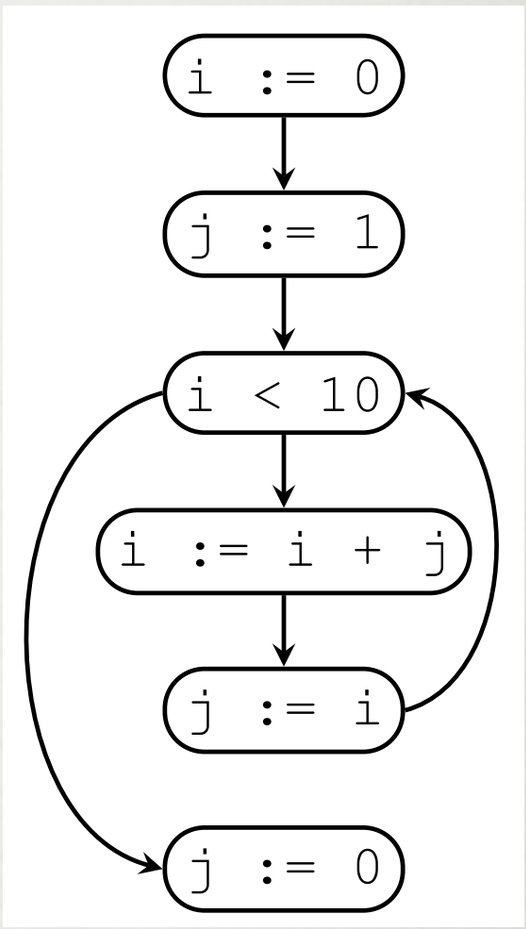
If both $\top$ and $\bot$ exist then the lattice is called complete.

A General Framework for Dataflow Analyses based on Basic Lattice Theory

Component 1:
Domains are (semi)-lattices of finite height!

Example: Live Variable Analysis

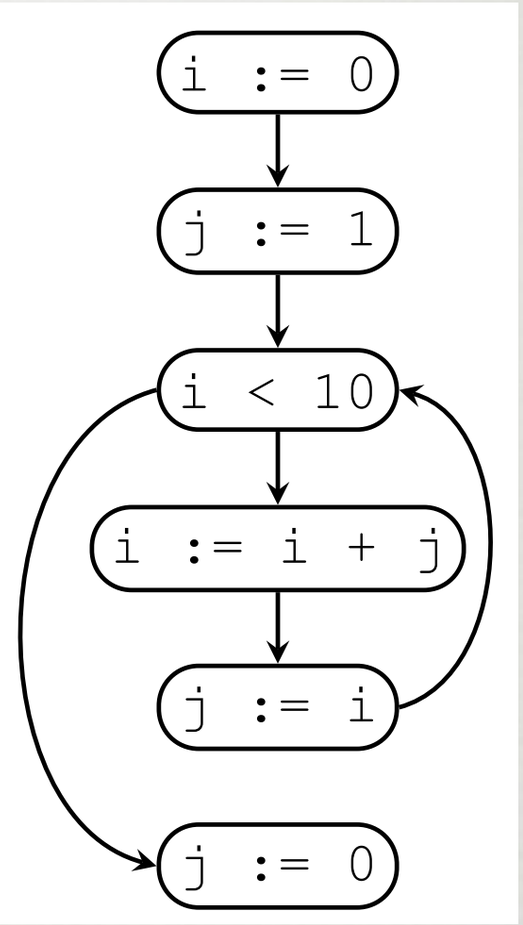
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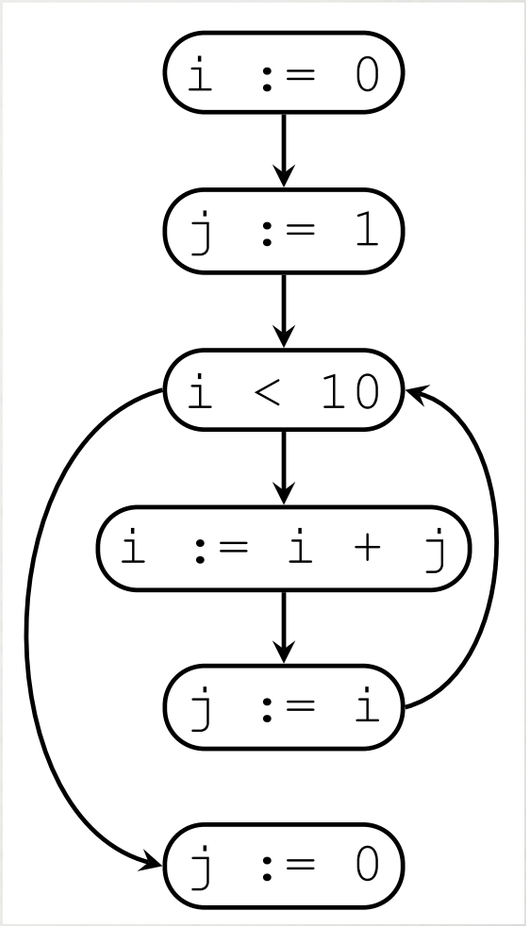
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Since the witness for variable being live is in the future, this is a backward dataflow problem.

