CSC410

Assignment 4

Due on December 6, 2023 at 11:59pm, worth 15% of the course mark

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Assignment Format and Guidelines on Submission

Submit a properly **typed** PDF on Markus. No handwritten assignment will be accepted. Unless otherwise specified, no English words will be marked.

This assignment is partially released and will be completed by November 23rd.

List of files to submit:

- `a4.pdf` which will include the cleanly **typed** solutions to the problems.

LTL Problems

Problem 1 – Warmup (12 points)

Alice and Bob work together in a workshop and share some tools. They also have different set of skills and may need each others' help to finish a project from time to time. Consider the following atomic propositions:

- at: Alice is using tool t.
- bt: Bob is using tool t.
- aw: Alice has requested Bob's help with a task and is *waiting* for help.
- bw: Bob has requested Alice's help with a task and is *waiting* for help.
- ah: Alice provides help to Bob.
- bh: Bob provides help to Alice.

Translate each of the following desirable properties of the workshop to an LTL formula:

(a) Whenever Alice is waiting for Bob's help, Alice is not using the tool t.

(b) If Bob requests help, then either Alice will eventually help or Bob will be waiting forever.

(c) If Alice requests Bob's help, then Bob cannot request Alice's help until he provides help to Alice.

(d) Neither Bob nor Alice can request the other person's help if their help has already been requested until they provide the requested help. To clarify: this is like adding a "vice versa" to (c) so that it states the same condition about Bob requesting help from Alice.

Problem 2 (20 points)

Below, use atomic propositions p_1, p_2, and p_3 to indicate that processes 1, 2, and 3 are respectively being executed by the scheduler. It is OK to assume that the scheduler can run a single process at any given time. Use req to indicate that a request is issued, and res for a response being issued.

(a) **Operating Systems**: processes 1, 2, and 3 are executed by a round-robin scheduler. That is, each process gets executed for one time step, and then we switch to the next.

(b) Modify the above to let the scheduler run each process for more than one step at a time, but only for finitely many steps before switching to the next process.

(c) **Client-Server Systems**: a response is only ever issued if there is a request pending. That is, no response is issued if there is no request or if the request has already received a response.

Below, assume that the light bulb can be of colours white (w), red (r), green (g), and blue (b) when it is on, or it can be off (o). The system changes state every second, and the status of the light accordingly changes (or remains the same). Note that it is assumed that the light is always exactly one colour.

Translate each of the following English specification for this simple system to an LTL formula.

(d) The light bulb turns red *at most* once. To clarify: turning red means the bulb should be a different colour in the previous step and change colour to red. It can then stay red for an arbitrary amount of time. But, if it changes colour to something else, then it cannot turn back to red again.

(e) The light bulb can only turn white if it has previously been at least once blue, once green, and once red (but not necessarily in that order).

Problem 3 (24 points)

Which one of the following equivalences hold? Give a formal proof for the correct ones and provide a counterexample for the incorrect ones. A counterexample is an infinite path that satisfies one side and not the other. You may not use any of the equivalences from the lecture/book as a boost. You are meant to prove these from scratch whenever they hold.

(a) $\varphi \, \mathtt{U} \, \neg\varphi \equiv true$

(b) $(\Diamond \Box \varphi_1) \wedge (\Diamond \Box \varphi_2) \equiv \Diamond(\Box\varphi_1 \wedge \Box\varphi_2)$

(c) $\Box\Diamond\varphi \implies \Box\Diamond\psi \equiv \Box(\varphi \implies \Diamond\psi)$

(d) $\varphi \, \mathtt{U}(\psi \vee \neg\varphi) \equiv \Box\varphi \implies \Diamond\psi$

Problem 4 (10 points)

Recall that satisfiability and validity of LTL formulas are defined in the same way as propositional logic formulas. An LTL formula φ is **satisfiable** if and only if there exists a path π that satisfies it ($\exists\pi : \pi \vDash \varphi$). An LTL formula φ is valid if and only if all paths π satisfy it ($\forall\pi : \pi \vDash \varphi$). Note that validity of φ can also be reformulated as the equality $\varphi \equiv true$.

For the formulas below, determine if the formula is satisfiable, unsatisfiable, or valid. Formally justify your answer.

(a) $\Diamond b \implies (a \, \mathtt{U} \, b)$.

