Lecture 4
Constraint Satisfaction Problems

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Contents

1 Learning Goals

By the end of the lecture, you should be able to

- Formulate a real-world problem as a constraint satisfaction problem.
- Trace the execution of the backtracking search algorithm.
- Verify whether a constraint is arc-consistent.
- Trace the execution of the AC-3 arc consistency algorithm.
- Trace the execution of the backtracking search algorithm with arc consistency.
- Trace the execution of the backtracking search algorithm with arc consistency and with heuristic for choosing variables and values.

2 Examples of CSP Problems

Let's first look at some examples of constraint satisfaction problems (CSP).

Example: Pascal van Hentenryck is one of the world leading experts on solving large real-world constraint satisfaction problems. He's worked on many different domains such as disaster recovery (http://videolectures.net/icaps2011_van_hentenryck_disaster/) and transportation planning (https://www.youtube.com/watch?v=SxvM0jG3qLA).

These problems are extremely challenging. They often require moving resources to where they're needed as quickly as possible. A typical approach to solving these problems is divide and conquer.

Example: Scheduling, whether for courses, meetings, or exams, is an example of CSPs used in the real world.

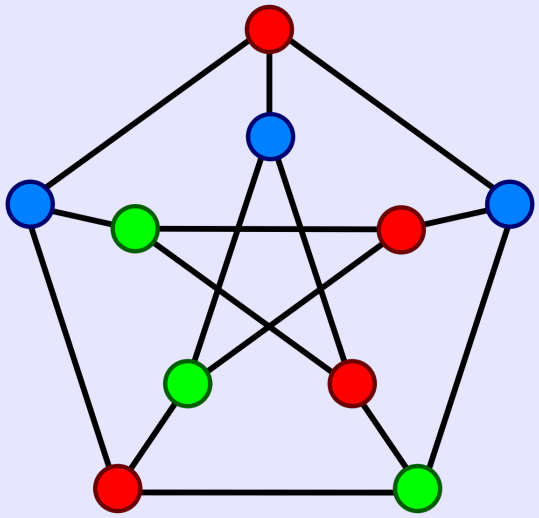
Often with scheduling, we only see the end product: course enrollment results, meeting schedules, or exam schedules. A lot of people complain about how poor the schedules are.

However, these are actually extremely challenging problems. If you ever get the chance to work on one of these problems, you may appreciate the effort people put in to solving these problems.

Example: Another example of CSPs being used in the real world are in crossword puzzles.

T	A	O	S	■	S	P	E	C	S	■	S	E	P	T
E	G	G	Y	■	A	L	A	A	P	■	I	L	I	A
T	H	R	E	E	P	E	N	N	Y	■	D	H	A	K
H	A	E	■	X	E	B	E	C	■	S	E	I	N	E
■	■	■	V	O	L	■	D	H	O	W	S	■	■	■
S	W	E	E	N	E	Y	■	A	N	A	T	M	A	N
P	R	Y	S	■	S	O	N	■	E	P	E	I	R	A
L	Y	R	I	C	■	N	O	S	■	S	P	A	E	R
I	L	I	C	E	S	■	R	U	C	■	P	O	C	K
T	Y	R	A	N	E	D	■	R	O	P	E	W	A	Y
■	■	■	T	S	A	R	S	■	C	U	D	■	■	■
B	A	G	I	E	■	O	I	N	K	S	■	T	S	K
O	L	E	O	■	S	W	E	E	P	S	T	A	K	E
Y	A	R	N	■	E	N	V	O	I	■	A	B	U	T
G	Y	M	S	■	I	D	E	N	T	■	G	U	G	A

Example: The graph colouring problem is a well known mathematical problem where we're given a graph and a set of colours, and we want to determine if we can colour all the vertices in the graph such that no two adjacent vertices have the same colour.



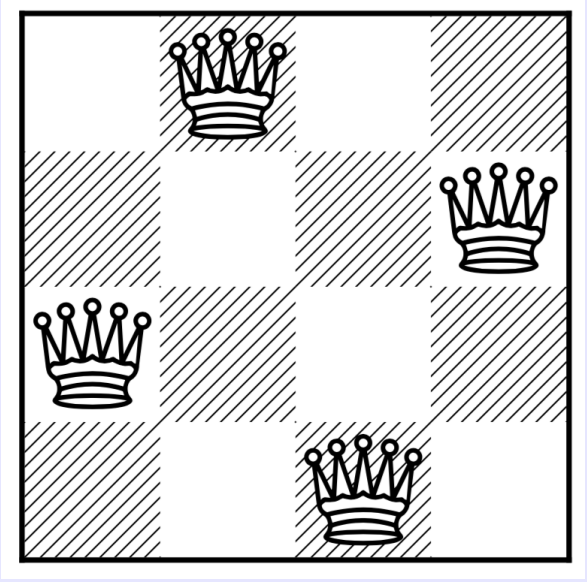
The basic form of this question looks a bit abstract and tedious, but it does have many applications in the real world. It can be used, for example, in designing seating plans or exam scheduling.

Example: The Sudoku puzzle is sort of like a crossword puzzle, but with numbers.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

In a Sudoku puzzle, the squares must be filled with numbers from 1 - 9 such that the numbers are different in every row, every column, and every 3x3 grid.

Example: The 4-queens problem is a specific implementation of the N-queens problem we saw before.



We're going to use the 4-queens problem as a running example to talk about the details of the algorithm to solve a CSP.

3 Formulating a CSP

3.1 Why study CSPs separately?

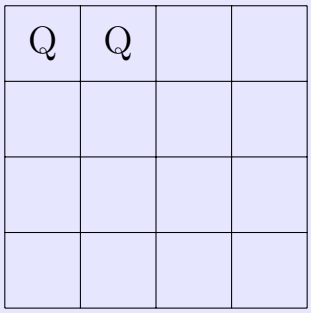
We have spent quite a bit of time discussing search algorithms. Given this, why are we devoting a separate lecture for constraint satisfaction problems? How are CSPs different from other search problems?

Let's take a look at how we can solve 4-queens problem using a form of depth-first search. Start with an empty board and add queens to the board one by one. Once we've added four queens, the algorithm tests whether the state is a goal state or not. If it's not the goal state, we backtrack. We'll change the position of the last queen placed on the board. Whenever we have a full board, we will run the goal test again. This form of depth-first search is also called backtracking search.

Backtracking search treats each state as a black box. It does not know what is inside each state. Backtracking search can perform two tasks: (1) test whether a state is a goal, and (2) generate successors of a state.

Let me show you one reason why we may not be happy with backtracking search.

Example:



Consider this partial state above. The two queens are in the same row. To us, it is clear that this state does not lead to a solution since the two queens already violate the row constraint. We should backtrack immediately and put the second queen in a different row.

Unfortunately, backtracking search does not know this. It only knows that this state is not a goal. Backtracking search needs to explore the entire sub-tree starting from this state to realize that this partial state does not lead to a solution.

For this example, it is very useful for the search algorithm to understand the internal structure of the state. In particular, if it knows and understands the positions of the queens, it may be able to prune more states complete the search much more quickly. Given this, when we are solving a CSP as a search problem, we will explicitly model the internal structure of each state, and we will discuss specialized algorithms that can solve CSPs efficiently.

3.2 Components of a CSP

Let's take a look at the components of a CSP. A CSP is one type of a search problem. Like any other search problem, we need to define

- The states
- The initial state.
- The goal states.
- The successor function.
- The cost function.

In addition, we will model the internal structure of each state. A state includes three parts, the variables, a domain for each variable, and a set of constraints. Each constraint involves one or more variables and specifies the combinations of allowable values for these variables. A solution to a CSP will include an assignment of values to all the variables such that this assignment satisfies all the constraints.

3.3 Formulating 4-Queens Problem as a CSP

Let's formulate the four queens problem as a CSP. I will focus on defining the components of a state.

How should we define the variables, the domains, and the constraint? Here is an idea that came to me first. We can define one variable for each queen. The variable stores the row and the column positions of the queen. The domain for each variable contains 16 possible positions for this queen. Our constraints are, no two queens can be in the same row, the same column, or the same diagonal.