General Construction Rule

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- Define the (semi) lattice: dataflow facts and how to combine them!

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- Decide on the direction of the analysis: forward or backward!

Component 2:
Transfer Functions

Transfer Functions

A *transfer function* models, for a particular data flow analysis problem, the effect of the programming language constructs as a mapping from the lattice (used in the analysis) to itself).

$$\forall st \in \text{Statements},\ f_{st} :\ \mathbf{L} \mapsto \mathbf{L}$$

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backward: would reverse for forward!

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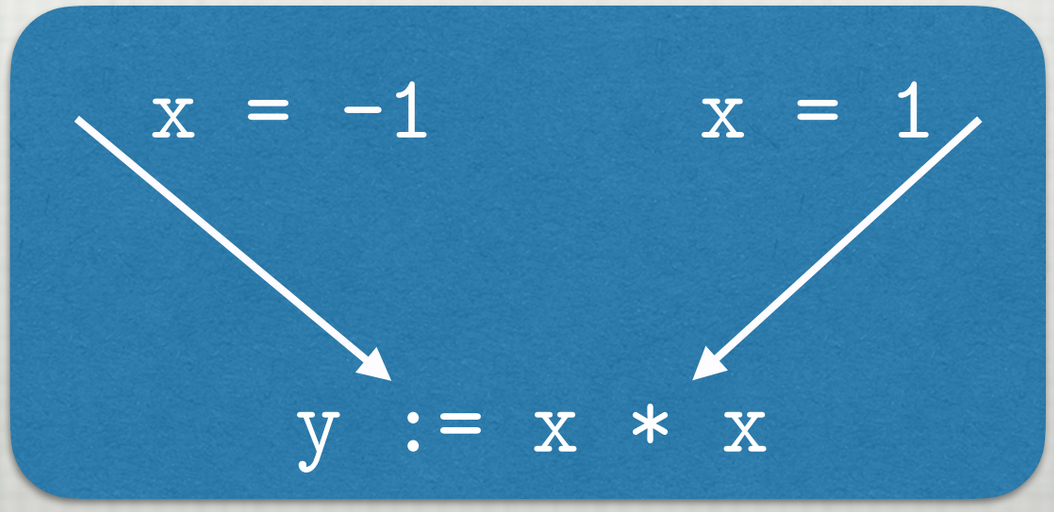
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Constant Propagation:



Component 3:
The Computation

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meet-over-all-paths (MOP) solutions.

- Start from the beginning (entry node, or exist note for backward flow problems) with some initial information.

- Walk down a path and apply transfer functions along these paths to each node in the flow graph.

- For each node, compute the *meet* of all paths to this point.

Formally

For a path $\pi = init \ldots \ell' \ \ell$

$$f_\pi = f_{init} \circ \cdots \circ f_{\ell'}$$

$$MOP_\circ(\ell) = \bigsqcap_{\pi \in Path(\ell)} f_\pi(\iota)$$

$$MOP_\bullet(\ell) = f_\ell(MOP_\circ(\ell))$$

Formally

For a path $\pi = init \ldots \ell' \ \ell$

Transfer Function for location l'

$$f_\pi = f_{init} \circ \cdots \circ f_{\ell'}$$

Set of all paths to l

Initial information at "init"

$$MOP_\circ(\ell) = \bigsqcap_{\pi \in Path(\ell)} f_\pi(\iota)$$

$$MOP_\bullet(\ell) = f_\ell(MOP_\circ(\ell))$$

Can this solution be computed effectively?

