Propositional Logic: Semantics

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Lecture 4, September 19, 2017

Announcements

The roadmap of propositional logic

FCC spectrum auction — an application of propositional logic

To repurpose radio spectrums

2 auctions:

- one to buy back spectrums from broadcasters
- the other to sell spectrums to telecoms

A computational problem in the buy back auction: If I pay these broadcasters to go off air, could I repackage the spectrums and sell to telecoms? Could I lower your price and still manage to get useful spectrums to sell to telecoms?

The problem comes down to, how many satisfiability problems can I solve in a very short amount of time? (determine that a formula is satisfiable or determine that it is unsatisfiable.)

Talk by Kevin Leyton-Brown
`https://www.youtube.com/watch?v=u1-jJOivP70`

Learning goals

By the end of this lecture, you should be able to

- Evaluate the truth value of a formula
    - Define a (truth) valuation.
    - Determine the truth value of a formula by using truth tables.
    - Determine the truth value of a formula by using valuation trees.
- Determine and prove whether a formula has a particular property
    - Define tautology, contradiction, and satisfiable formula.
    - Compare and contrast the three properties (tautology, contradiction, and satisfiable formula).
    - Prove whether a formula is a tautology, a contradiction, or satisfiable, using a truth table and/or a valuation tree.
    - Describe strategies to prove whether a formula is a tautology, a contradiction or a satisfiable formula.

The meaning of well-formed formulas

To interpret a formula, we have to give meanings to the propositional variables and the connectives.

A propositional variable has no intrinsic meaning; it gets a meaning via a valuation.

A (truth) valuation is a function $t : \mathcal{P} \mapsto \{\mathtt{F}, \mathtt{T}\}$ from the set of all proposition variables $\mathcal{P}$ to $\{\mathtt{F}, \mathtt{T}\}$. It assigns true/false to every propositional variable.

Two notations: $t(p)$ and p^t both denote the truth value of p under the truth valuation t.

Truth tables for connectives

The unary connective $\neg$:

α	$(\neg\alpha)$
T	F
F	T

The binary connectives $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$:

α	β	$(\alpha \wedge \beta)$	$(\alpha \vee \beta)$	$(\alpha \rightarrow \beta)$	$(\alpha \leftrightarrow \beta)$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

Truth value of a formula

Fix a truth valuation t. Every formula α has a value under t, denoted α^t, determined as follows.

1. $p^t = t(p)$.
2. $(\neg\alpha)^t = \begin{cases} \texttt{T} & \text{if } \alpha^t = \texttt{F} \\ \texttt{F} & \text{if } \alpha^t = \texttt{T} \end{cases}$
3. $(\alpha \wedge \beta)^t = \begin{cases} \texttt{T} & \text{if } \alpha^t = \beta^t = \texttt{T} \\ \texttt{F} & \text{otherwise} \end{cases}$
4. $(\alpha \vee \beta)^t = \begin{cases} \texttt{T} & \text{if } \alpha^t = \texttt{T} \text{ or } \beta^t = \texttt{T} \\ \texttt{F} & \text{otherwise} \end{cases}$
5. $(\alpha \to \beta)^t = \begin{cases} \texttt{T} & \text{if } \alpha^t = \texttt{F} \text{ or } \beta^t = \texttt{T} \\ \texttt{F} & \text{otherwise} \end{cases}$
6. $(\alpha \leftrightarrow \beta)^t = \begin{cases} \texttt{T} & \text{if } \alpha^t = \beta^t \\ \texttt{F} & \text{otherwise} \end{cases}$

Evaluating a formula using a truth table

Example. The truth table of $((p \lor q) \rightarrow (q \land r))$.

p	q	r	$(p \lor q)$	$(q \land r)$	$((p \lor q) \rightarrow (q \land r))$
F	F	F	F	F	T
F	F	T	F	F	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	F	F
T	T	F	T	F	F
T	T	T	T	T	T

Evaluating a formula using a truth table

Build the truth table of $\big((p \to (\neg q)) \to (q \vee (\neg p))\big)$.

Understanding the disjunction and the biconditional

α	β	$(\alpha \vee \beta)$	Exclusive OR	Biconditional
F	F	F	F	T
F	T	T	T	F
T	F	T	T	F
T	T	T	F	T

- What is the difference between an inclusive OR (the disjunction) and an exclusive OR?
- What is the relationship between the exclusive OR and the biconditional?