Example:

- Variable: x_i represents the i-th queen. $i \in \{0, 1, 2, 3\}$.
- Domains: $D_i = \{0, \ldots, 15\}$.
- Constraints: no two queens are in the same row, column, or diagonal.

This formulation works, but I will show you a better one. The second formulation keeps track of less information and has fewer constraints. We will make an assumption that there is exactly one queen each column. Having assumed this, we no longer need to keep track of the column positions of the queens. We also don't have to worry about the column constraints.

Here are the details. I will define four variables, x0 to x3. The subscript i refers to the column for each queen. For example, x2 is the queen placed in the third column from the left. The value of each variable is the row position of that queen. For example, if x0 has the

value 0, the leftmost queen is in the top row. The domain of each variable contains the four possible row positions. The constraints are, no two queens can be in the same row or in the same diagonal.

Example:

- Variables: There is exactly one queen in each column. x_0, x_1, x_2, x_3 where x_i is the row position of the queen in column i, where $i \in \{0, 1, 2, 3\}$.
- Domains: $D_{x_i} = \{0, 1, 2, 3\}$ for all x_i.
- Constraints: No two queens are in the same row or diagonal.

$$\forall_i \, \forall_j \, (i \neq j) \rightarrow (\, (x_i \neq x_j) \wedge (|x_i - x_j| \neq |i - j|) \,)$$

The description of the constraints in words is quite clear. However, if you prefer, you can express the constraints as a logical formula. See the example above. The first part encodes the row constraint, and the second part encodes the diagonal constraint. If the queens are in the same diagonal, then the difference in their row positions must be equal to the difference in their column positions. We must require these two differences to be not equal. For example, to encode that the leftmost two queens cannot be in the same row and also cannot be in the same diagonal, we have x0 is not equal to x1 and the absolute difference between x0 and x1 is not 1.

4 Backtracking Search

CSPs can be solved using backtracking search, which is a special type of DFS. We'll use the 4-queens problem again to illustrate backtracking search:

In order to apply a search algorithm to the 4-queens problem, we need to specify the components of the search problem. For the 4-queens problem, we'll use an incremental formulation, which means the state will be built step-by-step.

Recall that for the 4-queens problem, we start with an empty 4x4 board, and we add queens one-by-one until there are 4 queens on the board. At that point, we test whether the state is a goal state or not. If it's not a goal state, we backtrack by removing some of the queens, and putting them in different positions.

In the previous example, we focused on defining the components of a state when we defined the CSP: variables, domains, and constraints. In this example, we'll focus on defining the rest of the components of a search problem as a CSP.

We'll first make two assumptions that will simplify the formulation and avoid looking at some states unnecessarily.

The first assumption is that we are going to add queens onto the board from left to right. This assumption allows us to avoid looking at duplicate states.

When we are adding two queens, it would either be:

- Adding the left one first and then the right one
- Adding the right one first and then the left one

Either way, we will end up in the same state, and it is redundant to look at these duplicate states.

The second assumption is that we are always going to ensure that the constraints are satisfied. There is a trade-off with doing this.

Making this assumption means possibly more work when generating successors. It is because when generating successors, we are trying to add a queen onto the board, and it is necessary to find a position that doesn't violate any constraints with the existing queens on the board.

This means once all valid successors are generated, there will be fewer successors to look over.

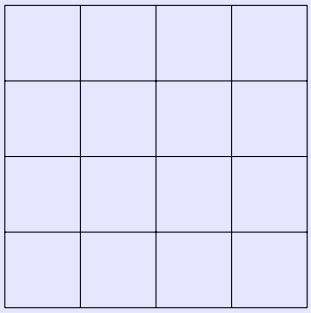
On the flip side, if we do not ensure constraints are satisfied, generating successors would be easier. However, we will have more successors to look over, and we also need to look through them to get rid of the successors that violate the constraints.

Given these assumptions, we can create the incremental formulation:

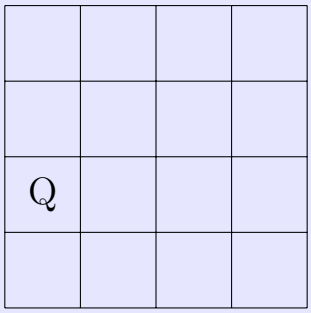
- State: one queen per column in the leftmost k columns with no pair of queens attacking each other.
- Initial state: no queens on the board.

- Goal state: 4 queens on the board. No pair of queens are attacking each other.
- Successor function: add a queen to the leftmost empty column such that it is not attacked by any other existing queen.

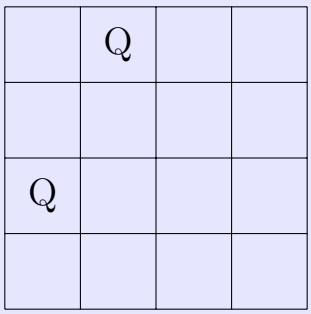
Example: To make sure that a state we generated is valid and does not violate any constraint, it's easier to look at a 4x4 board and try to draw the state. The empty board is a valid state. This is the state _ _ _ _ :



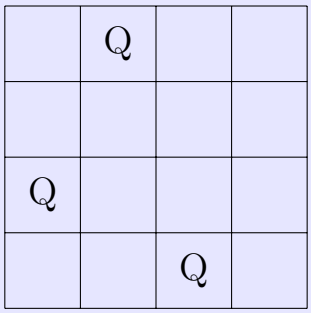
We can add the first queen in row 2. This is the state 2 _ _ _ :



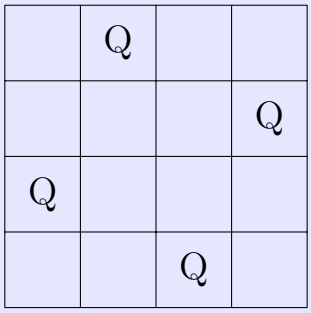
Then we can add the second queen in row 0. This is the state 2 0 _ _ :



Then we can add the third queen in row 3. This is the state 2 0 3 _ :



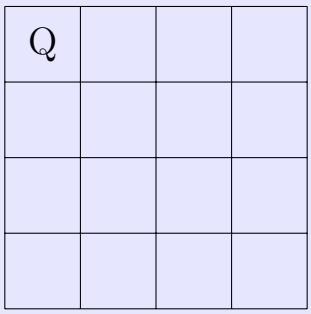
Finally, we can add the fourth queen in row 1. This is the state 2 0 3 1 :



We have now arrived at a goal state. None of the queens are attacking each other, and all 4 queens are on the board.

Let's try another example.

Example: Suppose we start with this state where we put the first queen in row zero. This is the state 0 _ _ _ :



Let's figure out what are this state's successors, and what are the successors of its successors. So when a queen is in row 0, there cannot be any other queens in that row. Also, there cannot be any other queens in that diagonal:

Q	X	X	X
	X		
		X	
			X

Therefore, the second queen can only be put into row 2 or row 3. Therefore, this state has two successors: 0 2 _ _ and 0 3 _ _. Let's suppose we put the second queen in row 2:

Q	X	X	X
	X		
	Q	X	
			X

There cannot be any other queen in row 2, and there cannot be any other queen in the diagonals of the second queen:

Q	X	X	X
X	X	X	
X	Q	X	X
X		X	X

Notice that the third queen has no possible position. This means that the state 0 2 _ _ has no successors.

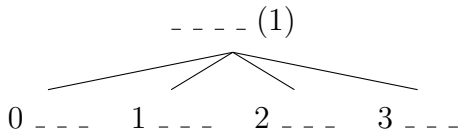
Taking the components of the search problem we defined for the 4-queen problem in this example with the components of a state we defined in the previous example, we have a complete incremental formulation of the 4-queens problem, and we can apply the backtracking algorithm.

