

Propositional Logic: Semantic Entailment

Alice Gao
Lecture 6

Based on work by J. Buss, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Outline

Propositional Logic: Semantic Entailment

- Learning goals

- Satisfying a set of formulas

- Definition of entailment

- Proving/Disproving an entailment

- Subtleties of an entailment

- Revisiting the learning goals

Learning goals

By the end of this lecture, you should be able to

(Semantic entailment)

- ▶ Determine if a set of formulas is satisfiable.
- ▶ Define semantic entailment.
- ▶ Explain subtleties of semantic entailment.
- ▶ Prove that a semantic entailment holds/does not hold by using the definition of semantic entailment, and/or truth tables.

Logical Deduction and Semantic Entailment

- ▶ Logic is the science of reasoning.
- ▶ The process of logical deduction is formalized by the notion of semantic entailment.
- ▶ Is a conclusion true based on a set of premises that we assume to be true?

Satisfaction of a Set of Formulas

Let Σ be a set of premises. Let φ be the conclusion.

A truth valuation t satisfies Σ (denoted $\Sigma^t = \top$):

For any formula α , if $\alpha \in \Sigma$, then $\alpha^t = \top$.

CQ 28 and 29 Is Sigma Satisfiable?

Semantic Entailment

Let Σ be a set of premises. Let φ be the conclusion.

Σ (semantically) entails φ (denoted $\Sigma \models \varphi$):

For any truth valuation t , if $\Sigma^t = T$, then $\varphi^t = T$.

Σ does not entail φ (denoted $\Sigma \not\models \varphi$):

There exists a truth valuation t such that $\Sigma^t = T$ and $\varphi^t = F$.

Prove an entailment

Consider the entailment $\Sigma \models \varphi$.

To prove that the entailment holds, we need to consider

- (A) Every truth valuation t under which $\Sigma^t = \text{T}$.
- (B) Every truth valuation t under which $\Sigma^t = \text{F}$.
- (C) One truth valuation t under which $\Sigma^t = \text{T}$.
- (D) One truth valuation t under which $\Sigma^t = \text{F}$.

Disprove an entailment

Consider the entailment $\Sigma \models \varphi$.

To prove that the entailment does NOT hold, we need to consider

- (A) Every truth valuation t under which $\Sigma^t = \text{T}$ and $\varphi = \text{T}$.
- (B) Every truth valuation t under which $\Sigma^t = \text{T}$ and $\varphi = \text{F}$.
- (C) One truth valuation t under which $\Sigma^t = \text{T}$ and $\varphi = \text{T}$.
- (D) One truth valuation t under which $\Sigma^t = \text{T}$ and $\varphi = \text{F}$.

Proving/disproving an entailment using a truth table

Let $\Sigma = \{(\neg(p \wedge q)), (p \rightarrow q)\}$, $x = (\neg p)$, and $y = (p \leftrightarrow q)$. Based on the truth table, which of the following statements is true?

- A) $\Sigma \models x$ and $\Sigma \models y$.
- B) $\Sigma \models x$ and $\Sigma \not\models y$.
- C) $\Sigma \not\models x$ and $\Sigma \models y$.
- D) $\Sigma \not\models x$ and $\Sigma \not\models y$.

| p | q | $(\neg(p \wedge q))$ | $(p \rightarrow q)$ | $(\neg p)$ | $(p \leftrightarrow q)$ |
|-----|-----|----------------------|---------------------|------------|-------------------------|
| F | F | T | T | T | T |
| F | T | T | T | T | F |
| T | F | T | F | F | F |
| T | T | F | T | F | T |

Proving/disproving an entailment using the definition

Exercise. Show that $\{(\neg(p \wedge q)), (p \rightarrow q)\} \models (\neg p)$.

Exercise. Show that $\{(\neg(p \wedge q)), (p \rightarrow q)\} \not\models (p \leftrightarrow q)$.

Subtleties of an entailment

Consider the entailment $\Sigma \models \varphi$.

Does the entailment hold under each of the following conditions?

1. Σ is the empty set.
2. Σ is not satisfiable.
3. φ is a tautology.
4. φ is a contradiction.

CQ 34-37 Subtleties of an entailment

Revisiting the learning goals

By the end of this lecture, you should be able to

(Semantic entailment)

- ▶ Determine if a set of formulas is satisfiable.
- ▶ Define semantic entailment.
- ▶ Explain subtleties of semantic entailment.
- ▶ Prove that a semantic entailment holds/does not hold by using the definition of semantic entailment, and/or truth tables.