Propositional Logic: Semantics

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Lecture 4

Based on work by J. Buss, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Outline

Admin stuff

FCC spectrum auction

- 20 billion revenue.
- Goal is to re-purpose radio spectrums.
- 2 auctions
- A computational problem in the buy back auction
- Satisfiability problems

Talk by Kevin Leyton-Brown
https://www.youtube.com/watch?v=u1-jJOivP70

Learning goals

By the end of this lecture, you should be able to

(Truth valuation, truth table, and valuation tree)

- Define a truth valuation.
- Determine the truth value of a formula given a truth valuation.
- Give a truth valuation under which a formula is true or false.
- Draw a truth table given a formula.
- Draw the valuation tree given a formula.

Learning goals (continued)

By the end of this lecture, you should be able to

(Properties of formulas)

- Define tautology, contradiction, and satisfiable formula.
- Determine if a given formula is a tautology, a contradiction, and/or a satisfiable formula.

The meaning of well-formed formulas

To interpret a formula, we have to give meanings to the propositional variables and the connectives.

A truth valuation assigns true/false to every propositional variable.

What is a truth valuation, intuitively?

Have you watched the TV series “Fringe”?

A truth valuation defines a parallel universe.

Our universe: The sky is blue and pigs do not fly.
Parallel universe 1: The sky is red, and pigs do not fly.
Parallel universe 2: The sky is blue, and pigs fly.

Definition of a truth valuation

A (truth) valuation is a function $t : \mathcal{P} \mapsto \{\mathsf{F}, \mathsf{T}\}$ from the set of all proposition variables $\mathcal{P}$ to the set $\{\mathsf{F}, \mathsf{T}\}$.

Consider a truth valuation t.

- For a propositional variable p,
  the value of p under t is denoted by $t(p)$ and p^t.
- For a well-formed formula φ (which is not necessarily a propositional variable), the value of φ under t is denoted by φ^t.

Pro tip: Always use φ^t.

Truth tables for connectives

Every line in a truth table corresponds to a truth valuation.

The unary connective $\neg$:

α	$(\neg\alpha)$
T	F
F	T

The binary connectives $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$:

α	β	$(\alpha \wedge \beta)$	$(\alpha \vee \beta)$	$(\alpha \rightarrow \beta)$	$(\alpha \leftrightarrow \beta)$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

A structural induction example

Theorem: Fix a truth valuation t. The truth value of every formula α is in $\{\mathsf{F}, \mathsf{T}\}$.

Prove this yourself as an exercise.

CQs 14-15 Evaluating the truth value of a formula

Inclusive OR, Exclusive OR, and Biconditional

α	β	$(\alpha \vee \beta)$	$((\alpha \wedge (\neg\beta)) \vee ((\neg\alpha) \wedge \beta))$	$(\alpha \leftrightarrow \beta)$
F	F	F	F	T
F	T	T	T	F
T	F	T	T	F
T	T	T	F	T

- Difference between the inclusive OR and the exclusive OR?
- Relationship between the exclusive OR and the bi-conditional?

CQ 16 Understanding an implication

Assume that the following proposition is true.

If Alice is rich, she pays your tuition.

Assuming that Alice is rich, does she pay your tuition?

(A) Yes

(B) No

(C) Maybe

CQ 17 Understanding an implication

Assume that the following proposition is true.

If Alice is rich, she pays your tuition.

Assuming that Alice is not rich, does she pay your tuition?

(A) Yes

(B) No

(C) Maybe

Proving that an implication is true

Consider the following implications.

(A) "If Alice is rich, she pays your tuition."

(B) $((p \wedge q) \to (p \vee q))$.

1. How do you prove that the implication is false?

2. How do you prove that the implication is true?

Review questions on the implication

- Think of an implication as a promise that someone made to you. In what case can you prove that the promise has been broken (i.e. the implication is false)?
- When the premise is true, what is the relationship between the truth value of the conclusion and the truth value of the implication?
- When the premise is false, the implication is vacuously true. Could you come up with an intuitive explanation for this?
- If the conclusion is true, is the implication true or false?
- The implication $(a \rightarrow b)$ is logically equivalent to $((\neg a) \vee b)$. Does this equivalent formula make sense to you? Explain.

Tautology, Contradiction, Satisfiable

A formula α is a tautology:

For every truth valuation t, $\alpha^t = \mathsf{T}$.

A formula α is a contradiction:

For every truth valuation t, $\alpha^t = \mathsf{F}$.

A formula α is satisfiable:

There exists a truth valuation t such that $\alpha^t = \mathsf{T}$.

Properties and truth tables

- Tautology: Each formula is
  true in EVERY/AT LEAST ONE/NO row of its truth table
  false in EVERY/AT LEAST ONE/NO row of its truth table.
- Satisfiable but not a tautology: Each formula is
  true in EVERY/AT LEAST ONE/NO row of its truth table
  false in EVERY/AT LEAST ONE/NO row of its truth table.
- Contradiction: Each formula is
  true in EVERY/AT LEAST ONE/NO row of its truth table
  false in EVERY/AT LEAST ONE/NO row of its truth table.

Is a formula a tautology, contradiction, and/or satisfiable?

Three approaches:

- Reasoning to get a quick answer
- Truth table
- Valuation tree: a more compact truth table

Reasoning to get a quick answer

I found a valuation for which the formula is true. Does the formula have each property below?

▶ Tautology	YES	NO	MAYBE
▶ Contradiction	YES	NO	MAYBE
▶ Satisfiable	YES	NO	MAYBE

I found a valuation for which the formula is false. Does the formula have each property below?

▶ Tautology	YES	NO	MAYBE
▶ Contradiction	YES	NO	MAYBE
▶ Satisfiable	YES	NO	MAYBE

CQ 21 Getting a quick answer

Simplifying a formula

Rather than filling out an entire truth table, we can simplify a formula in many situations:

$$
\begin{array}{ccc}
 & & p \to T \equiv \\
p \wedge T \equiv & p \vee T \equiv & p \to F \equiv \\
p \wedge F \equiv & p \vee F \equiv & T \to p \equiv \\
p \wedge p \equiv & p \vee p \equiv & F \to p \equiv \\
 & & p \to p \equiv
\end{array}
$$

We can evaluate a formula by constructing a valuation tree using these rules.

A valuation tree

Show that $(((p \land q) \to (\neg r)) \land (p \to q)) \to (p \to (\neg r)))$ is a tautology by using a valuation tree.

Case 1: Suppose $t(p) = \mathrm{T}$.
The formula becomes $(((q \to (\neg r)) \land q) \to (\neg r))$.
If $t(q) = \mathrm{T}$, the formula is T (Check!).
If $t(q) = \mathrm{F}$, the formula is T (Check!).

Case 2: Suppose $t(p) = \mathrm{F}$. The formula is T. (Check!).

The formula is true for every valuation and is a tautology.

Note: We never had to consider the truth value of r in our analysis.

Additional exercises

Determine if each formula is a tautology, a contradiction, or satisfiable but not a tautology. Justify your answer.

1. $((((p \land q) \rightarrow r) \land (p \rightarrow q)) \rightarrow (p \rightarrow r))$
2. $(p \land (\neg p))$

Revisiting the Learning goals

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CQ 18 and 19 Properties of formulas

CQ 20 If I found a valuation under which the formula is true/false