We'll trace the backtracking algorithm on the 4-queens problem. We'll generate the search tree, and keep track of how the states change on the 4x4 board. We'll expand nodes in lexicographical order, and we'll number the states as we expand them:

_ _ _ _ (1)

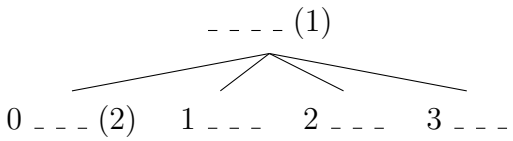
	x_0	x_1	x_2	x_3
0				
1				
2				
3				

The initial state is the empty state, and it has 4 possible successors since there are 4 possible rows to put the first queen on the board:



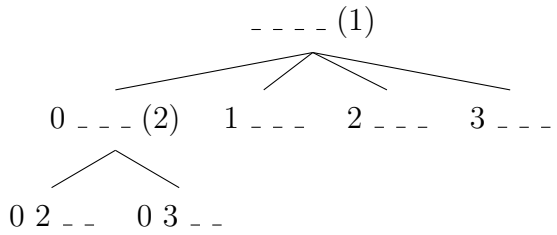
	x_0	x_1	x_2	x_3
0				
1				
2				
3				

Since we're expanding nodes in lexicographical order, then next node to expand is 0 _ _ _. We'll add the queen to the board, and also cross out all the positions that the other queens cannot be in:



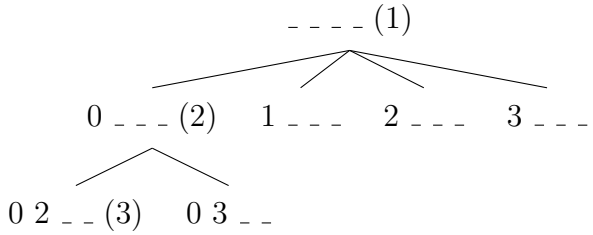
	x_0	x_1	x_2	x_3
0	Q	x	x	x
1	x	x		
2	x		x	
3	x			x

So the second queen can only be in row 2 or row 3, which means the state 0 _ _ _ has two successors:



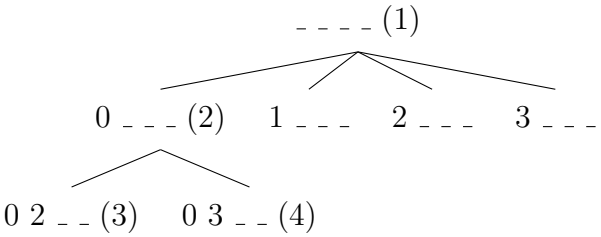
	x_0	x_1	x_2	x_3
0	Q	x	x	x
1	x	x		
2	x		x	
3	x			x

We expand state 0 2 _ _. We add the queen to the board, and also cross out all the positions that the other queens cannot be in:



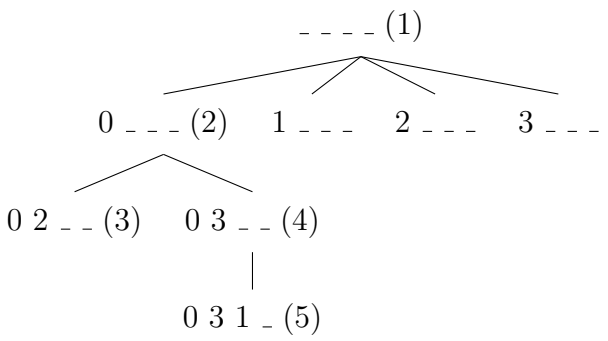
	x_0	x_1	x_2	x_3
0	Q	x	x	x
1	x	x	x	
2	x	Q	x	x
3	x	x	x	x

Unfortunately, adding the second queen in row 2 has resulted in no possible positions for the third queen, and therefore there are no successors for the state 0 2 _ _. At this point, we need to backtrack to the parent node, and try the other successors of the parent node. There is only one other successor of the parent node 0 _ _ _ and that is 0 3 _ _, so we expand this state next. We also cross out all the positions that the other queens cannot be in:



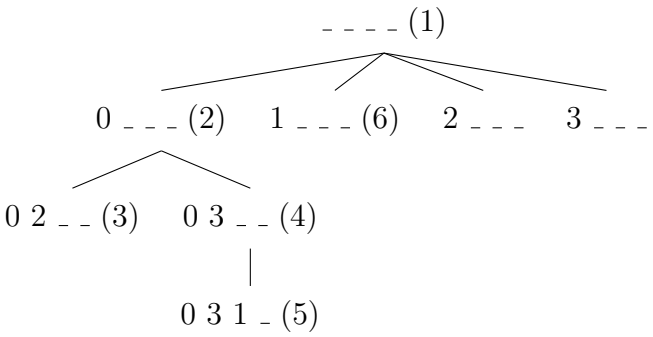
	x_0	x_1	x_2	x_3
0	Q	x	x	x
1	x	x		x
2	x	x	x	
3	x	Q	x	x

The third queen can only be in row 1. We add the queen to the board, and also cross out all the positions that the remaining queens cannot be in:



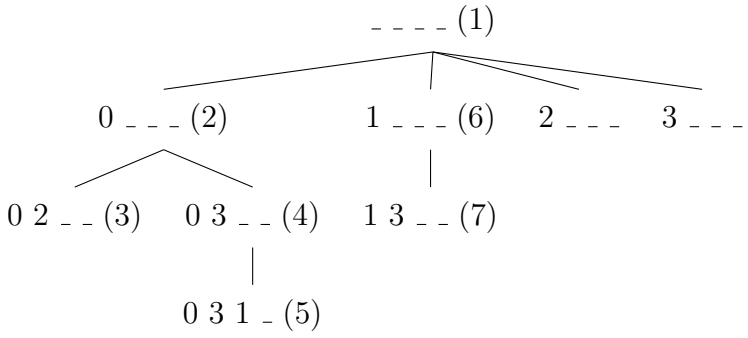
	x_0	x_1	x_2	x_3
0	Q	x	x	x
1	x	x	Q	x
2	x	x	x	x
3	x	Q	x	x

Unfortunately, there's no possible position for the last queen, so the state 0 3 1 _ does not have a successor. The search algorithm will backtrack all the way to the state _ _ _ _ and explore the next successor of that initial state, which is 1 _ _ _ :



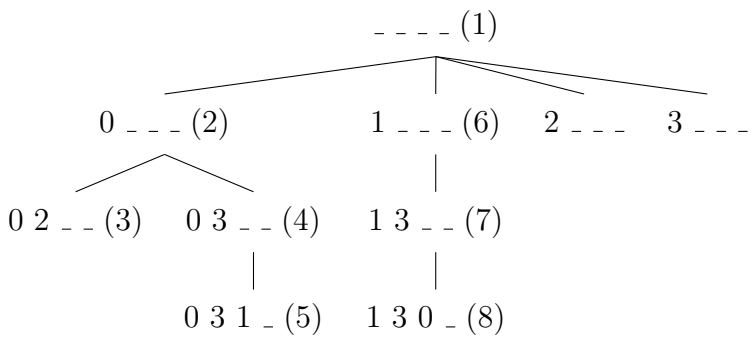
	x_0	x_1	x_2	x_3
0	x	x		
1	Q	x	x	x
2	x	x		
3	x		x	

There is only one possible place for the second queen, which is in row 3. We add the queen to the board, and also cross out all the positions that the remaining queens cannot be in:



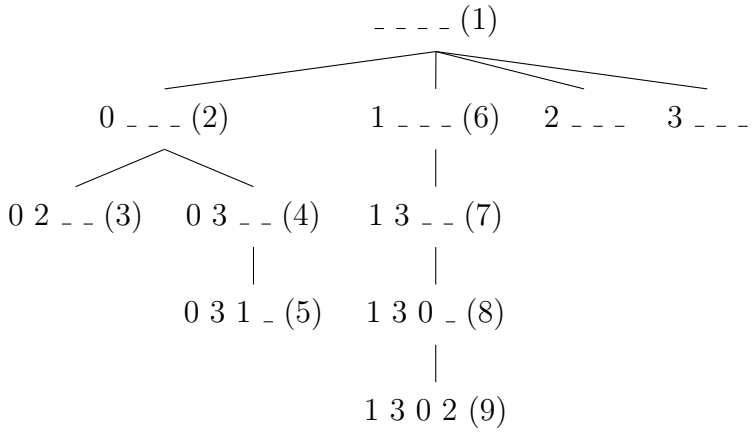
	x_0	x_1	x_2	x_3
0	x	x		
1	Q	x	x	x
2	x	x	x	
3	x	Q	x	x

There is only one possible place for the third queen, which is in row 0. We add the queen to the board, and also cross out all the positions that the remaining queens cannot be in:



	x_0	x_1	x_2	x_3
0	x	x	Q	x
1	Q	x	x	x
2	x	x	x	
3	x	Q	x	x

Finally, the only possible position for the last queen is in row 2. We add the queen to the board:



	x_0	x_1	x_2	x_3
0	x	x	Q	x
1	Q	x	x	x
2	x	x	x	Q
3	x	Q	x	x

At this point, we've reached a goal node and we stop searching.