Bad News!

For an arbitrary data flow analysis problem where transfer functions are only monotone, one can show that there may be no algorithm to compute the MOP solution.

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Lemma

The MOP solution for Constant Propagation is undecidable.

Proof: Let $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ be strings over the alphabet $\{1,\cdots,9\}$; let $\mid u \mid$ denote the length of u; let $[\![u]\!]$ be the natural number denoted.

The Modified Post Correspondence Problem is to determine whether or not $u_{i_1} \cdots u_{i_m} = v_{i_1} \cdots v_{i_n}$ for some sequence $i_1, \cdots, i_m$ with $i_1 = 1$.

```
x:=[[u_1]]; y:=[[v_1]];
while [···] do
   (if [···] then x:=x * 10^|u_1| + [[u_1]]; y:=y * 10^|v_1| + [[v_1]] else
    ⋮
    if [···] then x:=x * 10^|u_n| + [[u_n]]; y:=y * 10^|v_n| + [[v_n]] else skip)
[z:=abs((x-y)*(x-y))]^ℓ
```

Then $MOP_\bullet(\ell)$ will map `z` to 1 if and only if the Modified Post Correspondence Problem has no solution. This is undecidable.

So, what do we do?

Good News

Instead, compute the *maximal fixed point solution* (MFP).

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Is it unique?

Algebra brings it all together!

Fixed Point Solutions

Theorem: (Knaster-Tarski Fixpoint Theorem)

(proof on the board)

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Let $\mathbf{L}$ be a complete lattice and $F : \mathbf{L} \to \mathbf{L}$ be a monotone function. Then, the set of fixpoints of F in $\mathbf{L}$ is also a complete lattice.

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Corollary: We have a set of solutions (fixed points), with a guarantee for the existence of a maximal (also minimal) solution.

How do we compute it?

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DFA Algorithm

$\forall k \in N \,.\, \mathrm{IN}_k = \mathrm{OUT}_k = \top$
repeat
  foreach $k \in N$ **do** {
    $\mathrm{IN}_k = \sqcap \{ \mathrm{OUT}_p \mid p \in \mathit{pred}(k) \}$
    $\mathrm{OUT}_k = F_k(\mathrm{IN}_k)$
  }
while solution changes

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If transfer functions are distributive:

$$MOP_{\circ}(l) = MFP_{\circ}(l) \qquad MOP_{\bullet}(l) = MFP_{\bullet}(l)$$

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Less Precise!

The fixpoint solution over-approximates the result!

If transfer functions are distributive:

$$MOP_\circ(l) = MFP_\circ(l) \qquad MOP_\bullet(l) = MFP_\bullet(l)$$

The two solutions coincide!

Proof of their relation

If transfer functions are monotone:

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(exercise for you: prove the two definitions are equivalent)

Let's make another instance of our framework!

Very Busy Expressions

$$\texttt{if } [a>b]^1 \texttt{ then } ([x:=b-a]^2;[y:=a-b]^3) \texttt{ else } ([y:=b-a]^4;[x:=a-b]^5)$$

Very Busy Expressions

$$\texttt{if [a>b]}^{1}\ \texttt{then ([x:= b-a]}^{2}\texttt{;[y:= a-b]}^{3}\texttt{) else ([y:= b-a]}^{4}\texttt{;[x:= a-b]}^{5}\texttt{)}$$



$$\texttt{[t1:= b-a]}^{A}\texttt{;[t2:= b-a]}^{B}\texttt{;}$$
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An expression is very busy at the exit from a label if, no matter what path is taken from the label, the expression is always used before any of the variables occurring in it are redefined.