Understanding the conditional $\rightarrow$

Assume that proposition p defined below is true.

p: If Alice is rich, she will pay your tuition.

If Alice is rich, will she pay your tuition?

a. Yes
b. No
c. Maybe

If Alice is not rich, will she pay your tuition?

a. Yes
b. No
c. Maybe

Understanding the conditional $\rightarrow$

Alice is rich	Alice will pay your tuition.	If Alice is rich, she will pay your
F	F	T
F	T	T
T	F	F
T	T	T

- Suppose that the implication is a promise that I made. How can you show that I broke my promise?
- If the premise is false, is the statement true or false? (Will the statement ever be contradicted?)
- When the conclusion is true, is the statement true or false?
- When the premise is true, how does the truth value of the statement compare to the truth value of the conclusion?
- Convert $p \rightarrow q$ into a logically equivalent formula which only uses the connectives $\wedge$, $\vee$ and $\neg$. Does this alternative formula make sense?

Another example of structural induction

Theorem: Fix a truth valuation t. Every formula α has a value α^t in $\{\mathtt{F}, \mathtt{T}\}$.

Proof: The property for $R(\alpha)$ is "α has a value α^t in $\{\mathtt{F}, \mathtt{T}\}$".

1. If α is a propositional variable, then t assigns it a value of T or F (by the definition of a truth valuation).
2. If α has a value in $\{\mathtt{F}, \mathtt{T}\}$, then $(\neg\alpha)$ also does, as shown by the truth table of $(\neg\alpha)$.
3. If α and β each has a value in $\{\mathtt{F}, \mathtt{T}\}$, then $(\alpha \star \beta)$ also does for every binary connective $\star$, as shown by the corresponding truth tables.

By the principle of structural induction, every formula has a value.

By the unique readability of formulas, we have proved that a formula has **only one** truth value under any truth valuation t. QED

Tautology, Contradiction, Satisfiable

A formula α is a *tautology* if and only if
for every truth valuation t, $\alpha^t = \texttt{T}$.

A formula α is a *contradiction* if and only if
for every truth valuation t, $\alpha^t = \texttt{F}$.

A formula α is *satisfiable* if and only if
there exists a truth valuation t such that $\alpha^t = \texttt{T}$.

Relationships among the properties

Divide the set of all formulas into 3 mutually exclusive and exhaustive sets. We know two things about these sets:

- A formula is in set 1 if and only if the formula is true in every row of the formula's truth table.
- A formula is in set 3 if and only if it is a contradiction.

Which of the following statements is true?

a. In set 3, every formula is false in every row of the formula's truth table.
b. In set 2, every formula is true in at least one row and false in at least one row of the formula's truth table.
c. Sets 2 and 3 contain exactly the set of satisfiable formulas.
d. Two of (a), (b), and (c) are true.
e. All of (a), (b), and (c) are true.

Examples

1. $\Big(\big(((p \wedge q) \rightarrow (\neg r)) \wedge (p \rightarrow q)\big) \rightarrow (p \rightarrow (\neg r))\Big)$
2. $\Big(\big(((p \wedge q) \rightarrow r) \wedge (p \rightarrow q)\big) \rightarrow (p \rightarrow r)\Big)$
3. $(p \vee q) \leftrightarrow ((p \wedge (\neg q) \vee ((\neg p) \wedge q))$
4. $(p \wedge (\neg p))$

How to determine the properties of a formula

- Truth table
- Valuation tree
- Reasoning

Valuation Tree

Rather than fill out an entire truth table, we can analyze what happens if we plug in a truth value for one variable.

$\neg\mathtt{T}$	$\mathtt{F}$
$\neg\mathtt{F}$	$\mathtt{T}$

$p \land \mathtt{T}$	p
$p \land \mathtt{F}$	$\mathtt{F}$
$p \land p$	p

$p \lor \mathtt{T}$	$\mathtt{T}$
$p \lor \mathtt{F}$	p
$p \lor p$	p

$p \to \mathtt{T}$	$\mathtt{T}$
$p \to \mathtt{F}$	$\neg p$
$\mathtt{T} \to p$	p
$\mathtt{F} \to p$	$\mathtt{T}$
$p \to p$	$\mathtt{T}$

We can evaluate a formula by using these rules to construct a *valuation tree*.