Notice that backtracking search is just a special type of DFS. We go deep into the tree until we hit a dead end or a solution. If we hit a dead end, then we backtrack to the previous parent and try to explore the other successors of the parent.

Also notice that when we reach the state 0 2 _ _, we already realize that this state cannot possibly lead to any solutions, so we backtrack immediately and explore the other branches of the search tree. This makes backtracking search very efficient.

In contrast, the search algorithms that we used before like DFS treated the state like a black box. Those search algorithms could not recognize that the state 0 2 _ _ could not possibly lead to a solution. These search algorithms would keep exploring the successors of the state 0 2 _ _.

The backtracking search algorithm does much better than DFS since it prunes large portions of the search tree.

5 The Arc Consistency Definition

5.1 Motivation

First, why do we want to use arc consistency? It provides another way for us to solve a CSP.

In the previous section, I solved the 4-queens problem using backtracking search. In the leftmost sub-tree, we started by putting the first queen in row 0 and later on realized that this assignment does not lead to a solution. It turns out that, if we make use of arc consistency, we can detect this dead end without expanding the leftmost sub-tree at all.

Let me illustrate the idea of arc consistency using an example.

Example: Suppose that we put the first queen x0 in row 0. After this, due to the row and diagonal constraints, queen x1 cannot be 0 or 1. What if we put queen x1 in row 2? How can we figure out that this assignment does not lead to a solution?

	x_0	x_1	x_2	x_3
0	Q	X		
1		X		
2		Q		
3				

Since queen x0 is in row 0, we cannot put any queen in the top row. We also cannot put any queen in the diagonal containing x0.

	x_0	x_1	x_2	x_3
0	Q	X	X	X
1		X		
2		Q	X	
3				X

Similarly, since queen x1 is in row 2, we cannot put any queen in row 2, and we cannot put any queen in the two diagonals containing x2.

	x_0	x_1	x_2	x_3
0	Q	X	X	X
1		X	X	
2	X	Q	X	X
3			X	X

At this point, all possible positions for queen x2 have been eliminated. Therefore, the assignment $x0 = 0$ and $x1 = 2$ does not lead to a solution.

This example illustrates the idea behind arc-consistency. By examining each constraint, we can eliminate certain values that cannot satisfy the constraint from some domains. Reducing the domains this way can help us get closer to finding a solution or proving that a solution does not exist.

5.2 Handling Different Types of Constraints

I will introduce an arc-consistency algorithm which handles binary constraints only. We will categorize each constraint based on the number of variables in the constraint. A unary constraint involves one variable. A binary constraint involves two variables. There are constraints involving 3 or more variables.

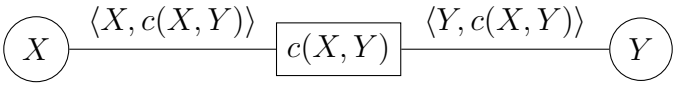
Restricting to binary constraints does not limit the applicability of our algorithm. In fact, this algorithm can solve any CSP. How does it handle other types of constraints? Handling a unary constraint is straightforward. A unary constraint says whether a variable can take a value or not. If a constraint does not allow a value, we can simply remove it from the variable's domain. What about a constraint involving 3 or more variables? It turns out that we can convert the constraint into an equivalent set of binary constraints by defining additional variables. How to convert these constraints is beyond the scope of this course. You can look it up if you are interested.

5.3 Notation for an Arc

Before I introduce the arc-consistency definition, let me introduce some notation for defining an arc.

Let X and Y be two variables. Let $c(X, Y)$ be a binary constraint. Depending on the meaning of the constraint, $c(X, Y)$ and $c(Y, X)$ may or may not represent the same constraint. For example, if the constraint is $X + Y = 5$, then $c(X, Y)$ and $c(Y, X)$ have the same meaning. If the constraint is X is divisible by Y, then $c(X, Y)$ and $c(Y, X)$ have different meanings.

Each constraint has two arcs associated with it. Take a look at the picture.



Let's look at the left arc: $\langle X, c(X,Y)\rangle$. The arc is denoted by a tuple in angle brackets. The second element in the tuple is the constraint. The first element in the tuple is the primary variable in the arc.

The primary variable tells us which arc we are looking at. For our example, if the primary variable is X, we are looking at the one on the left. If the primary variable is Y, we are looking at the one on the right.

5.4 Arc Consistency Definition

Let's look at the arc consistency definition.

Definition (Arc Consistency)**.** An arc $\langle X, c(X,Y)\rangle$ is consistent if and only if for every value v in D_X, there is a value w in D_Y such that (v, w) satisfies the constraint $c(X,Y)$.

There are two variables X and Y with domains D_X and D_Y. c(X,Y) is binary constraint. Consider the arc with X as the primary variable. This arc is consistent if and only if, for every value v in the domain of X, there is a value w in the domain of Y, such that v and w satisfy the constraint c.

To help you understand the definition, let's translate the definition into a predicate logic formula. Stop reading and write down the logical formula yourself. Then, keep reading for the answer.

$$\forall v \in D_X, \exists w \in D_Y, (v, w) \text{ satisfies } c(X,Y)$$

In predicate logic, the definition becomes the following. For all v in the domain of X, there exists w in the domain of Y, such that v and w satisfy the constraint c. We have mixed quantifiers here. The first one is universal and the second one is existential. When we verify the statement, we need to consider the quantifiers in order. First, we need to pick a value v in the domain of X. Then, we need to find a value w in the domain of Y such that the v and w satisfy the constraint.

Let's look at an example of applying the arc consistency definition.

Problem: Consider the constraint $X < Y$. Let $D_X = \{1, 2\}$ and $D_Y = \{1, 2\}$. Is the arc $\langle X, X < Y\rangle$ consistent?

(A) Yes.

(B) No.

(C) I don't know.

Solution: The correct answer is (B). The arc is NOT consistent.

To verify the definition, we start by picking any value of X and try to find a value of Y such that the two values satisfy the constraint.

If X = 1, then Y = 2 satisfy the constraint.

If X = 2, then no value of Y satisfies the constraint. Since we cannot find a corresponding value of Y for X = 2, this arc is not consistent. We can go ahead and remove 2 from the domain of X.

In general, if an arc is not consistent, we can remove at least one value from the primary variable's domain. Doing this will not rule out any solution. Reducing the domains is helpful as it takes us closer to finding a solution or showing that a solution does not exist.

6 The AC-3 Arc Consistency Algorithm

In the previous section, I introduced the Arc Consistency definition. The AC-3 algorithm uses the arc-consistency definition to reduce the variables' domains and bring us closer to solving the CSP.

The AC-3 algorithm was proposed in 1977 by Alan Mackworth, who is a professor emeritus in Computer Science at UBC. He's also one of the two authors in the Poole and Mackworth textbook. Also, you should know that Peter Van Beek, a CS professor at UW worked extensively on constraint satisfaction problems. The Russell and Norvig textbook has many citations of research by Professor Van Beek. I think it's very exciting that we learning concepts that are developed by our professors and colleagues.

Algorithm 1 The AC-3 Algorithm

```
put every arc in the set S.
while S is not empty do
    select and remove ⟨X, c(X, Y)⟩ from S
    remove every value in D_X that doesn't have a value in D_Y that satisfies the constraint c(X, Y)
    if D_X was reduced then
        if D_X is empty then return false
        for every Z ≠ Y, add ⟨Z, c'(Z, X)⟩ to S
return true
```

The goal of the AC-3 algorithm is to make every arc consistent by reducing the variables' domains.