Check List

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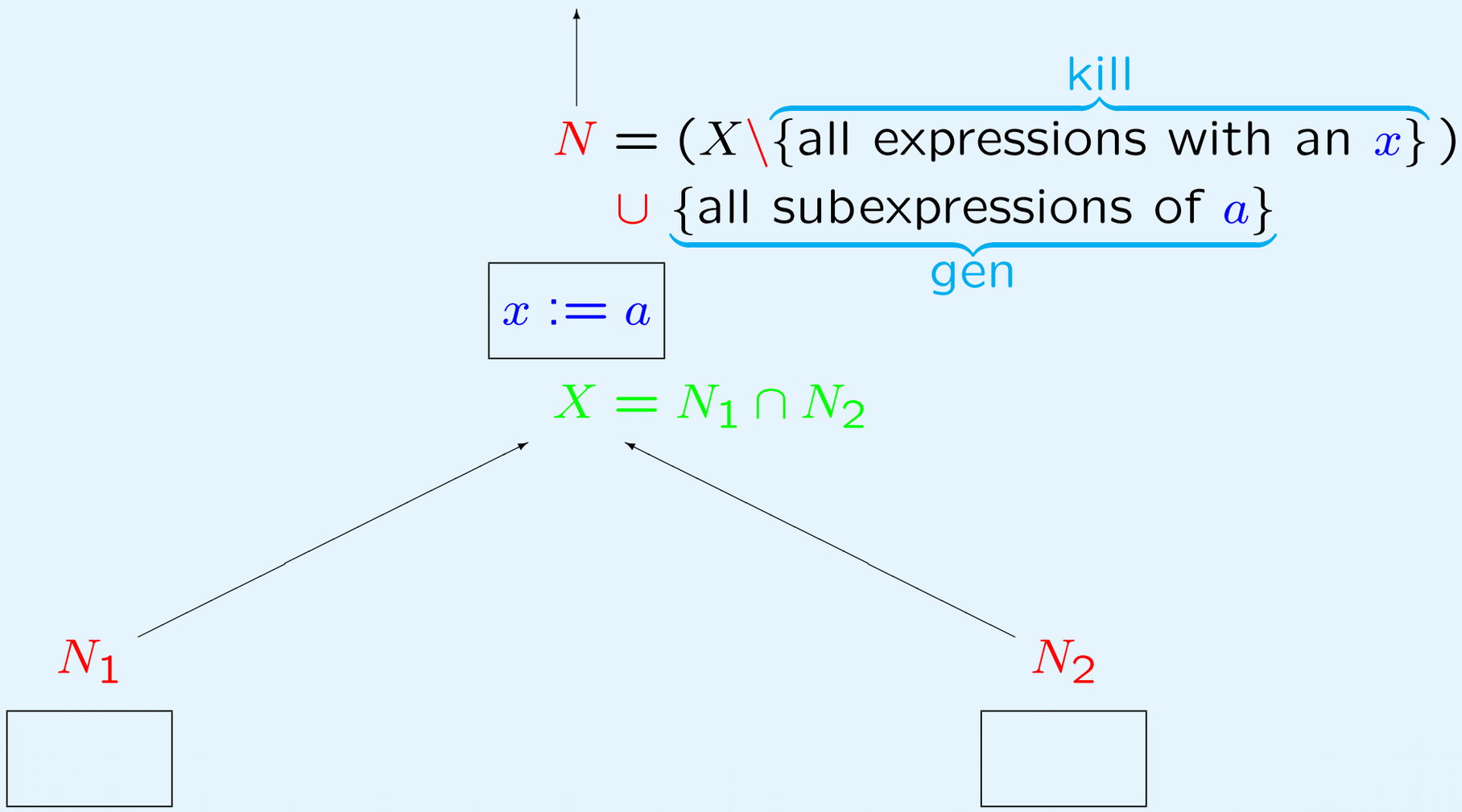
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- Design the transfer functions:
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    - Sanity check: Monotonicity!

The Design Process



Very Busy Expressions: Formal Setup

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$$VB_\bullet(l) = \begin{cases} \emptyset & l = exit \\ \bigcap_{(l,l') \in flow} VB_\bullet(l') & \text{otherwise} \end{cases}$$

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$$VB_\circ(l) = VB_\bullet(l) \setminus \{exp |\ var(exp) \cap write(l) \neq \emptyset\} \cup computed(l)$$