Example of a valuation tree

Example. Show that $\big(((p \wedge q) \to (\neg r)) \wedge (p \to q)\big) \to (p \to (\neg r))\big)$ is a tautology by using a valuation tree.

Suppose $t(p) = \mathtt{T}$. We put $\mathtt{T}$ in for p:

$$\big(((\mathtt{T} \wedge q) \to (\neg r)) \wedge (\mathtt{T} \to q)\big) \to \big(\mathtt{T} \to (\neg r)\big) \enspace .$$

Based on the truth tables for the connectives, the formula becomes $\Big(((q \to (\neg r)) \wedge q\big) \to (\neg r)\Big)$.

If $t(q) = \mathtt{T}$, this yields $((\neg r) \to (\neg r))$ and then $\mathtt{T}$. (Check!).
If $t(q) = \mathtt{F}$, it yields $(\mathtt{F} \to (\neg r))$ and then $\mathtt{T}$. (Check!).

Suppose $t(p) = \mathtt{F}$. We get

$$\big(((\mathtt{F} \wedge q) \to (\neg r)) \wedge (\mathtt{F} \to q)\big) \to \big(\mathtt{F} \to (\neg r)\big) \enspace ,$$

Simplification yields $\big((\mathtt{F} \to (\neg r)) \wedge \mathtt{T}\big) \to \mathtt{T}$ and eventually $\mathtt{T}$.

Thus every valuation makes the formula true, as required.

Reasoning about the properties

I found a valuation for which the formula is true. Does the formula have each property below?

- Tautology YES NO MAYBE
- Contradiction YES NO MAYBE
- Satisfiable YES NO MAYBE

I found a valuation for which the formula is false. Does the formula have each property below?

- Tautology YES NO MAYBE
- Contradiction YES NO MAYBE
- Satisfiable YES NO MAYBE

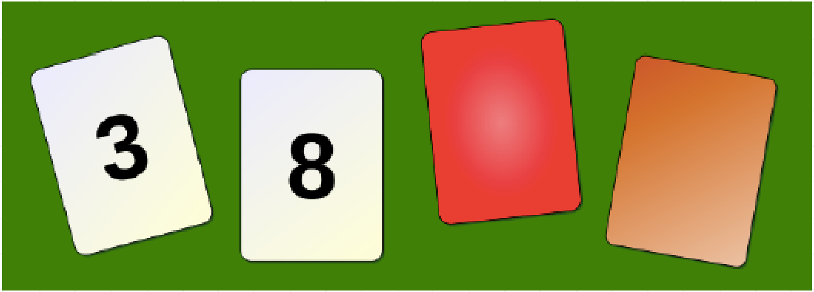
Examples

1. $\Big(\big(((p \wedge q) \to (\neg r)) \wedge (p \to q)\big) \to (p \to (\neg r))\Big)$
2. $\Big(\big(((p \wedge q) \to r) \wedge (p \to q)\big) \to (p \to r)\Big)$
3. $(p \vee q) \leftrightarrow ((p \wedge (\neg q)) \vee ((\neg p) \wedge q))$
4. $\big(p \wedge (\neg p)\big)$

A logic puzzle

Each of the four cards has a number on one side and a color on the other side. How many cards do you have to turn over to test whether this statement is true: “if a card has an even number on one side, then its opposite side is red”?

Your answer is (a) 0 (b) 1 (c) 2 (d) 3 (e) 4.



Propositional Logic: Semantics

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Lecture 5, September 21, 2017

Announcements

Clickers

- What are they for? Active learning, engagement
- Why do you ask us to answer twice? Peer instruction
- If you choose to be here, please participate!

Outline for today

- How can we prove that two formulas have the same meaning? (Logical equivalence)
- Which set of connectives is sufficient to express all possible formulas? (Adequate set of connectives)

Learning goals

Logical equivalence of formulas:

- Prove that the logical equivalence of formulas using truth tables and/or logical identities.
- Describe strategies to prove logical equivalence using logical identities.
- Translate a condition in a block of code into a propositional logic formula.
- Simplify code using truth tables and logical identities.
- Determine whether a piece of code is live or dead using truth tables and logical identities.

Adequate set of connectives:

- Prove that a set of connectives is an adequate set for propositional logic by using truth tables and logical identities.
- Prove that a set of connectives is not an adequate set for propositional logic.