The algorithm starts by putting every arc into a set called S. Remember that each binary constraint has two arcs associated with it. We need to put both arcs into S.

Next, we'll go into a loop. While the set S is not empty, select and remove one arc from the set. The order of removing arcs is not important. Any order will lead us to the same solution.

Suppose that the arc we removed is between X and Y, where X is the primary variable. Next, check whether this arc is consistent. If the arc is consistent, we can go back to the beginning of the loop and select another arc to remove.

If the arc is not consistent, we will reduce the domain of the primary variable (X's domain) until the arc becomes consistent. After reducing X's domain, if the domain becomes empty, we know that the problem has no solution and the algorithm can terminate.

If X's domain is not empty, we may need to add one or more arcs back to S. At this step, consider any arc where X is the secondary variable and Y is not the primary variable. If any such arc is not in S, we will add the arc to S.

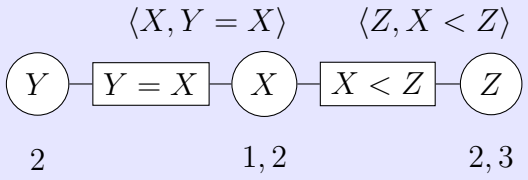
6.1 Why do we need to add arcs back to S?

The last few lines in the algorithm deserves some further discussion. In particular, after reducing a variable's domain, why do we need to add these arcs back into the set?

The reason is that, reducing a variable's domain may cause a previously consistent arc to become inconsistent. If this is a possibility, we need to add the arc back to S so that we will check it again.

Let's look at an example.

Example: The diagram shows three variables: Y, X, and Z. There are two binary constraints: $Y = X$, and $X < Z$. Consider two arcs, $\langle X, Y = X\rangle$, and $\langle Z, X < Z\rangle$.

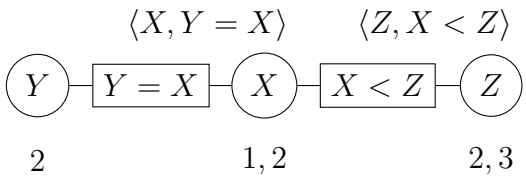


Right now, the right arc $\langle Z, X < Z\rangle$ is consistent, but the left arc $\langle X, Y = X\rangle$ is NOT consistent. Let's verify this quickly. For the right arc $\langle Z, X < Z\rangle$, if $Z = 2$, we can choose $X = 1$ and X is less than Z. If $Z = 3$, we can choose $X = 2$ and X is less than Z. Therefore, the arc is consistent. For the left arc $\langle X, Y = X\rangle$, if X is 2, we can choose $Y = 2$. If $X = 1$, no value of Y can satisfy the constraint.

To make the left arc consistent, we need to remove 1 from X's domain. After removing 1 from X's domain, the left arc $\langle X, Y = X\rangle$ becomes consistent. What about the right arc $\langle Z, X < Z\rangle$? The right arc becomes inconsistent because of this change. Let's verify it. If Z = 3, we can choose $X = 2$ and X is less than Z. If $Z = 2$, no value of X can satisfy the constraint $X < Z$.

Before we reduced X's domain, the right arc $\langle Z, X < Z\rangle$ was consistent. After reducing X's domain, the right arc $\langle Z, X < Z\rangle$ becomes inconsistent. In this case, we need to add $\langle Z, X < Z\rangle$ back to S so that we make it consistent by reducing Z's domain.

Finally, there is an edge case to consider. Let's look at the diagram again.



To make $\langle X, Y = X\rangle$ consistent, we need to reduce X's domain. After this change, we do not need to add the arc $\langle Y, Y = X\rangle$ to S. In other words, we do not need to add back the other arc for the same constraint. I will leave it as an exercise for you. Below is the exact statement that you will want to prove.

Problem: Consider two variables X and Y, and a binary constraint $c(X, Y)$. Assume that $\langle X, c(X, Y)\rangle$ is NOT consistent and $\langle Y, c(X, Y)\rangle$ is consistent. Assume that after removing one value from X's domain, $\langle X, c(X, Y)\rangle$ becomes consistent. Prove that after reducing X's domain, $\langle Y, c(X, Y)\rangle$ remains consistent.

6.2 Tracing the AC-3 Algorithm with x0 = 0

When I solved the 4-queens problems using backtracking search. I showed you that, assigning x0 = 0 does not lead to a solution. You can see this on the leftmost branch of the search tree. In this example, I will show you that, we can arrive at the same conclusion by running the AC-3 algorithm.

See the constraint graph for the 4-queens problem below. Let's start by keeping only 0 in the domain of x0. This means that we fix x0 to be 0.

There are 6 constraints, and there are 2 arcs per constraints. Therefore, at the start of the AC-3 algorithm, we will add 12 arcs to the set S.

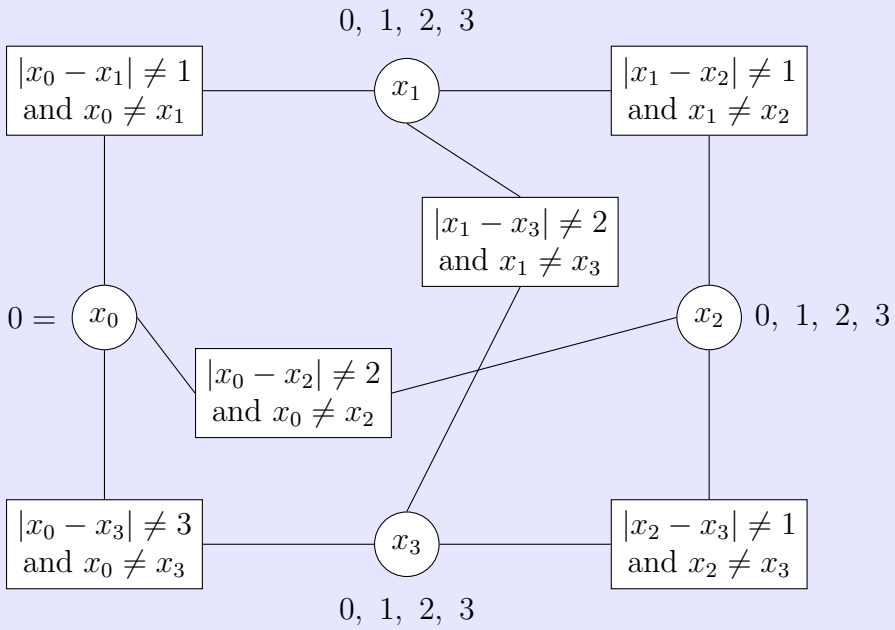
We will describe each step in words and also using a picture. The domain of each variable x_i is shown beside the corresponding node. We will indicate arcs in the set S with solid lines, and arcs not in the set S with dashed lines. The changes at each step are indicated by corresponding numbers.

Example:

To start, the set S has 12 arcs: $\langle x_0, c(x_0, x_1)\rangle$ $\langle x_1, c(x_0, x_1)\rangle$ $\langle x_0, c(x_0, x_2)\rangle$ $\langle x_2, c(x_0, x_2)\rangle$ $\langle x_0, c(x_0, x_3)\rangle$ $\langle x_3, c(x_0, x_3)\rangle$ $\langle x_1, c(x_1, x_2)\rangle$ $\langle x_2, c(x_1, x_2)\rangle$ $\langle x_1, c(x_1, x_3)\rangle$ $\langle x_3, c(x_1, x_3)\rangle$

$\langle x_2, c(x_2, x_3)\rangle$ $\langle x_3, c(x_2, x_3)\rangle$

The domains of the variables are: $x_0 \in \{0\}, x_1 \in \{0,1,2,3\}, x_2 \in \{0,1,2,3\}, x_3 \in \{0,1,2,3\}$.



Step 1: Remove $\langle x_1, c(x_0, x_1)\rangle$ from the set S.

$\langle x_1, c(x_0, x_1)\rangle$ is NOT arc-consistent. We need to remove 0 and 1 from the domain of x_1.

After reducing the domain of x_1, we possibly want to add back the following arcs: $\langle x_0, c(x_0, x_1)\rangle$ $\langle x_2, c(x_1, x_2)\rangle$ $\langle x_3, c(x_1, x_3)\rangle$.

