Consider a server with three clients. Clients can issue requests, and the server can provide answers. You may assume that the system can only process one event at a time; that is, at a given time step, either no action is made, or a single request/answer is issued.

Let $\text{req}_i$ denote that Client $i$ has issued a request, and let $\text{ans}_i$ denote that the server has issued an answer to Client $i$. 
Formalizing Specifications

Formalize the following specification in LTL:

“If Client 1 issues a request, then Clients 2 and 3 will not receive answers until Client 1 is answered.”

\[ \square (\text{req}_1 \implies \neg (\text{ans}_2 \lor \text{ans}_3) \mathcal{U} \text{ans}_1) \]
Now add additional variables $\text{wait}_i$ to denote that client $i$ is waiting for an answer. Formalize the following specification in LTL:

“After issuing a request, Client 2 will be waiting until it receives an answer.”

Note that an answer is not guaranteed to arrive.

$$\Box (\text{req}_2 \implies (\text{wait}_2 \mathcal{U} \text{ans}_2) \lor \Box \text{wait}_2)$$
Formalize the following specification in LTL:

“If a request is received from both Clients 1 and 2 before either Client is answered, then Client 2 will be answered before Client 1. Both clients will be answered.”

\[
\Box \left( \text{req}_1 \land (\neg \text{ans}_1 U \text{req}_2) \lor \text{req}_2 \land (\neg \text{ans}_2 U \text{req}_1) \Rightarrow (\neg \text{ans}_1 U \text{ans}_2) \land \Diamond \text{ans}_1 \right)
\]
For each of the following, either prove or disprove the equivalence. If it is not true, provide a counterexample.
Prove or disprove:

$$\square (\phi \lor \psi) \equiv \square \phi \lor \square \psi$$
Prove or disprove:

$$\Diamond \Box (\phi \lor \psi) \equiv \Diamond \Box \phi \lor \Diamond \Box \psi$$
Prove or disprove:

\[ \Diamond \phi \equiv \phi \lor \bigcirc \Diamond \phi \]
Prove or disprove:

$$\square(\phi \lor \neg \psi) \equiv \neg \diamond (\neg \phi \land \psi)$$
1 Equivalences

Prove or disprove the following LTL equivalences. If they do not hold, provide a counterexample.

1. $\Box(\phi \lor \psi) \equiv \Box \phi \lor \Box \psi$

   Fix a set of atomic propositions $AP$. If this is true, this means that
   
   $\forall \pi \in \mathcal{P}(AP)\omega. \pi \models \Box(\phi \lor \psi) \iff \pi \models \Box \phi \lor \Box \psi$

   We can see that the reverse direction is true:

   Assume $\pi \models \Box \phi \lor \Box \psi$. Proceed by cases analysis.

   Case 1: $\pi \models \Box \phi$. We need to prove $\pi \models \Box(\pi \lor \psi)$. That is, we need to prove
   
   $\forall k \in \mathbb{N}. \pi[k..] \models \phi \lor \psi$

   Fix such a $k$. By our assumption, $\forall i \in \mathbb{N}. \pi[i..] \models \phi$. Then in particular, $\pi[k..] \models \phi$. And hence $\pi[k..] \models \phi \lor \psi$.

   Case 2: Symmetric with case 1.

   So by case analysis, $\pi \models \Box \phi \lor \Box \psi \implies \pi \models \Box(\phi \lor \psi)$

   However, the reverse direction does not hold. Suppose that $\pi \models \Box(\phi \lor \psi)$. We need to prove $\pi \models \Box \phi \lor \Box \pi \models \Box \pi$. 


Consider fixing $AP = \{a, b\}$, and consider the path $\pi = (\{a\}\{b\})^\omega$. Then $\pi \models \Box(a \lor b)$. However, $\pi \not\models \Box a \lor \Box b$. For instance, $\pi[1..] \not\models a$ since $\pi[1] = \{b\}$, so $\pi \not\models \Box a$. Likewise, $\pi[2..] \not\models b$ since $\pi[2] = \{a\}$. Since $\pi \not\models \Box a$ and $\pi \not\models \Box b$, we have $\pi \not\models \Box \phi \lor \Box \phi$.

2. Prove or disprove $\Diamond \Box (\phi \lor \psi) \equiv \Diamond \Box \phi \lor \Diamond \Box \psi$.

You should be able to see that this is basically the same problem as before, just slightly generalized. We could even use the same counterexample, but I’ll use a different one anyway.

