Proving Undecidability via Reductions

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Lecture 24

Based on work by J. Buss, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Outline

Learning Goals

By the end of this lecture, you should be able to:
Proving that a problem is undecidable by a reduction from the halting problem

- ▶ Define reduction.
- ▶ Describe at a high level how we can use reduction to prove that a decision problem is undecidable.
- ▶ Prove that a decision problem is undecidable by using a reduction from the halting problem.

Proving that other problems are undecidable

We proved that the Halting problem is undecidable.
How do we prove that another problem is undecidable?

- We could prove it from scratch, or
- We could prove that it is as difficult as the halting problem.
  Hence, it must be undecidable.

Proving undecidability via reductions

We will prove undecidability via reductions.
Problem A is reducible to problem B.

- Given an algorithm for solving B, we could use it to solve A.
- If B is decidable, then A is decidable.
- If A is undecidable, then B is undecidable.

Proving undecidability via reductions

Theorem: Problem X is undecidable.

Proof by Contradiction.

Assume that there is an algorithm B, which solves problem X.

We will construct algorithm A, which uses algorithm B to solve the halting problem. (Describe algorithm A.)

Since algorithm B solves problem X, algorithm A solves the halting problem.

This contradicts with the fact that the halting problem is undecidable. Therefore, algorithm B does not exist. $\qquad\square$

Example 1 of reduction proofs

The halting-no-input problem:

> *Given a program P which takes no input, does P halt?*

Theorem: The halting-no-input problem is undecidable.

Example 2 of reduction proofs

The both-halt problem:

> *Given two programs P_1 and P_2 which take no input, do both programs halt?*

Theorem: The both-halt problem is undecidable.

CQ 1 Does this reduction work?

CQ 1: To prove that the both-halt problem is undecidable, does the following reduction work?

Let P_1 run P with input I. Let P_2 do nothing and terminate immediately.

(A) Yes

(B) No

(C) I don't know.

CQ 2 Does this reduction work?

CQ 2: To prove that the both-halt problem is undecidable, does the following reduction work?

Let P_1 run P with input I. Let P_2 run an infinite loop and never terminate.

(A) Yes

(B) No

(C) I don't know.

Example 3 of reduction proofs

The exists-halting-input problem

Given a program P, does there exist an input I such that P halts with input I?

Theorem The exists-halting-input problem is undecidable.

Example 4 of reduction proofs

The partial-correctness problem

> *Given a Hoare triple $(\!|\,P\,|\!)\,C\,(\!|\,Q\,|\!)$,*
> *is the triple satisfied under partial correctness?*

Theorem: The partial-correctness problem is undecidable.

Revisiting the learning goals

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- ▶ Define reduction.
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- ▶ Prove that a decision problem is undecidable by using a reduction from the halting problem.