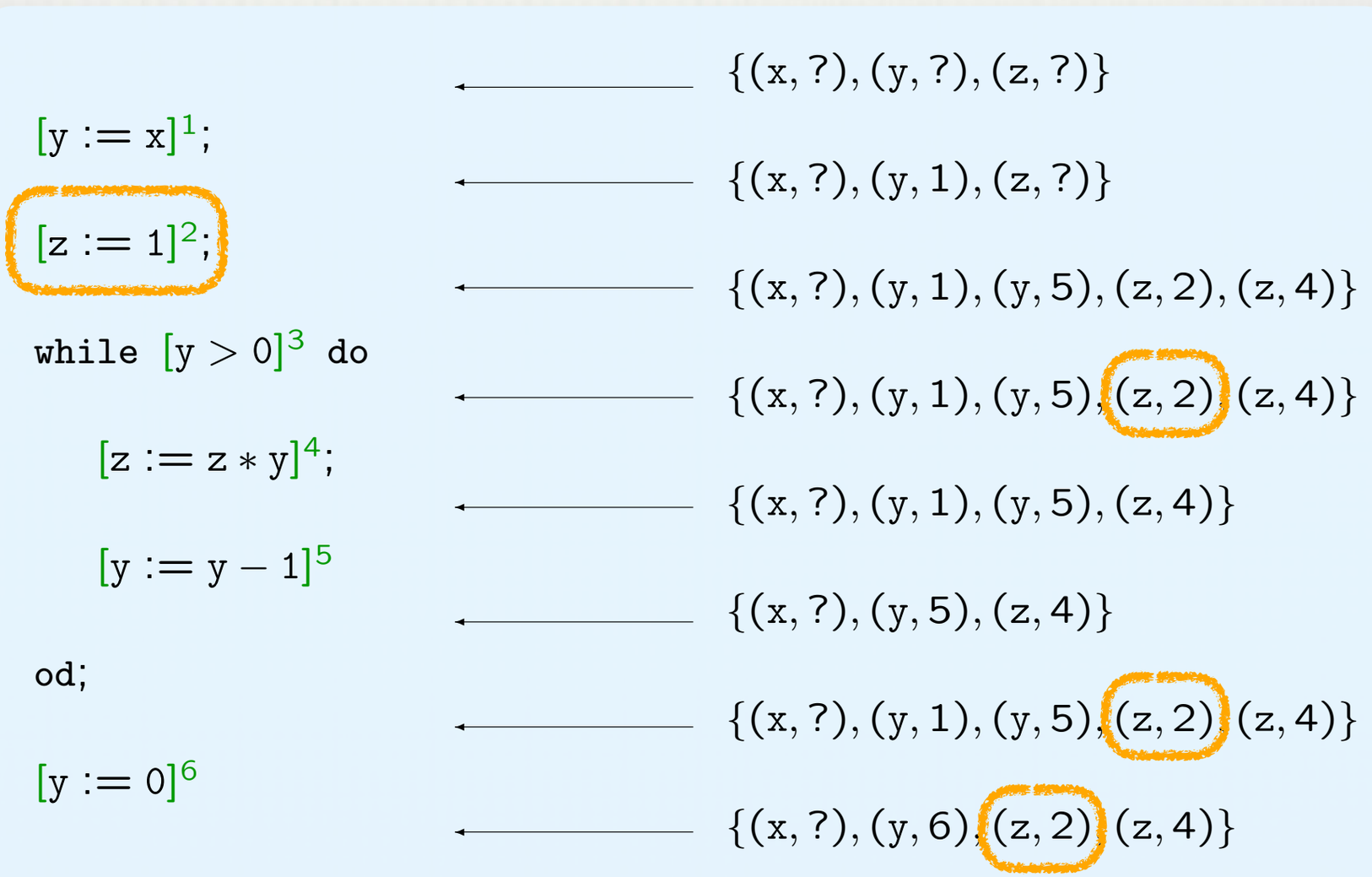
Reaching Definitions

Formal definition: A definition of var at label l reaches a label l′ if on some CFG path from l to l′, var is not overwritten.

	$\{(x,?),(y,?),(z,?)\}$
$[y := x]^1;$	$\{(x,?),(y,1),(z,?)\}$
$[z := 1]^2;$	$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$
`while` $[y > 0]^3$ `do`	$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$
$[z := z * y]^4;$	$\{(x,?),(y,1),(y,5),(z,4)\}$
$[y := y - 1]^5$	$\{(x,?),(y,5),(z,4)\}$
`od;`	$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$
$[y := 0]^6$	$\{(x,?),(y,6),(z,2),(z,4)\}$

