Propositional Logic: Tautological Consequence and Translations

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Lecture 6

Outline

Learning goals

By the end of this lecture, you should be able to

- Determine if a set of formulas is satisfiable.
- Define tautological consequence. Explain subtleties of tautological consequence.
- Prove that a tautological consequence holds/does not hold by using the definition of tautological consequence, and/or truth tables.
- Translate an English sentence with no logical ambiguity into a propositional formula.
- Translate an English sentence with logical ambiguity into multiple propositional formulas and prove that the propositional formulas are not tautologically equivalent.

Logical Deduction and Tautological Consequence

- Logic is the science of reasoning.
- The process of logical deduction is formalized by the notion of tautological consequence.
- Can we deduce a conclusion based on a set of premises?

Outline

Satisfying a Set of Formulas

Let Σ denote any set of formulas.

$$\Sigma^t = \begin{cases} 1, & \text{if for each } B \in \Sigma, B^t = 1, \\ 0, & \text{otherwise} \end{cases}$$

What does $\Sigma^t = 0$ mean? It means that there exists a formula $B \in Sigma$ such that $B^t = 0$ (at least one formula in Σ is false), not that for all $B \in Sigma$, $B^t = 0$ (every formula in Σ is false).

Definition (Satisfiability)

Σ is satisfiable if and only if there is some truth valuation t such that $\Sigma^t = 1$. When $\Sigma^t = 1$, t is said to satisfy Σ.

CQ Is Sigma Satisfiable?

Outline

Tautological Consequence

Definition (Tautological Consequence)

Suppose $\Sigma \subseteq Form(L^p)$ and $A \in Form(L^p)$.

A is a tautological consequence of Σ (that is, of the formulas in Σ), written as $\Sigma \models A$, if and only if
for any truth valuation t, $\Sigma^t = 1$ implies $A^t = 1$.

Σ does not logically imply A (denoted $\Sigma \not\models A$):
There exists a truth valuation t such that $\Sigma^t = 1$ and $A^t = 0$.

Tautological Equivalence

A and B are (tautologically) equivalent if and only if $A \models\!\!\!\!\!\!\!\dashv B$ holds.

$A \models\!\!\!\!\!\!\!\dashv B$ denotes $A \models B$ and $B \models A$.

Outline

CQ: Prove a tautological consequence

CQ: Consider the tautological consequence $\Sigma \vDash A$.
To prove that the tautological consequence holds, we need to consider

(A) Every truth valuation t under which $\Sigma^t = 1$.
(B) Every truth valuation t under which $\Sigma^t = 0$.
(C) One truth valuation t under which $\Sigma^t = 1$.
(D) One truth valuation t under which $\Sigma^t = 0$.

CQ: Disprove a tautological consequence

CQ: Consider the tautological consequence $\Sigma \vDash A$.
To prove that the tautological consequence does NOT hold, we need to consider

(A) Every truth valuation t under which $\Sigma^t = 1$ and $A^t = 1$.
(B) Every truth valuation t under which $\Sigma^t = 1$ and $A^t = 0$.
(C) One truth valuation t under which $\Sigma^t = 1$ and $A^t = 1$.
(D) One truth valuation t under which $\Sigma^t = 1$ and $A^t = 0$.

CQ: Proving/disproving a tautological consequence using a truth table

CQ: Let $\Sigma = \{\neg(p \wedge q), p \to q\}$, $x = \neg p$, and $y = p \leftrightarrow q$.
Based on the truth table, which of the following statements is true?

A) $\Sigma \models x$ and $\Sigma \models y$.

B) $\Sigma \models x$ and $\Sigma \not\models y$.

C) $\Sigma \not\models x$ and $\Sigma \models y$.

D) $\Sigma \not\models x$ and $\Sigma \not\models y$.

p	q	$(\neg(p \wedge q))$	$(p \to q)$	$(\neg p)$	$(p \leftrightarrow q)$
0	0	1	1	1	1
0	1	1	1	1	0
1	0	1	0	0	0
1	1	0	1	0	1

Prove a tautological consequence using the definition

Exercise. Show that $\{(\neg(p \wedge q)), (p \to q)\} \models (\neg p)$.
Consider any truth valuation t under which $(\neg(p \wedge q))^t = 1$ and $(p \to q)^t = 1$. We want to prove that $(\neg p)^t = 1$.