Definition of logical equivalence

Two formulas α and β are logically equivalent if and only if they have the same value under any valuation.

- $\alpha^t = \beta^t$, for every valuation t.
- α and β must have the same final column in their truth tables.
- $(\alpha \leftrightarrow \beta)$ is a tautology.

Why do we care about logical equivalence?

- Will I lose marks if I provide a solution that is syntactically different but logically equivalent to the provided solution?
- Do these two circuits behave the same way?
- Do these two pieces of code fragments behave the same way?

You already know one way of proving logical equivalent. What is it?

Theorem: $(((\neg p) \land q) \lor p) \equiv (p \lor q)$.

Logical Identities

Commutativity

$$(\alpha \land \beta) \equiv (\beta \land \alpha)$$
$$(\alpha \lor \beta) \equiv (\beta \lor \alpha)$$

Associativity

$$\big(\alpha \land (\beta \land \gamma)\big) \equiv \big((\alpha \land \beta) \land \gamma\big)$$
$$\big(\alpha \lor (\beta \lor \gamma)\big) \equiv \big((\alpha \lor \beta) \lor \gamma\big)$$

Distributivity

$$\big(\alpha \lor (\beta \land \gamma)\big) \equiv \big((\alpha \lor \beta) \land (\alpha \lor \gamma)\big)$$
$$\big(\alpha \land (\beta \lor \gamma)\big) \equiv \big((\alpha \land \beta) \lor (\alpha \land \gamma)\big)$$

Idempotence

$$(\alpha \lor \alpha) \equiv \alpha$$
$$(\alpha \land \alpha) \equiv \alpha$$

Double Negation

$$\big(\neg(\neg\alpha)\big) \equiv \alpha$$

De Morgan's Laws

$$\big(\neg(\alpha \land \beta)\big) \equiv \big((\neg\alpha) \lor (\neg\beta)\big)$$
$$\big(\neg(\alpha \lor \beta)\big) \equiv \big((\neg\alpha) \land (\neg\beta)\big)$$

Logical Identities, cont'd

Simplification I (Absorbtion)

$$(\alpha \land \texttt{T}) \equiv \alpha$$
$$(\alpha \lor \texttt{T}) \equiv \texttt{T}$$
$$(\alpha \land \texttt{F}) \equiv \texttt{F}$$
$$(\alpha \lor \texttt{F}) \equiv \alpha$$

Simplification II

$$\big(\alpha \lor (\alpha \land \beta)\big) \equiv \alpha$$
$$\big(\alpha \land (\alpha \lor \beta)\big) \equiv \alpha$$

Implication

$$(\alpha \to \beta) \equiv ((\neg\alpha) \lor \beta)$$

Contrapositive

$$(\alpha \to \beta) \equiv \big((\neg\beta) \to (\neg\alpha)\big)$$

Equivalence

$$(\alpha \leftrightarrow \beta) \equiv \big((\alpha \to \beta) \land (\beta \to \alpha)\big)$$

Excluded Middle

$$\big(\alpha \lor (\neg\alpha)\big) \equiv \texttt{T}$$

Contradiction

$$\big(\alpha \land (\neg\alpha)\big) \equiv \texttt{F}$$

A logical equivalence proof

Theorem: $(((\neg p) \wedge q) \vee p) \equiv (p \vee q)$.

Proof:

$(((\neg p) \wedge q) \vee p)$	
$\equiv (((\neg p) \vee p) \wedge (q \vee p))$	Distributivity
... to be filled in ...	
$\equiv (q \vee p)$	Simplification I
$\equiv (p \vee q)$	Commutativity

QED

A logical equivalence proof

What is missing from the proof?

a. $(q \vee p)$
b. $(\texttt{F} \wedge (q \vee p))$
c. $(q \wedge (q \vee p))$
d. None of these, but I know what it is.
e. None of these, and there's not enough information to tell.

Theorem: $(((\neg p) \wedge q) \vee p) \equiv (p \vee q)$.

Proof:

$(((\neg p) \wedge q) \vee p)$	
$\equiv (((\neg p) \vee p) \wedge (q \vee p))$	Distributivity
... to be filled in ...	
$\equiv (q \vee p)$	Simplification I
$\equiv (p \vee q)$	Commutativity

QED

A practice problem

"If it is sunny, I will play golf, provided that I am relaxed."

s: it is sunny. g: I will play golf. r: I am relaxed.

