Semantic entailment. Sept 26.

Show that $\{(p \to q), (q \to r)\} \models (p \to r)$

Proof ①:

p	q	r	$p \to q$	$q \to r$	$p \to r$	
0	0	0	1	1	1	*
0	0	1	1	1	1	*
0	1	0	1	0	1	
0	1	1	1	1	1	*
1	0	0	0	1	0	
1	0	1	0	1	1	
1	1	0	1	0	0	
1	1	1	1	1	1	*

The * marks all the rows in which $p \to q$ and $q \to r$ are both true. $(p \to r)$ is true in all of the * rows, So the entailment holds

QED.

Proof ②: We prove this by contradiction.
Assume that the entailment does not hold.
There is a truth valuation t such that

$$(p \to q)^t = T, \quad (q \to r)^t = T \text{ and } (p \to r)^t = F.$$

If $(p \to r)^t = F$, then it has to be that $p^t = T$ and $r^t = F$.

If $(p \to q)^t = T$ and $p^t = T$, then $q^t = T$.
If $(q \to r)^t = T$ and $r^t = F$, then $q^t = F$
This is a contradiction.

Our assumption is false and the entailment holds

QED.

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Show that

$$\{((p \to (\neg q)) \vee r),\ (q \wedge (\neg r)),\ (p \Leftrightarrow r)\} \not\models (p \wedge (q \to r))$$

Proof: Consider a truth valuation t such that
$p^t = F$, $q^t = T$, and $r^t = F$.

$$((p \to (\neg q)) \vee r)^t = ((F \to F) \vee F) = T$$
$$(q \wedge (\neg r))^t = (T \wedge T) = T$$
$$(p \Leftrightarrow r)^t = (F \Leftrightarrow F) = T$$

$$(p \wedge (q \to r))^t = (F \wedge (T \to F)) = F$$

The premises are true but the conclusion is false,
so the entailment does not hold.

QED