Constraint Satisfaction Problems: Backtracking Search and Arc Consistency

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Lecture 5

Based on work by K. Leyton-Brown, K. Larson, and P. van Beek

Outline

Learning Goals

Examples of CSP Problems

Introduction to CSPs

Formulating Problems as CSPs

The AC-3 Arc Consistency Algorithm

Arc Consistency

Revisiting the Learning goals

Learning Goals

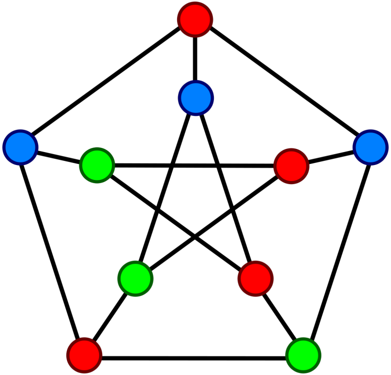
By the end of the lecture, you should be able to

- Formulate a real-world problem as a constraint satisfaction problem.
- Verify whether a variable is arc-consistent with respect to another variable for a constraint.
- Trace the execution of and implement the AC-3 arc consistency algorithm.

Example: Crossword Puzzles

T	A	O	S		S	P	E	C	S		S	E	P	T
E	G	G	Y		A	L	A	A	P		I	L	I	A
T	H	R	E	E	P	E	N	N	Y		D	H	A	K
H	A	E		X	E	B	E	C		S	E	I	N	E
			V	O	L		D	H	O	W	S			
S	W	E	E	N	E	Y		A	N	A	T	M	A	N
P	R	Y	S		S	O	N		E	P	E	I	R	A
L	Y	R	I	C		N	O	S		S	P	A	E	R
I	L	I	C	E	S		R	U	C		P	O	C	K
T	Y	R	A	N	E	D		R	O	P	E	W	A	Y
			T	S	A	R	S		C	U	D			
B	A	G	I	E		O	I	N	K	S		T	S	K
O	L	E	O		S	W	E	E	P	S	T	A	K	E
Y	A	R	N		E	N	V	O	I		A	B	U	T
G	Y	M	S		I	D	E	N	T		G	U	G	A

Example: Graph Coloring Problem



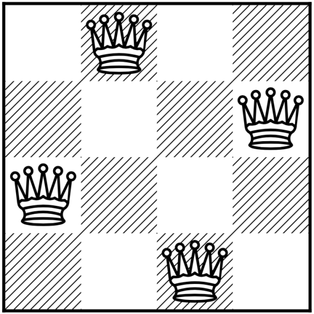
Applications:

- Designing seating plans
- Exam scheduling
- ...

Example: Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Example: 4-Queens Problem



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Introduction to CSPs

- So far, search algorithms are unaware of the structure of the states.
- Can we do better by taking advantage of the structure of states?

Definition of a CSP

Each state contains

- A set X of variables: $\{X_1, X_2, ..., X_n\}$
- A set D of domains: D_i is the domain for variable X_i, $\forall i$.
- A set C of constraints specifying allowable combinations of values

A solution is an assignment of values to all the variables that satisfy all the constraints.

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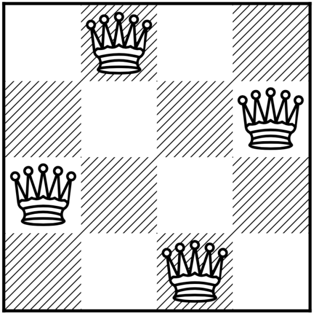
Example: 4-Queens Problem



CQ: Defining Constraints as a Formula

CQ: How should we encode the following constraint as a propositional formula?

> *The two queens in columns 0 and 2 are not in the same row or diagonal.*

(A) $(x_0 \neq x_2)$

(B) $((x_0 \neq x_2) \wedge ((x_0 - x_2) \neq 1))$

(C) $((x_0 \neq x_2) \wedge ((x_0 - x_2) \neq 2))$

(D) $((x_0 \neq x_2) \wedge (|x_0 - x_2| \neq 1))$

(E) $((x_0 \neq x_2) \wedge (|x_0 - x_2| \neq 2))$

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Solving a CSP - Search and Inference

When solving a CSP, we can combine

- Backtracking search, and
- Inference using the arc-consistency algorithm.

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Definition of Arc Consistency

Definition (Arc Consistency)

The variable X_i is arc-consistent with respect to another variable X_j if and only if for every value v_i in D_i, there is a value v_j in D_j such that (v_i, v_j) satisfies the constraint (X_i, X_j).

If X_i is not arc-consistent with the variable X_j, we can make it consistent by removing values in D_i that is not consistent with any value on D_j. This removal can never rule out any solution.

CQ: Definition of Arc Consistency

CQ: Consider the constraint "X is divisible by Y" between two variables X and Y. X is arc-consistent with respect to Y in how many of the four scenarios below?

1. $dom(X) = \{10, 12\}, dom(Y) = \{3, 5\}$
2. $dom(X) = \{10, 12\}, dom(Y) = \{2\}$
3. $dom(X) = \{10, 12\}, dom(Y) = \{3\}$
4. $dom(X) = \{10, 12\}, dom(Y) = \{3, 5, 8\}$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

CQ: Is Arc-Consistency Symmetric?

CQ: True or False:

If X is arc-consistent with respect to Y,
then Y is arc-consistent with respect to X.

(A) True
(B) False
(C) Not enough information to tell

CQ: Effect of Removing a Value on Arc Consistency

CQ: Assume that X is arc-consistent with respect to Y.
Remove one value from the domain of Y.
Is X still arc-consistent with respect to Y?

(A) Yes

(B) No

(C) Not enough information to tell

CQ: Effect of Removing a Value on Arc Consistency

CQ: Assume that X is arc-consistent with respect to Y.
Remove one value from the domain of X.
Is X still arc-consistent with respect to Y?

(A) Yes
(B) No
(C) Not enough information to tell

Making (X_i, C) arc-consistent

Let C be a constraint between the variables X_i and X_j.

Algorithm 1 Revise(X_i, C)

revised $\leftarrow$ false
for x in $dom(X_i)$ **do**
    if $\neg\exists\, y \in dom(X_j)$ s.t. (x, y) satisfies the constraint C **then**
        remove x from $dom(X_i)$
        revised $\leftarrow$ true
    end if
end for
return revised

The AC-3 Arc Consistency Algorithm

Algorithm 2 The AC-3 Algorithm

```
Put (v, C) in the set S for every variable v and every constraint involving v.
while S is not empty do
    remove (X_i, C_ij) from S (C_ij is a constraint between X_i and X_j.)
    if Revise(X_i, C_ij) then
        if dom(X_i) is empty then return false
        for X_k where C_ki is a constraint between X_k and X_i do
            add (X_k, C_ki) to S
        end for
    end if
end while
return true
```

Trace the execution of AC-3 algorithm

Properties of the AC-3 Algorithm

- Does the order in which arcs are considered matter?
- Three possible outcomes of the arc consistency algorithm:
- Time complexity:
  n variables, c binary constraints, and the size of each domain is at most d.

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