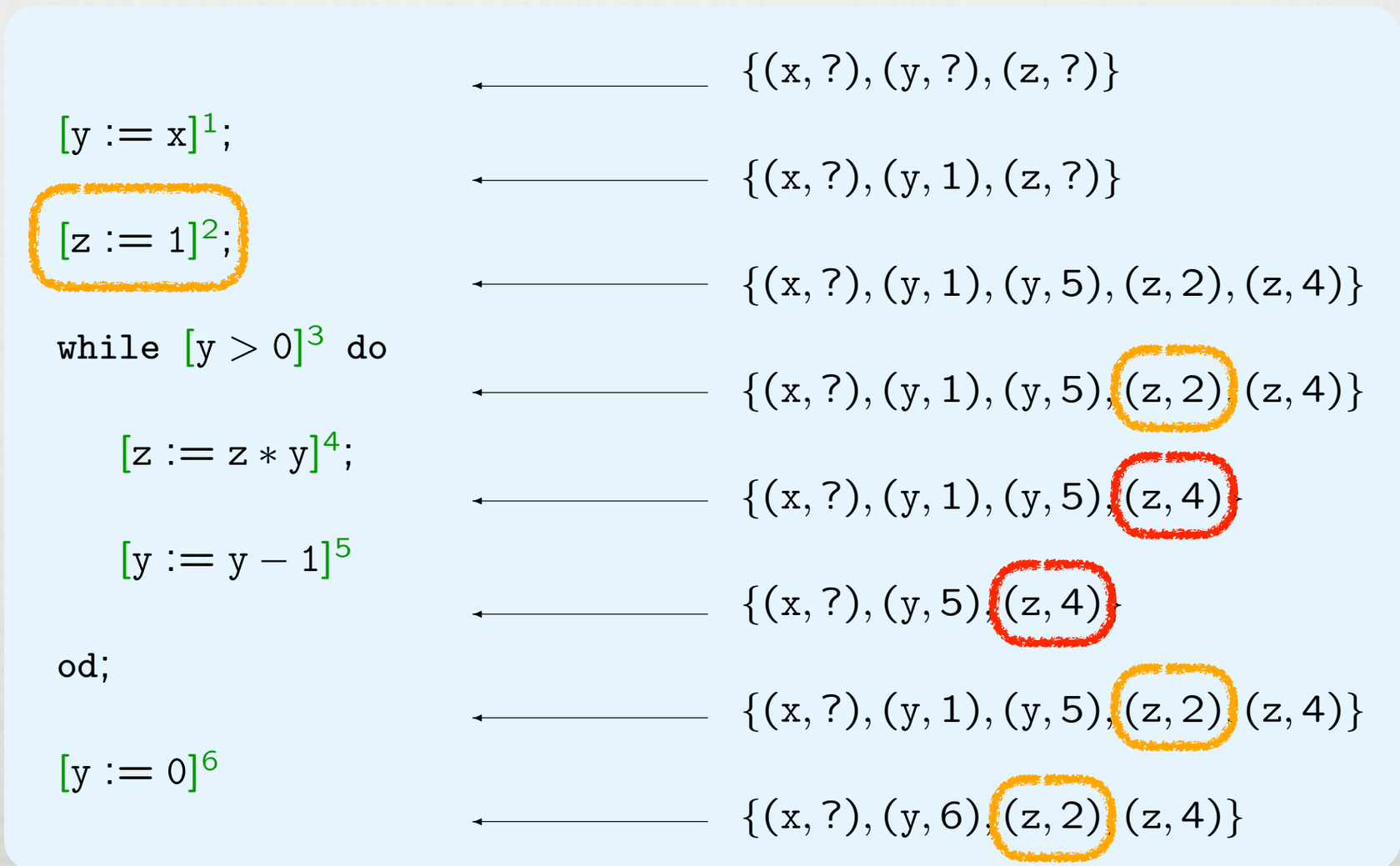
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$$RD_\bullet(l) = RD_\circ(l) \setminus write(l) \times Locs \cup write(l) \times \{l\}$$

All Analyses

	Live Variables	Reaching Definitions	Very Busy Expressions
$\mathbf{L}$	$\mathcal{P}(Var)$	$\mathcal{P}(Var \times Loc)$	$\mathcal{P}(Exp)$
$\sqcap$	$\cup$	$\cup$	$\cap$
$\sqsubseteq$	$\supseteq$	$\supseteq$	$\subseteq$
$\bot$	$\emptyset$	$\emptyset$	Exp
	Backward	Forward	Backward