# Tutorial 8: Linear Temporal Logic

CSC410

November 3, 2023

# Formalizing Specifications

Consider a server with three clients. Clients can issue requests, and the server can provide answers. You may assume that the system can only process one event at a time; that is, at a given time step, either no action is made, or a single request/answer is issued.

Let $\texttt{req}_i$ denote that Client $i$ has issued a request, and let $\texttt{ans}_i$ denote that the server has issued an answer to Client $i$.

# Formalizing Specifications

Formalize the following specification in LTL:

"If Client 1 issues a request, then Clients 2 and 3 will not receive answers until Client 1 is answered."

$$\Box(\mathtt{req}_1 \implies \neg(\mathtt{ans}_2 \vee \mathtt{ans}_3)\,\mathcal{U}\,\mathtt{ans}_1)$$

# Formalizing Specifications

Now add additional variables $\texttt{wait}_i$ to denote that client $i$ is waiting for an answer. Formalize the following specification in LTL:

> "After issuing a request, Client 2 will be waiting until it receives an answer."

Note that an answer is not guaranteed to arrive.

$$\Box(\texttt{req}_2 \implies (\texttt{wait}_2 \,\mathcal{U}\, \texttt{ans}_2) \vee \Box\, \texttt{wait}_2)$$

# Formalizing Specifications

Formalize the following specification in LTL:

"If a request is received from both Clients 1 and 2 before either Client is answered, then Client 2 will be answered before Client 1. Both clients will be answered."

$$
\begin{aligned}
&\Box( \\
&\quad \mathtt{req}_1 \wedge (\neg \mathtt{ans}_1 \,\mathcal{U}\, \mathtt{req}_2) \quad \vee \quad \mathtt{req}_2 \wedge (\neg \mathtt{ans}_2 \,\mathcal{U}\, \mathtt{req}_1) \\
&\quad\; \implies (\neg \mathtt{ans}_1 \,\mathcal{U}\, \mathtt{ans}_2) \wedge \Diamond \mathtt{ans}_1 \\
&\quad )
\end{aligned}
$$

# LTL Equivalences

For each of the following, either prove or disprove the equivalence. If it is not true, provide a counterexample.

Prove or disprove:

$$\Box(\phi \lor \psi) \equiv \Box\phi \lor \Box\psi$$

# LTL Equivalences

Prove or disprove:

$$\Diamond\Box(\phi \vee \psi) \equiv \Diamond\Box\phi \vee \Diamond\Box\psi$$

# LTL Equivalenecs

Prove or disprove:

$$\Diamond\phi \equiv \phi \vee \bigcirc\Diamond\phi$$

# LTL Equivalences

Prove or disprove:

$$\Box(\phi \vee \neg\psi) \equiv \neg\Diamond(\neg\phi \wedge \psi)$$

# LTL Tutorial

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## 1 Equivalences

Prove or disprove the following LTL equivalences. If they do not hold, provide a counterexample.

1. $\Box(\phi \lor \psi) \equiv \Box\phi \lor \Box\phi$

   Fix a set of atomic propositions $AP$. If this is true, this means that

   $$\forall \pi \in \mathcal{P}(AP)^\omega .\, \pi \vDash \Box(\phi \lor \psi) \iff \pi \vDash \Box\phi \lor \Box\psi$$

   We can see that the reverse direction is true:

   Assume $\pi \vDash \Box\phi \lor \Box\psi$. Proceed by cases analysis.

   Case 1: $\pi \vDash \Box\phi$. We need to prove $\pi \vDash \Box(\pi \lor \psi)$. That is, we need to prove

   $$\forall k \in \mathbb{N}.\, \pi[k..] \vDash \phi \lor \psi$$

   Fix such a $k$. By our assumption, $\forall i \in \mathbb{N}. \pi[i..] \vDash \phi$. Then in particular, $\pi[k..] \vDash \phi$. And hence $\pi[k..] \vDash \phi \lor \psi$.

   Case 2: Symmetric with case 1.

   So by case analysis, $\pi \vDash \Box\phi \lor \Box\psi \implies \pi \vDash \Box(\phi \lor \psi)$

   However, the reverse direction does not hold. Suppose that $\pi \vDash \Box(\phi \lor \psi)$. We need to prove $\pi \vDash \Box\phi \lor \pi \vDash \Box\pi$.

Consider fixing $AP = \{a, b\}$, and consider the path $\pi = (\{a\}\{b\})^\omega$. Then $\pi \models \Box(a \vee b)$. However, $\pi \not\models \Box a \vee \Box b$. For instance, $\pi[1..] \not\models a$ since $\pi[1] = \{b\}$, so $\pi \not\models \Box a$. Likewise, $\pi[2..] \not\models b$ since $\pi[2] = \{a\}$. Since $\pi \not\models \Box a$ and $\pi \not\models \Box b$, we have $\pi \not\models \Box\phi \vee \Box\phi$.