(b) $\bigcirc(a \vee \Diamond a) \implies \Diamond a$

CTL Problems

Problem 5 (16 points)

Recall the setup of Problems 1 and 2. We will reuse them for this problem to write a few more properties in CTL.

(a) Bob cannot ask for Alice's help unless he has already helped Alice at least once. Note that Bob does not have to ask for Alice's help at all; but, if he does, it should be after having helped Alice before.

(b) If Alice asks for Bob's help, then it is a future possibility (but not necessity) that the tool remains available from this moment (that the help was requested) until the help is delivered.

(c) The light bulb has a possible future in which it is never indefinitely stuck on any one colour.

(d) If the light bulb has ever switched from white to blue in the past, then it cannot switch from blue to white in the future.

Problem 6 (16 points)

Let TS be a finite transition system (over AP) without terminal states (i.e. every state has an outgoing transition), and Φ and Ψ be CTL state formulae (over AP). Prove or disprove: $TS \models \exists(\Phi \mathbin{\mathsf{U}} \Psi)$ if and only if $TS' \models \exists\Diamond\Psi$ where TS' is obtained from TS by eliminating all outgoing transitions from states s such that $s \models \Psi \vee \neg\Phi$.

Problem 7 (20 points)

Which one of the following equivalences hold? Give a formal proof for the correct ones and provide a counterexample for the incorrect ones. Proofs should be from scratch, and not by referencing other equalities.

(a) $\forall \Box \exists\Diamond(\varphi_1 \wedge \varphi_2) \equiv \forall \Box \exists\Diamond\varphi_1 \wedge \forall \Box \exists\Diamond\varphi_2$

(b) $\exists \Box \forall\Diamond(\varphi_1 \wedge \varphi_2) \equiv \exists \Box \forall\Diamond\varphi_1 \wedge \exists \Box \forall\Diamond\varphi_2$

(c) $\exists \Box \varphi \equiv \varphi \wedge \exists\bigcirc\exists \Box \varphi$

Model Checking

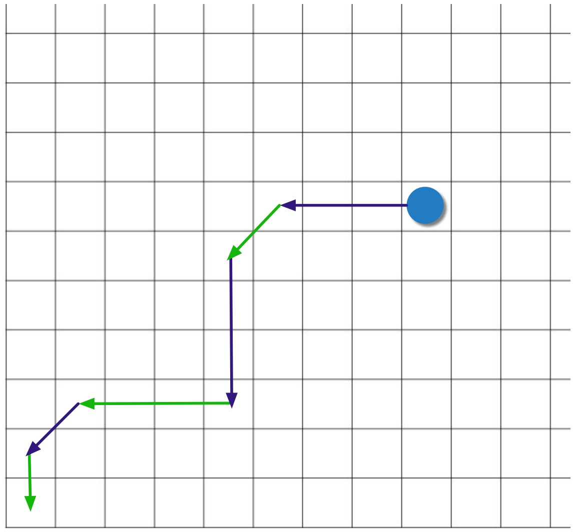
One more problem left on model checking, which will be released in synch with class on November 23rd. The grad problem will also be released by that date.

Problem 8 (30 points)

This problem is for undergraduates only, and its goals is to ensure the understanding of the ideas behind the CTL model checking algorithm, specifically the way universal and existential *until* is computed through fixpoints.

There is a game played on a grid of squares with one piece which is initially located at a position (m, n) of a grid (with $m, n \in \mathbb{N}$). The grid's origin $(0, 0)$ is at the bottom left corner and it is arbitrarily large including all squares with pairs of natural number coordinates.

The game is played between two players, who take alternate turns to move this game piece towards the origin. The valid moves for the piece are like a chess queen, as long as the direction of the move is towards the origin, i.e. left, down or diagonally towards left-down. Like a chess queen, the piece is allowed to travel as far as the player chooses in a valid direction during the one move. The player that moves the piece to the origin wins the game. Below is an example play of the game:



played from the initial location $(8, 6)$ where the first player loses the game.

We say a player has a *winning strategy* for a game iff there is play for this player to win this game independent of the choices that the opponent makes. For example, the first player always has a winning strategy from any location $(n, 0)$, $(0, n)$, or (n, n) because the player can move the piece in one move to the origin and win.