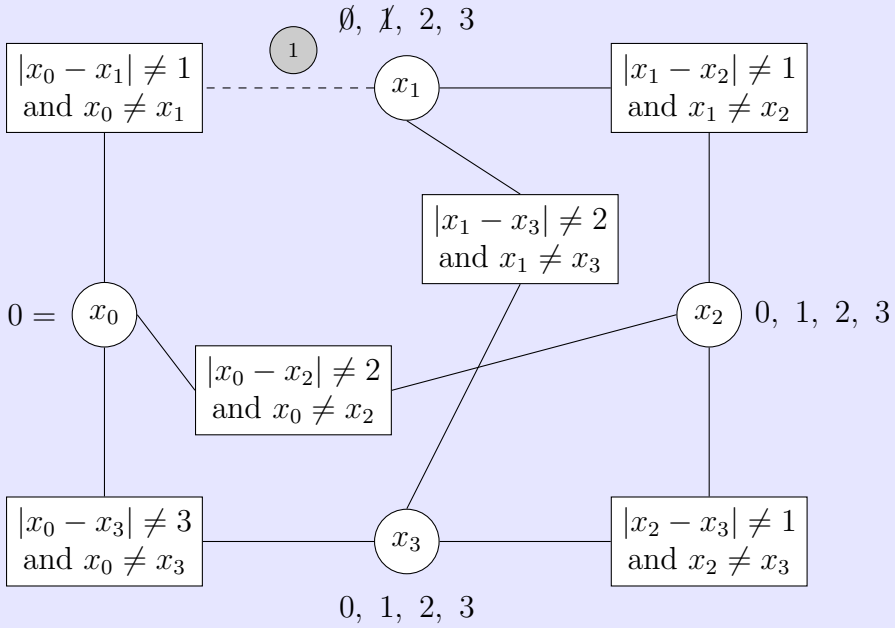
Due to the special case in the algorithm, we do not need to add back the arc for the same constraint where the other variable is the primary variable. Thus, we do not want to add back $\langle x_0, c(x_0, x_1)\rangle$.

We should add the other 2 arcs back to S. However, both are already in S and S is a set with no duplicates. Thus, we do not need to add them again.

The set S becomes: $\langle x_0, c(x_0, x_1)\rangle$ $\langle x_0, c(x_0, x_2)\rangle$ $\langle x_2, c(x_0, x_2)\rangle$ $\langle x_0, c(x_0, x_3)\rangle$ $\langle x_3, c(x_0, x_3)\rangle$ $\langle x_1, c(x_1, x_2)\rangle$ $\langle x_2, c(x_1, x_2)\rangle$ $\langle x_1, c(x_1, x_3)\rangle$ $\langle x_3, c(x_1, x_3)\rangle$ $\langle x_2, c(x_2, x_3)\rangle$ $\langle x_3, c(x_2, x_3)\rangle$

The domains of the variables become: $x_0 \in \{0\}, x_1 \in \{2,3\}, x_2 \in \{0,1,2,3\}, x_3 \in$

$\{0, 1, 2, 3\}$.



Step 2: Remove $\langle x_2, c(x_0, x_2)\rangle$ from the set S.

$\langle x_2, c(x_0, x_2)\rangle$ is NOT arc-consistent. We need to remove 0 and 2 from the domain of x_2.

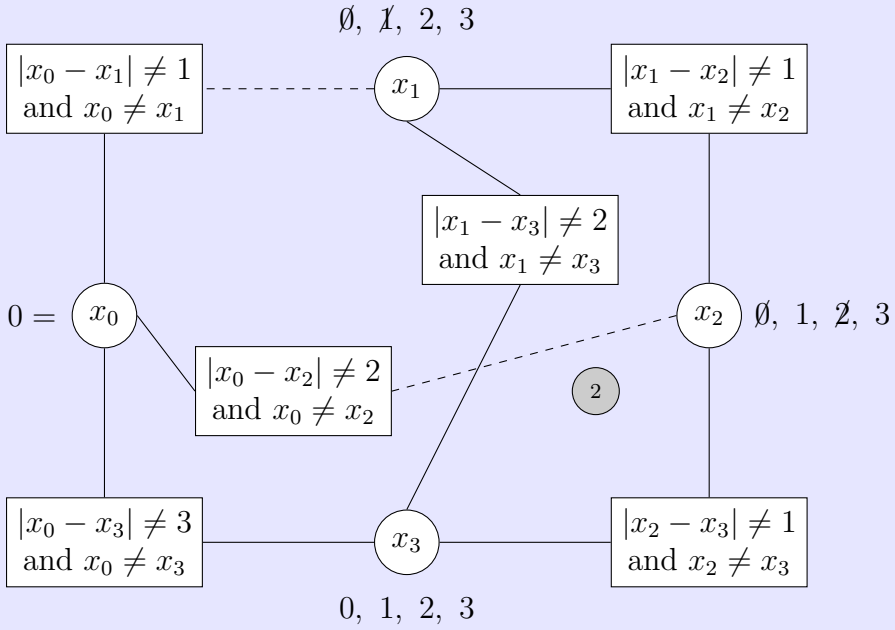
After reducing the domain of x_1, we possibly want to add back the following arcs: $\langle x_0, c(x_0, x_2)\rangle$ $\langle x_1, c(x_1, x_2)\rangle$ $\langle x_3, c(x_2, x_3)\rangle$.

Due to the special case in the algorithm, we do not need to add back the arc for the same constraint where the other variable is the primary variable. Thus, we do not want to add back $\langle x_0, c(x_0, x_2)\rangle$.

We should add the other 2 arcs back to S. However, both are already in S and S is a set with no duplicates. Therefore, we do not need add any arc back to S.

The set S becomes: $\langle x_0, c(x_0, x_1)\rangle$ $\langle x_0, c(x_0, x_2)\rangle$ $\langle x_0, c(x_0, x_3)\rangle$ $\langle x_3, c(x_0, x_3)\rangle$ $\langle x_1, c(x_1, x_2)\rangle$ $\langle x_2, c(x_1, x_2)\rangle$ $\langle x_1, c(x_1, x_3)\rangle$ $\langle x_3, c(x_1, x_3)\rangle$ $\langle x_2, c(x_2, x_3)\rangle$ $\langle x_3, c(x_2, x_3)\rangle$

The domains of the variables become: $x_0 \in \{0\}, x_1 \in \{2, 3\}, x_2 \in \{1, 3\}, x_3 \in \{0, 1, 2, 3\}$.



Step 3: Remove $\langle x_3, c(x_0, x_3)\rangle$ from the set S.

$\langle x_3, c(x_0, x_3)\rangle$ is NOT arc-consistent. We need to remove 0 and 3 from the domain of x_3.

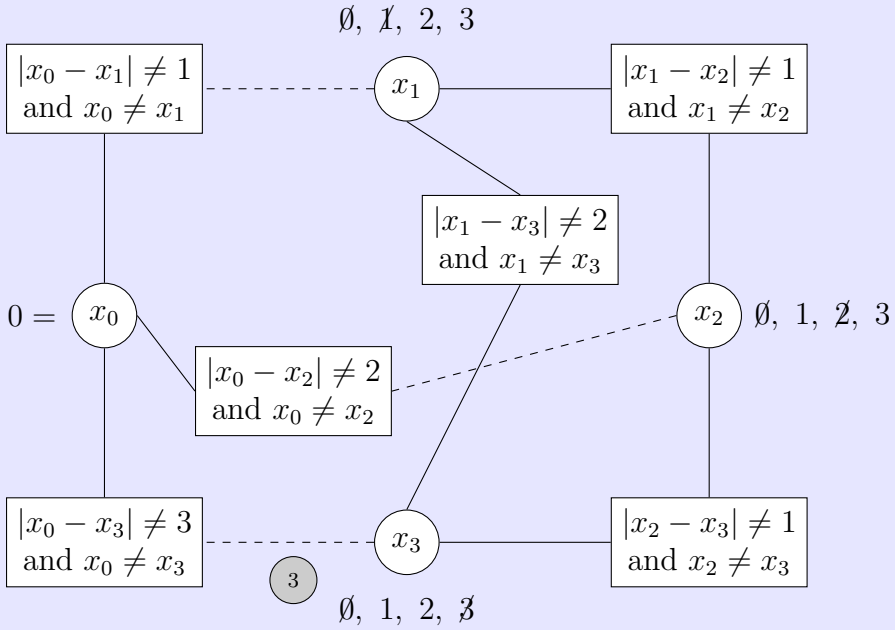
After reducing the domain of x_1, we possibly want to add back the following arcs: $\langle x_0, c(x_0, x_3)\rangle$ $\langle x_1, c(x_1, x_3)\rangle$ $\langle x_2, c(x_2, x_3)\rangle$.