2. Prove or disprove $\Diamond\Box(\phi \vee \psi) \equiv \Diamond\Box\phi \vee \Diamond\Box\psi$.

   You should be able to see that this is basically the same problem as before, just slightly generalized. We could even use the same counterexample, but I'll use a different one anyway.

   Let us fix $AP = \{a, b\}$ and some number – say 3. The choice of the number doesn't matter. Let's define $\pi$ such that for all $j < 3$, $\pi[j] = \emptyset$, $\pi[3] = \{a\}$, and for all $i$, if $i$ is even, $\pi[3+i] = \{a\}$, and if $i$ is odd, then $\pi[3+i] = \{b\}$. We could also write this as $\pi = \emptyset\emptyset\emptyset(\{a\}\{b\})^\omega$. One can easily prove that $\pi \models \Diamond\Box(a \vee b)$.

   It can also prove, from our construction, that for arbitrary $k$, if $\pi[k] \models a$, then $\pi[k+1] \not\models a$, and likewise for $b$.

   However, it is not the case that $\pi \models \Diamond\Box a$. By way of contradiction, assume there is some $k$ such that $\pi[k..] \models \Box a$.

   So $a$ holds in every time point after $k$. Then in particular we have $\pi[k..] \models a$, and also $\pi[k+j+1..] \models a$. But this contradicts our construction of $\pi$, as noted above. The argument is symmetric to show why $\pi \not\models \Diamond\Box a \vee b$. Since neither of these disjuncts are satisfied by $\pi$, it serves as a counterexample to the equivalence.

3. Prove or disprove $\Diamond\phi \equiv \phi \vee \bigcirc\Diamond\phi$

   **Forward Direction:**

   Assume $\pi \vDash \Diamond\phi$. Then $\exists k.\, \pi[k..] \vDash \phi$. Fix such a $k$.

   Then $k = 0 \vee k = k' + 1$ for some $k' \in \mathbb{N}$.

   **Case 1.** $k = 0$. Then $\pi[0..] = \pi \vDash \phi$, and therefor $\pi \vDash \phi \vee \bigcirc\Diamond\phi$.

   **Case 2.** $k = k' + 1$. Then $\pi[k' + 1.. \vDash \phi]$, so $\pi[1..][k'..] \vDash \phi$.

   Hence $\exists k' \in \mathbb{N}.\, \pi[1..][k'..] \vDash \phi$. Then $\pi[1..] \vDash \Diamond\phi$. And so $\pi \vDash \bigcirc\Diamond\phi$, and hence $\pi \vDash \phi \vee \bigcirc\Diamond\phi$.

   **Reverse Direction**:

   Assume $\pi \vDash \phi \vee \bigcirc\Diamond\phi$. We proceed by case analysis.

   **Case 1**. Assume $\pi \vDash \phi$. Then use the witness 0 to prove $\exists k.\, \pi[k..] \vDash \phi$.

   **Case 2.** Assume $\pi \vDash \bigcirc\Diamond\phi$. Then $\pi[1..] \vDash \Diamond\phi$, which means $\exists j \in \mathbb{N}.\, \pi[1..][j..] \vDash \phi$. That is, we have $j$ such that $\pi[j + 1..] \vDash \phi$. So we use $j + 1$ as the witness for $\exists k \in \mathbb{N}.\, \pi[k..] \vDash \phi$

4. Prove or disprove: $\Box(\phi \vee \neg\psi) \equiv \neg\Diamond(\neg\phi \wedge \psi)$

   This one we can do as a series of rewrites. Fix a path $\pi$.

$$
\begin{aligned}
\pi \vDash \Box(\phi \vee \neg\psi) &\iff \forall i \in \mathbb{N}.\, \pi[i..] \vDash \phi \vee \neg\psi && \text{(Def. } \Box\text{)} \\
&\iff \forall i \in \mathbb{N}.\, \phi \vDash \neg\neg(\phi \vee \neg\psi) && \text{(Double Negation)} \\
&\iff \neg\exists i \in \mathbb{N}.\, \pi \vDash \neg(\phi \vee \neg\psi) && \text{(Distribute } \neg \text{ over } \forall\text{)} \\
&\iff \neg\exists i \in \mathbb{N}.\, \pi \vDash \neg\phi \wedge \psi && \text{(DeMorgan)} \\
&\iff \neg\pi \vDash \Diamond(\neg\phi \wedge \psi) && \text{(Def. } \Diamond\text{)} \\
&\iff \pi \vDash \neg\Diamond(\neg\phi \wedge \psi)
\end{aligned}
$$