A two-player game is called *determined* if from any given position, exactly one of the players has a winning strategy. The above game is determined. Given two excellent players and any location (m, n), either player one always wins the game from (m, n) or he always loses.

The goal of this exercise is to implement a *decision procedure*. The input will be the pair of numbers (m, n). The output is "1" if the first player has a winning strategy from this location, and "2" if the second player has a winning strategy from this location.

Note: this problem is not a random implementation problem. To come up with a solution that scales up, you are encouraged to think carefully about how checking for the existence of a winning strategy relates to the concepts of existential and universal path properties. You are also encouraged to think about the algorithm we discussed for *until* and how the idea behind that algorithm can hint at a nice solution for this problem.

Forbidden Implementation Tricks: In order to let your solution *win* on merit, rather than hacks, we explicitly forbid any sort of optimization trick. For example:

- You cannot boost your solution by giving it a lookup table for smaller values. For example, one can manually computer all solutions for a 10×10 grid, enter those values as a constant for the algorithm, and let further points to get to one of these points. **This will be considered cheating.**
- You cannot use an oracle like a SAT/SMT solver under the hood.

The list above is naturally not comprehensive, because it is impossible to a priori guess any trick you may have up your sleeves as an intelligent bunch. But, it should give you the idea that we are not looking for shortcuts, but elegant algorithmic solutions.

Format

You are free to implement this in the programming language of your choice. Submit your source files as one zipped directory `source.zip`. This directory should include an executable called `game` that runs on the CDF machines. The input is passed to your executable as a command line parameter, that is:

```
./game 2 1
```

should execute on a CDF machine and return "2", since the second player has a winning strategy from the location $(2, 1)$.

Grading

There is a naive algorithm to solve this which will scale very poorly. What does poorly mean? It means that the algorithm has exponential complexity and will likely take over a minute to process a location as small as $(12, 10)$. You may assume that this naive algorithm will get no marks. The reason is that this default naive solution is something anyone who knows programming can implement and has nothing to do with the material taught in this class.

A reasonable algorithm should handle the same location $(12, 10)$ in a small fraction of a second. It would be imprecise to put an exact number on this (since it will be hardware dependent), but think of it as around 0.01s. But, more importantly, you should not see a substantial jumps for small coordinate changes at these values, for example, between the times for $(12, 10)$ and $(13, 12)$. As an another example, think about your algorithm scaling up to around coordinates $(60, 60)$ with the execution time remaining under one minute. A solution like this will take the full mark. But, if you are truly careful with your solution, you should be able to solve any point in the 60×60 grid in under one second.

Note that we will not grade your source code. We ask you to submit it for insurance, that is, in case something goes wrong with the executables and you would like to reclaim your mark through the original material submitted, and also glance at it to make sure there are no cheats.

Graduate Problem (30 points)

Graduate students do not need to submit problem 8. This problem is your replacement for problem 8. You will have a standard extension for handing this problem in one week later than the due date of this assignment. We will setup a different Markus entry for this with a different due date.

Linear temporal logic admits an axiomatization. This linked paper provides one such axiomatization in Section 3, as illustrated below:

Axiom schemas:

$$\vdash \Diamond p \equiv \neg \Box \neg p, \tag{A1}$$
$$\vdash \Box(p \supset q) \supset (\Box p \supset \Box q), \tag{A2}$$
$$\vdash \bigcirc \neg p \equiv \neg \bigcirc p, \tag{A3}$$
$$\vdash \bigcirc(p \supset q) \supset (\bigcirc p \supset \bigcirc q), \tag{A4}$$
$$\vdash \Box p \supset p \wedge \bigcirc p \wedge \bigcirc \Box p, \tag{A5}$$
$$\vdash \Box(p \supset \bigcirc p) \supset (p \supset \Box p), \tag{A6}$$
$$\vdash \Box p \supset p\, U\, q, \tag{A7}$$
$$\vdash p\, U\, q \equiv q \vee (p \wedge \bigcirc(p\, U\, q)). \tag{A8}$$

Inference rules:

If w is a propositional tautology, then $\vdash w$, (R1)

If $\vdash w_1 \supset w_2$ and $\vdash w_1$, then $\vdash w_2$, (R2)

If $\vdash w$, then $\vdash \Box w$, (R3)

There are others. Your task in this problem is to take one such axiomatization and write a small theorem prover based on it that can prove goals in LTL. This provides an alternative way of proving goals, compared to how we did things in class through classic proofs.