Let us fix $AP = \{a, b\}$ and some number – say 3. The choice of the number doesn’t matter. Let’s define $\pi$ such that for all $j < 3$, $\pi[j] = \emptyset$, $\pi[3] = \{a\}$, and for all $i$, if $i$ is even, $\pi[3+i] = \{a\}$, and if $i$ is odd, then $\pi[3+i] = \{b\}$. We could also write this as $\pi = \emptyset\emptyset\emptyset(\{a\}\{b\})^\omega$. One can easily prove that $\pi \models \Diamond \Box (a \lor b)$.

It can also prove, from our construction, that for arbitrary $k$, if $\pi[k] \models a$, then $\pi[k+1] \not\models a$, and likewise for $b$.

However, it is not the case that $\pi \models \Diamond \Box a$. By way of contradiction, assume there is some $k$ such that $\pi[k..] \models \Box a$.

So $a$ holds in every time point after $k$. Then in particular we have $\pi[k..] \models a$, and also $\pi[k+j+1..] \models a$. But this contradicts our construction of $\pi$, as noted above. The argument is symmetric to show why $\pi \not\models \Diamond \Box a \lor b$. Since neither of these disjuncts are satisfied by $\pi$, it serves as a counterexample to the equivalence.
3. Prove or disprove $\Diamond \phi \equiv \phi \lor \Box \Diamond \phi$

**Forward Direction:**
Assume $\pi \vDash \Diamond \phi$. Then $\exists k. \pi[k..] \vDash \phi$. Fix such a $k$.

Then $k = 0 \lor k = k' + 1$ for some $k' \in \mathbb{N}$.

**Case 1.** $k = 0$. Then $\pi[0..] = \pi \vDash \phi$, and therefor $\pi \vDash \phi \lor \Box \Diamond \phi$.

**Case 2.** $k = k' + 1$. Then $\pi[k' + 1..] \vDash \phi$, so $\pi[1..][k'..] \vDash \phi$.

Hence $\exists k' \in \mathbb{N}$. $\pi[1..][k'..] \vDash \phi$. Then $\pi[1..] \vDash \Diamond \phi$. And so $\pi \vDash \Box \Diamond \phi$, and hence $\pi \vDash \phi \lor \Box \Diamond \phi$.

**Reverse Direction:**
Assume $\pi \vDash \phi \lor \Box \Diamond \phi$. We proceed by case analysis.

**Case 1.** Assume $\pi \vDash \phi$. Then use the witness 0 to prove $\exists k. \pi[k..] \vDash \phi$.

**Case 2.** Assume $\pi \vDash \Box \Diamond \phi$. Then $\pi[1..] \vDash \Diamond \phi$, which means $\exists j \in \mathbb{N}. \pi[1..][j..] \vDash \phi$. That is, we have $j$ such that $\pi[j + 1..] \vDash \phi$. So we use $j + 1$ as the witness for $\exists k \in \mathbb{N}. \pi[k..] \vDash \phi$.

4. Prove or disprove: $\Box (\phi \lor \neg \psi) \equiv \neg \Diamond (\neg \phi \land \psi)$

   This one we can do as a series of rewrites. Fix a path $\pi$.

\[
\begin{align*}
\pi \vDash \Box (\phi \lor \neg \psi) &\iff \forall i \in \mathbb{N}. \pi[i..] \vDash \phi \lor \neg \psi & \text{(Def. $\Box$)} \\
&\iff \forall i \in \mathbb{N}. \phi \vDash \neg (\phi \lor \neg \psi) & \text{(Double Negation)} \\
&\iff \forall i \in \mathbb{N}. \pi \vDash \neg (\phi \lor \neg \psi) & \text{(Double Negation)} \\
&\iff \neg \exists i \in \mathbb{N}. \pi \vDash (\phi \lor \neg \psi) & \text{(Distribute $\neg$ over $\forall$)} \\
&\iff \neg \exists i \in \mathbb{N}. \pi \vDash \neg \phi \land \psi & \text{(De Morgan)} \\
&\iff \neg \pi \vDash \Diamond (\neg \phi \land \psi) & \text{(Def. $\Diamond$)} \\
&\iff \pi \vDash \neg \Diamond (\neg \phi \land \psi) 
\end{align*}
\]