Since $(\neg(p \wedge q))^t = 1$, we have that $(p \wedge q)^t = 0$ by the truth table of $\neg$. By the truth table of $\wedge$, there are three possible cases: (1) $p^t = 0, q^t = 1$, (2) $p^t = 1, q^t = 0$, and (3) $p^t = 0, q^t = 0$.

We assumed that $(p \to q)^t = 1$. By the truth table of $\to$, it is impossible that $p^t = 1$ and $q^t = 0$. Therefore, the only two cases left are: $p^t = 0, q^t = 1$, and $p^t = 0, q^t = 0$. In both cases, $p^t = 0$, and therefore $(\neg p)^t = 1$.
QED

Prove a tautological consequence using the definition

Exercise. Show that $\{(\neg(p \wedge q)), (p \rightarrow q)\} \vDash (\neg p)$.
We prove this by contradiction. Assume that there exists a truth valuation such that $(\neg(p \wedge q))^t = 1$, $(p \rightarrow q)^t = 1$, and $(\neg p)^t = 0$.

Since $(\neg p)^t = 0$, $p^t = 1$ by the truth table of $\neg$. Since $(p \rightarrow q)^t = 1$, we have that $q^t = 1$ by the truth table of $\rightarrow$. By the truth table of $\vee$, $(p \wedge q)^t = 1$. By the truth table of $\neg$, $(\neg(p \wedge q))^t = 0$, which contradicts with our assumption that $(\neg(p \wedge q))^t = 1$.

Therefore, the tautological consequence holds.
QED

Disprove a tautological consequence using the definition

Exercise. Show that $\{(\neg(p \wedge q)), (p \rightarrow q)\} \not\models (p \leftrightarrow q)$.
We need to find a truth valuation such that $(\neg(p \wedge q))^t = 1$, $(p \rightarrow q)^t = 1$ and $(p \leftrightarrow q)^t = 0$.

Consider the truth valuation $p^t = 0$ and $q^t = 1$.

$(p \wedge q)^t = 0$ by the truth table of $\wedge$. $(\neg(p \wedge q))^t = 1$ by the truth table of $\neg$. $(p \rightarrow q)^t = 1$ by the truth table of $\rightarrow$. $(p \leftrightarrow q)^t = 0$ by the truth table of $\leftrightarrow$.
QED

Disproving propositional logical consequence

A student is trying to prove that $\{(A \to B)\} \not\models (B \to A)$ where A and B are well-formed predicate formulas. The student starts the proof by writing down the following sentence.

> *There exists a truth valuation t such that $B^t = 1$ and $A^t = 0$.*

Is the above sentence true (a valid claim)?

(A) Yes, it is true.

(B) No, it is false.

(C) There is not enough information to tell.

Outline

Subtleties of a Tautological Consequence

Consider the tautological consequence $\Sigma \vDash A$.
Does the tautological consequence hold
under each of the following conditions?

1. Σ is the empty set.
2. Σ is not satisfiable.
3. A is a tautology.
4. A is a contradiction.

CQ Subtleties of a Tautological Consequence

Outline

English sentences with no logical ambiguity

Translate the following sentences to propositional formulas.

1. Nadhi eats a fruit if the fruit is an apple.
   Nadhi eats every apple in the world.
   Nadhi may eat other fruits, such as bananas.
   Nadhi eats a fruit only if the fruit is an apple.
   The only fruit that Nadhi eats is apple. Nadhi does not eat bananas.
   Nadhi may not eat every apple in the world.
2. Soo-Jin will eat an apple or an orange but not both.
3. If it is sunny tomorrow, then I will play golf, provided that I am relaxed.
   What does "provided" modify? (1) The entire sentence. (2) only the play golf part (3) modifies playing golf together with it is sunny tomorrow.

English sentences with logical ambiguity

Give multiple translations of the following sentences into propositional logic.

1. Sidney will carry an umbrella unless it is sunny.
   Will Sidney carry an umbrella when it is sunny?
2. Pigs can fly and the grass is red or the sky is blue.

Translations: A reference page

- $\neg p$: p does not hold; p is false; it is not the case that p
- $p \land q$: p but q; not only p but q; p while q; p despite q; p yet q; p although q
  The English words express subtle emotions, but both parts are true.
- $p \lor q$: p or q or both; p and/or q;
- $p \to q$: p implies q; q if p; p only if q; q when p; p is sufficient for q; q is necessary for p
  If and only if. Sufficient and necessary.
- $p \leftrightarrow q$: p is equivalent to q; p exactly if q; p is necessary and sufficient for q

Revisiting the learning goals

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