Inferences in Bayesian Networks
Variable Elimination Algorithm

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Lecture 15

Based on work by K. Leyton-Brown, K. Larson, and P. van Beek

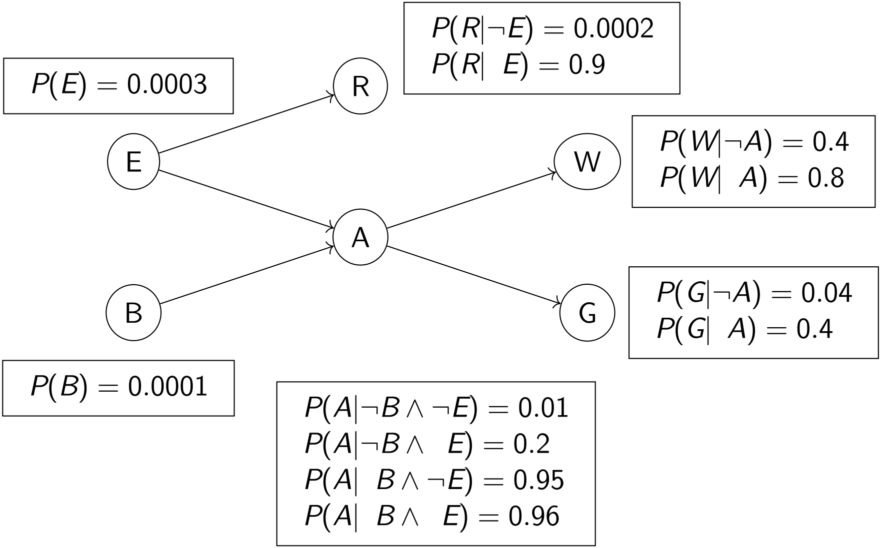
Outline

Learning Goals

By the end of the lecture, you should be able to

- Define factors. Manipulate factors using operations restrict, sum out, multiply and normalize.
- Compute a prior or a posterior probability given a Bayesian network.
- Describe/trace/implement the variable elimination algorithm for calculating a prior or a posterior probability given a Bayesian network.

A Bayesian Network for the Holmes Scenario



Answering a Question

What is the probability that a burglary is happening given that Dr. Watson and Mrs. Gibbon both call?

$$P(B = b | W = t \land G = t), b \in \{t, f\}$$

- Query variables: B
- Evidence variables: W and G
- Hidden variables: A, E, and R.

Number of operations

How many addition and multiplication operations do we need to perform to evaluate the following expression?

$$
\begin{aligned}
&P(B = t \land W = t \land G = t) \\
&= \sum_{a \in \{t,f\}} \sum_{e \in \{t,f\}} \sum_{r \in \{t,f\}} P(B = t) P(E = e) P(A = a | B = t \land E = e) \\
&\qquad P(R = r | E = e) P(G = t | A = a) P(W = t | A = a)
\end{aligned}
$$

(A) Less than 10
(B) 10-25
(C) 26-40
(D) 41-55
(E) More than 55

Number of operations

How many addition and multiplication operations do we need to perform to evaluate the following expression?

$$
\begin{aligned}
&P(B = t \wedge W = t \wedge G = t) \\
&= P(B = t) \sum_{a \in \{t,f\}} P(G = t | A = a) P(W = t | A = a) \\
&\qquad \sum_{e \in \{t,f\}} P(E = e) P(A = a | B = t \wedge E = e)
\end{aligned}
$$

(A) Less than 10

(B) 10-25

(C) 26-40

(D) 41-55

(E) More than 55

Factors

- A factor is a representation of a function from a tuple of random variables into a number.
- We will write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$.
- A factor denotes a distribution over the given tuple of variables in some (unspecified) context
  - e.g., $P(X_1, X_2)$ is a factor $f(X_1, X_2)$
  - e.g., $P(X_1, X_2, X_3 = v_3)$ is a factor $f(X_1, X_2)$
  - e.g., $P(X_1, X_3 = v_3 | X_2)$ is a factor $f(X_1, X_2)$

Restrict a factor

Restrict a factor by assigning a value to the variable in the factor.

- $f(X_1 = v_1, X_2, \ldots, X_j)$, where $v_1 \in dom(X_1)$, is a factor on $X_2, \ldots, X_j$.
- $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number that is the value of f when each X_i has value v_i.

Restrict a factor

$f_1(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

- What is $f_2(Y, Z) = f_1(X = t, Y, Z)$?
- What is $f_3(Y) = f_2(Y, Z = f)$?
- What is $f_4() = f_3((Y = f)$?

Sum out a variable

Sum out a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$\begin{aligned}&(\sum_{X_1} f)(X_2, \ldots, X_j)\\ &= \quad f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)\end{aligned}$$

Sum out a variable

$f_1(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

What is $f_2(X, Z) = \sum_B f_1(X, Y, Z)$?

Multiplying factors

Multiply two factors together.

- The product of factor $f_1(X, Y)$ and $f_2(Y, Z)$, where Y are the variables in common, is the factor $(f_1 \times f_2)(X, Y, Z)$ defined by:

$$(f_1 \times f_2)(X, Y, Z) \quad = \quad f_1(X, Y) f_2(Y, Z).$$

Multiplying factors

$f_1(A, B)$:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$f_2(B, C)$:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

What is $f_1(A, B) \times f_2(B, C)$?

Variable elimination algorithm

To compute $P(X_q | X_{o_1} = v_1 \wedge \ldots \wedge X_{o_j} = v_j)$:

- Construct a factor for each conditional probability distribution.
- Restrict the observed variables to their observed values.
- Eliminate each hidden variable X_{h_j}.
  - Multiply all the factors that contain X_{h_j} to get new factor g_j.
  - Sum out the variable X_{h_j} from the factor g_j.
- Multiply the remaining factors.
- Normalize the resulting factor.

Revisiting the Learning Goals

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