For example, let's say we want to prove the validity of

$$\Box(p \implies q) \implies \Diamond p \implies \Diamond q$$

.

The derivation using the axioms would go as

$\vdash p \implies q \equiv \neg q \implies \neg p$	(R1)	(1)
$\vdash \Box(p \implies q) \equiv \Box(\neg q \implies \neg p)$	(R3)	(2)
$\vdash \Box(\neg q \implies \neg p) \implies \Box\neg q \implies \Box\neg p$	(A2)	(3)
$\vdash \Box\neg q \implies \Box\neg p \equiv \neg\Diamond q \implies \neg\Diamond p$	(A1 and R1)	(4)
$\vdash \neg\Diamond q \implies \neg\Diamond p \equiv \Diamond p \implies \Diamond q$	(A1 and R1)	(5)
$\vdash \Box(p \implies q) \implies \Diamond p \implies \Diamond q$	(2, 3, 4, 5, and R1)	(6)

As another example, let's consider proving the equality

$$\Box(p \wedge q) \equiv \Box p \wedge \Box q$$

.

The derivation using the axioms would go as

$$\vdash (p \land q) \implies p \quad \text{(R1)} \tag{1}$$
$$\vdash \Box(p \land q) \implies \Box p \quad \text{(R3)} \tag{2}$$
$$\vdash (p \land q) \implies q \quad \text{(R1)} \tag{3}$$
$$\vdash \Box(p \land q) \implies \Box q \quad \text{(R3)} \tag{4}$$
$$\vdash \Box(p \land q) \implies \Box p \land \Box q \quad \text{(2, 4, and R1)} \tag{5}$$
$$\vdash p \implies (q \implies (p \land q)) \quad \text{(R1)} \tag{6}$$
$$\vdash \Box p \implies \Box(q \implies (p \land q)) \quad \text{(R1)} \tag{7}$$
$$\vdash \Box(q \implies (p \land q)) \implies (\Box q \implies \Box(p \land q)) \quad \text{(R2)} \tag{8}$$
$$\vdash \Box p \implies (\Box q \implies \Box(p \land q)) \quad \text{(7, 8, and R1)} \tag{9}$$
$$\vdash \Box p \land \Box q \implies \Box(p \land q) \quad \text{(9 and R1)} \tag{10}$$
$$\vdash \Box p \land \Box q \equiv \Box(p \land q) \quad \text{(5 and 10)} \tag{11}$$

Your tool should be able to take (validity) goals of this type as an input, and produce proofs of the above form. Formally, you are proving the validity of an input LTL formula. For equivalences (in the style of our second example above), your tool proves the validity of the LTL formula

$$\Box(p \land q) \iff \Box p \land \Box q$$

Clearly, a proof search scheme comes with concerns about termination. As a graduate problem, we are intentionally leaving things open ended for you and treat this as a small research problem. You may need to think about a compromise between completeness and convergence. As such, we do not give you a very precise description on what your tool should and should not able to handle. You need to make some decisions about the type of compromises you want to make.

Accordingly, we do not expect you to submit a tool that will be able to prove everything, especially within a reasonable time bound. The examples listed here are supposed to inform, and not be treated as test cases that must definitely pass through the tool. As is often the case with research questions, once faced with a task that cannot be completed in its most perfect version, your job as the researchers is to define a reasonable subproblem that can be solved. We are asking you to define (and solve) the best subproblem that you can, describe your approach, and discuss its strengths and weaknesses.

It is noteworthy that the entire power of propositional reasoning is packed under the single axiom (R1). You may use the power of a SAT solver to check your propositional tautologies, rather than having to write your own theorem prover for propositional logic.

Submit your work as:

- A package `ltl-prover.zip` with a proper `Readme` on how to run it.
- A clearly mark examples directory under the package that would include samples of at least 10 proofs your tool can do.
- A one page PDF that explains your research solution to the tension between convergence and completeness, and a discussion of its strengths and shortcomings.

We will set aside some bonus marks for a tool that really does this well, and can solve many problems. You can use these bonus marks then globally to compensate for lost marks in other assignments.