Theorem 0: Every well-formed formula starts with a propositional variable or an opening bracket.

Theorem 1: Every well-formed formula has an equal number of opening and closing brackets.

Theorem 2: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

Theorem: There is a unique way to construct every well-formed formula.

Proof by structural induction:

Let x be a well-formed formula. We want to prove that there is a unique way to construct x as a well-formed formula.

Base case: x is a propositional variable.

<u>Can we construct x as (¬a) for a well-formed formula a by applying negation as the last step?</u>

If we construct a formula by applying negation as the last step, then it has to be of the form (¬a) and has to contain at least 3 symbols. However, the formula x only has 1 symbol. Therefore, we cannot construct x by applying negation as the last step.

<u>Can we construct x as (a*b) for well-formed formulas a and b by applying a binary connective * as the last step?</u>

If we construct a formula by applying a binary connective as the last step, then it has to be of the form (a*b) and has to contain at least 5 symbols. However, the formula x only has 1 symbol. Therefore, we cannot construct x by applying a binary connective as the last step.

Induction step:

Case 1: x is (¬a) for a well-formed formula a.

Induction hypothesis: Assume that there is a unique way to construct a.
We need to prove that there is a unique way to construct (¬a).

We already know one way to construct (¬a): construct a, and apply negation as the last step. We need to show that there is no other way to construct (¬a).

<u>Can we construct (¬a) as a propositional variable?</u>

If we construct a formula as a propositional variable, then it has to have 1 symbol. However, the formula (¬a) has at least 3 symbols. So we cannot construct (¬a) as a propositional variable.

<u>Can we construct (¬a) as (c*d) for well-formed formulas c and d by applying a binary connective * as the last step?</u>

Suppose that we can construct (¬a) as (c*d) for well-formed formulas c and d by applying a binary connective * as the last step. Then the first symbol in c must be ¬. By Theorem 0, c is not a well−formed formula, which contradicts with our assumption.

Therefore, we cannot construct (¬a) by applying a binary connective as the last step.

Case 2: x is (a*b) for well-formed formulas a and b where * is one of ∧, ∨, →, and ↔.

Induction hypothesis: assume that there is a unique way to construct a and b respectively. We need to prove that there is a unique way to construct (a*b).

We already know one way to construct (a*b): construct a and b separately, and apply * as the last step. We need to show that there is no other way to construct (a*b).

<u>Can we construct (a*b) as a propositional variable?</u>

If we construct a formula as a propositional variable, then it has to have 1 symbol. However, the formula (a*b) has at least 3 symbols. So we cannot construct (a*b) as a propositional variable.

<u>Can we construct (a*b) as (¬c) for well-formed formula c by applying negation as the last step?</u>

Suppose that we can construct (a*b) by applying negation as the last step. Then the first symbol of a has to be ¬. By Theorem 0, a is not a well-formed formula, which contradicts our assumption.

Therefore, we cannot construct (a*b) as (¬c) for well-formed formula c by applying negation as the last step.

<u>Can we construct (a*b) as (c@d) for well-formed formulas c and d by applying a binary connective @ that is different from * as the last step?</u>

Suppose that we can construct (a*b) as (c@d) for well-formed formulas c and d. Then the binary connective @ has to be either in a or in b.

If the binary connective @ is in a, then c is a proper prefix of a. By Theorem 2, c has more opening than closing brackets. Thus, c is not a well-formed formula, which contradicts with our assumption.

If the binary connective @ is in b, then let b = m@n. Then c = a*m and d = n.

Let op(x) and cl(x) denote the number of opening and closing brackets in a formula x. We will prove that c is not a well-formed formula.

a is a well-formed formula. By Theorem 1, op(a) = cl(a). m is a proper prefix of the well-formed formula b. By Theorem 2, op(m) > cl(m). Thus, we have that

op(c)
= op(a) + op(m)
= cl(a) + op(m) By Theorem 1
> cl(a) + cl(m) By Theorem 2
= cl(c).

Thus, c has more opening than closing brackets. By Theorem 2, c is not a well-formed formula.

QED