Due to the special case in the algorithm, we do not need to add back the arc for the same constraint where the other variable is the primary variable. Thus, we do not want to add back $\langle x_0, c(x_0, x_3)\rangle$.

We should add the other 2 arcs back to S. However, both are already in S and S is a set with no duplicates. Therefore, we do not need add any arc back to S.

The set S becomes: $\langle x_0, c(x_0, x_1)\rangle$ $\langle x_0, c(x_0, x_2)\rangle$ $\langle x_0, c(x_0, x_3)\rangle$ $\langle x_1, c(x_1, x_2)\rangle$ $\langle x_2, c(x_1, x_2)\rangle$ $\langle x_1, c(x_1, x_3)\rangle$ $\langle x_3, c(x_1, x_3)\rangle$ $\langle x_2, c(x_2, x_3)\rangle$ $\langle x_3, c(x_2, x_3)\rangle$

The domains of the variables become: $x_0 \in \{0\}, x_1 \in \{2, 3\}, x_2 \in \{1, 3\}, x_3 \in \{1, 2\}$.

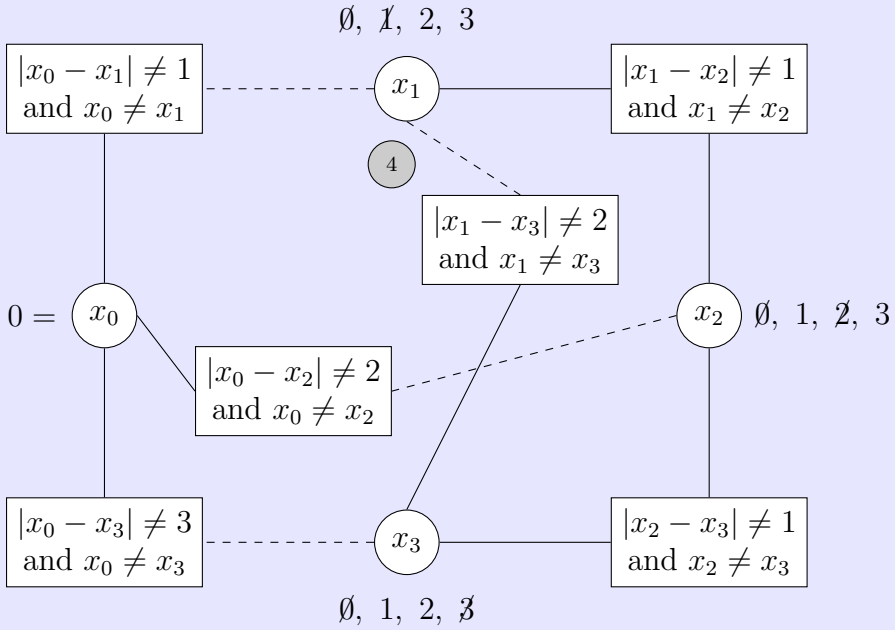


Step 4: Remove $\langle x_1, c(x_1, x_3)\rangle$ from the set S.

$\langle x_1, c(x_1, x_3)\rangle$ is arc-consistent. Thus, we do not reduce the domain of x_1, and we do not add any arc back to S.

The set S becomes: $\langle x_0, c(x_0, x_1)\rangle$ $\langle x_0, c(x_0, x_2)\rangle$ $\langle x_0, c(x_0, x_3)\rangle$ $\langle x_1, c(x_1, x_2)\rangle$ $\langle x_2, c(x_1, x_2)\rangle$ $\langle x_3, c(x_1, x_3)\rangle$ $\langle x_2, c(x_2, x_3)\rangle$ $\langle x_3, c(x_2, x_3)\rangle$

The domains of the variables become: $x_0 \in \{0\}, x_1 \in \{2, 3\}, x_2 \in \{1, 3\}, x_3 \in \{1, 2\}$.



Step 5: Remove $\langle x_2, c(x_2, x_3)\rangle$ from the set S.

$\langle x_2, c(x_2, x_3)\rangle$ is NOT arc-consistent. We need to remove 1 from the domain of x_2.

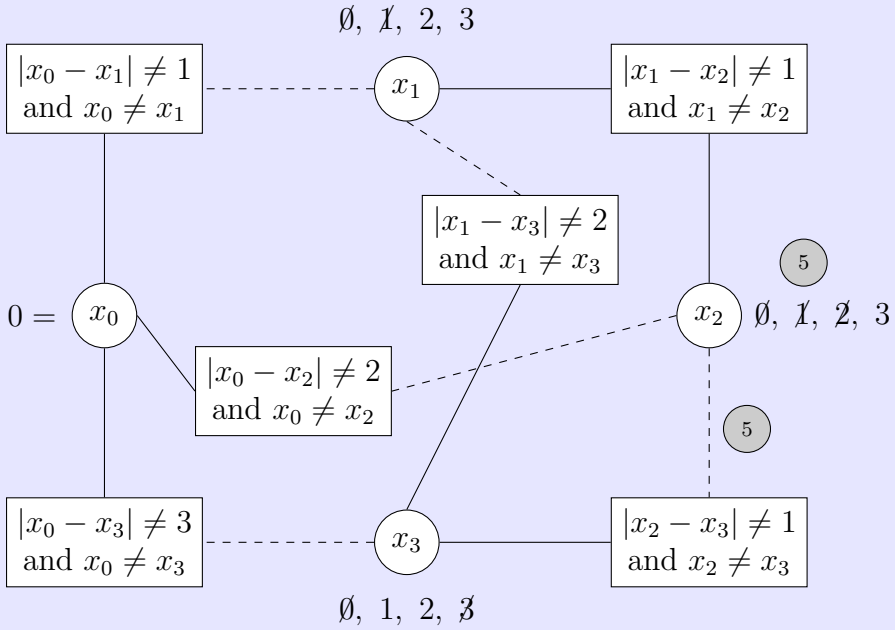
After reducing the domain of x_2, we possibly want to add back the following arcs: $\langle x_0, c(x_0, x_2)\rangle$ $\langle x_1, c(x_1, x_2)\rangle$ $\langle x_3, c(x_3, x_2)\rangle$. These are the arcs where x_2 is the secondary variable (x_2 is not the first element in the tuple).

We do not want to add back $\langle x_3, c(x_3, x_2)\rangle$ because there is no need to add back the same constraint for the other variable. (There is a proof for this.)

Therefore, we will add $\langle x_0, c(x_0, x_2)\rangle$ and $\langle x_1, c(x_1, x_2)\rangle$ back to S. These arcs are already in S and S is a set with no duplicates. Thus, S does not change.

The set S becomes: $\langle x_0, c(x_0, x_1)\rangle$ $\langle x_0, c(x_0, x_2)\rangle$ $\langle x_0, c(x_0, x_3)\rangle$ $\langle x_1, c(x_1, x_2)\rangle$ $\langle x_2, c(x_1, x_2)\rangle$ $\langle x_3, c(x_1, x_3)\rangle$ $\langle x_3, c(x_2, x_3)\rangle$

The domains of the variables become: $x_0 \in \{0\}, x_1 \in \{2, 3\}, x_2 \in \{3\}, x_3 \in \{1, 2\}$.



Step 6: Remove $\langle x_3, c(x_2, x_3)\rangle$ from the set S.

$\langle x_3, c(x_2, x_3)\rangle$ is NOT arc-consistent. We need to remove 2 from the domain of x_3.

After removing 2 from the domain of x_3, we possibly want to add back the following arcs: $\langle x_0, c(x_0, x_3)\rangle$ $\langle x_1, c(x_1, x_3)\rangle$ $\langle x_2, c(x_2, x_3)\rangle$.

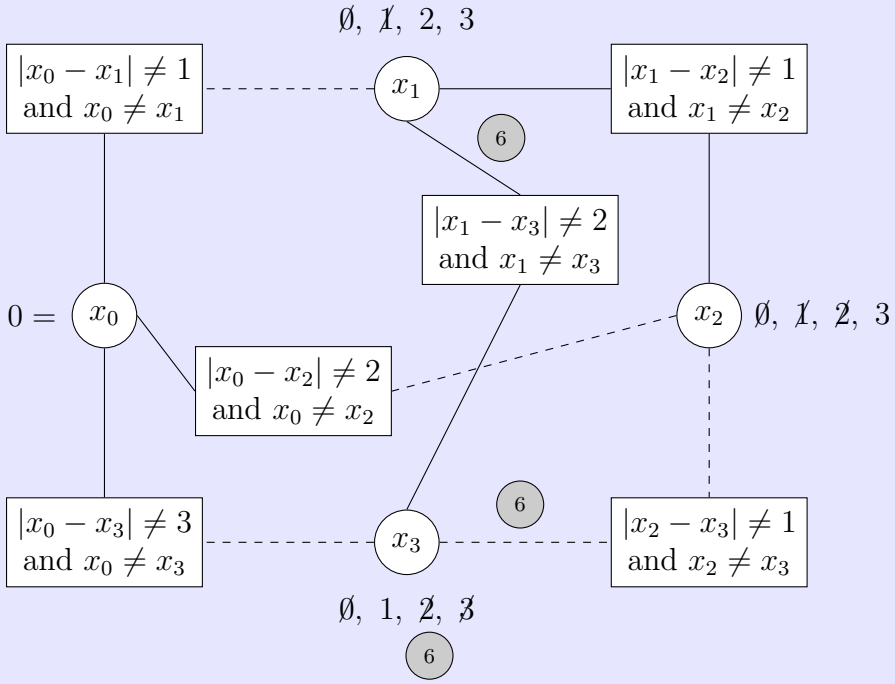
We do not want to add back $\langle x_0, c(x_0, x_3)\rangle$ to the set because it is already in S.

We do not want to add back $\langle x_2, c(x_2, x_3)\rangle$ because there is no need to add back the same constraint for the other variable. (There is a proof for this.)

Therefore, we will add $\langle x_1, c(x_1, x_3)\rangle$ back to S. S changes because it does not have this arc in it.

The set S becomes: $\langle x_0, c(x_0, x_1)\rangle$ $\langle x_0, c(x_0, x_2)\rangle$ $\langle x_0, c(x_0, x_3)\rangle$ $\langle x_1, c(x_1, x_2)\rangle$ $\langle x_2, c(x_1, x_2)\rangle$ $\langle x_3, c(x_1, x_3)\rangle$ $\langle x_1, c(x_1, x_3)\rangle$

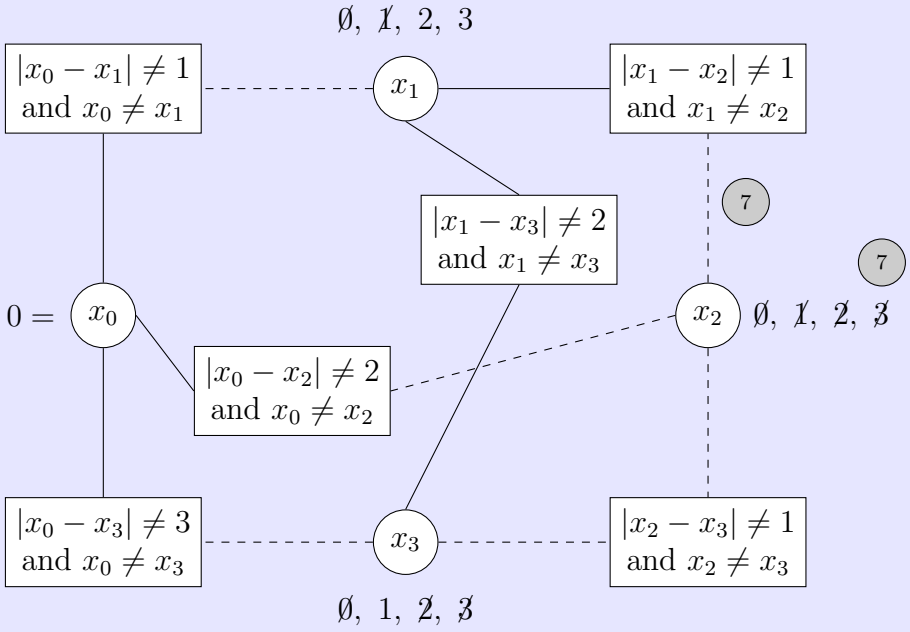
The domains of the variables become: $x_0 \in \{0\}, x_1 \in \{2, 3\}, x_2 \in \{3\}, x_3 \in \{1\}$.



Step 7: Remove $\langle x_2, c(x_1, x_2)\rangle$ from the set S.

$\langle x_2, c(x_1, x_2)\rangle$ is NOT arc-consistent. We need to remove 3 from the domain of x_2.

The domain of x_2 is empty. The AC-3 algorithm terminates and there is no solution (if we start with the initial assignment of $x_0 = 0$).



6.3 Properties of the AC-3 algorithm

Does the order in which the arcs are considered matter?

No. Any order will lead to the same solution.

Three possible outcomes of the AC-3 algorithm

1. A domain is empty. No solution exists.
2. Every domain has one value left. A unique solution exists.
3. Every domain has at least one value left and at least one domain has multiple values left. AC-3 inconclusive; need to do search or split domains to determine whether a solution exists.

Is AC-3 guaranteed to terminate?

Yes.

What is the complexity of the AC-3 algorithm?

Assume that the CSP has n variables, each variable's domain has at most d values, and there are c binary constraints.

At the beginning, there are $2c$ arcs in the set. For each arc, checking whether the arc is consistent requires $\mathcal{O}(d^2)$ time. (For each value in one domain, we need to look through each value in the other domain.)

After making an arc consistent and removing it from the set, we may need to add it back. Each arc may be added to the set d times. (To remove an arc from the set, we need to remove one value from a variable's domain. Since a domain has at most d values, we can only add an arc to the set at most d times.)

The overall run-time is $\mathcal{O}(c \cdot d^2 \cdot d) = \mathcal{O}(cd^3)$.

7 Practice Problems

1. Consider the constraint "X is divisible by Y" between two variables X and Y. For each scenario below, determine whether the arc $\langle X, c(X,Y)\rangle$ is consistent or not.

   (1) $D_X = \{10, 12\},\ D_Y = \{3, 5\}$

   (2) $D_X = \{10, 12\},\ D_Y = \{2\}$

   (3) $D_X = \{10, 12\},\ D_Y = \{3\}$

   (4) $D_X = \{10, 12\},\ D_Y = \{3, 5, 8\}$

2. (True or False) Let c be a binary constraint between X and Y. If $\langle X, c(X,Y)\rangle$ is arc-consistent, then $\langle Y, c(X,Y)\rangle$ is arc-consistent.

3. (True or False) Let c be a binary constraint between X and Y. If $\langle X, c(X,Y)\rangle$ is arc-consistent, then, after removing a value from D_Y, $\langle X, c(X,Y)\rangle$ is still arc-consistent.

4. (True or False) Let c be a binary constraint between X and Y. If $\langle X, c(X,Y)\rangle$ is arc-consistent, then, after removing a value from D_X, $\langle X, c(X,Y)\rangle$ is still arc-consistent.

5. Why don't we add the same constraint with the other variable as the primary variable back into the set S? Prove that there is no need to add the other arc — if the arc was consistent before the change, it is still consistent.

6. For the third outcome of AC-3, come up with an example where every domain has at least one value left, at least one domain has multiple values left and the problem has no solutions.

7. Prove or disprove: If $\langle X, c(X,Y)\rangle$ is arc-consistent, then $\langle Y, c(X,Y)\rangle$ is arc-consistent.