A few translations:

1. $(s \to (r \to g))$
2. $(r \to (s \to g))$
3. $((s \wedge r) \to g)$

Theorem: $(r \to (s \to g)) \equiv ((s \wedge r) \to g)$.

Proof:

... to be filled in ...

QED

How do you prove non-equivalence?

"If it snows then I won't go to class, but I will do my assignment."

s: it snows. c: I will go to class. a: I will do my assignment.

2 possible translations:

1. $((s \to (\neg c)) \land a)$
2. $(s \to ((\neg c) \land a))$

Theorem: $((s \to (\neg c)) \land a)$ and $(s \to ((\neg c) \land a))$ are not logically equivalent.

Which valuation t can we use to prove this theorem?

a. $s^t = \texttt{F}$, $(\neg c)^t = \texttt{F}$, $a^t = \texttt{F}$
b. $s^t = \texttt{F}$, $(\neg c)^t = \texttt{T}$, $a^t = \texttt{F}$
c. $s^t = \texttt{T}$, $(\neg c)^t = \texttt{T}$, $a^t = \texttt{T}$
d. Two of these.
e. All of these.

Collected Wisdom

- Try getting rid of $\rightarrow$ and $\leftrightarrow$.
- Try moving negations inward. $\neg(p \lor q) \equiv (\neg p) \land (\neg q)$.
- Work from the more complex side first, BUT
- Switch to different strategies/sides when you get stuck.
- In the end, write the proof in clean "one-side-to-the-other" form and double-check steps.

A piece of pseudo code

```
if ( (input > 0) or (not output) ) {
    if ( not (output and (queuelength < 100) ) ) {
        P_1
    } else if ( output and (not (queuelength < 100)) ) {
        P_2
    } else { P_3 }
} else { P_4 }
```

When does each piece of code get executed?

Let i: `input > 0`,
  u: `output`,
  q: `queuelength < 100`.

Simplifying the piece of pseudo code

```
if ( i or (not u) ) {
    if ( not (u and q) ) {
        P_1
    } else if ( u and (not q) ) {
        P_2
    } else { P_3 }
} else { P_4 }
```

Which one of the following is incorrect?

a. P_1 is executed when $(i \vee (\neg u)) \wedge (\neg(u \wedge q))$ is true.

b. P_2 is executed when $(i \vee (\neg u)) \wedge (u \wedge (\neg q))$ is true.

c. P_3 is executed when $((i \vee (\neg u)) \wedge (u \wedge q)) \wedge ((\neg u) \vee q)$ is true.

d. P_4 is executed when i is false and u is true.

e. All of them are correct.

A Code Example, cont'd

i	u	q	$(i \vee (\neg u))$	$(\neg(u \wedge q))$	$(u \wedge (\neg q))$	
T	T	T	T	F	F	P_3
T	T	F	T	T		P_1
T	F	T	T	T		P_1
T	F	F	T	T		P_1
F	T	T	F			P_4
F	T	F	F			P_4
F	F	T	T	T		P_1
F	F	F	T	T		P_1

Finding Dead Code

Prove that P_2 is dead code. That is, the conditions leading to P_2 is logically equivalent to F.

$$\Big(\Big((i \lor (\neg u)) \land (\neg(\neg(u \land q)))\Big) \land (u \land (\neg q))\Big)$$

... to be filled in ...

$\equiv$ F

Simplifying the above condition to F will necessarily use the following logical identities.

a. Simplification I

b. Excluded Middle

c. Contradiction

d. Two of the above

e. All of the above

Finding Live Code

Prove that P_3 is live code. That is, the conditions leading to P_3 is satisfiable.

Theorem:

$$\Big((i \vee (\neg u)) \wedge \Big((\neg(\neg(u \wedge q))) \wedge \big(\neg(u \wedge (\neg q))\big)\Big)\Big) \equiv ((i \wedge u) \wedge q)$$

Proof:

$$\Big((i \vee (\neg u)) \wedge \Big((\neg(\neg(u \wedge q))) \wedge \big(\neg(u \wedge (\neg q))\big)\Big)\Big)$$

... to be filled in ...

$$\equiv ((i \wedge u) \wedge q)$$

QED

Two pieces of code: Are they equivalent?

Fragment 1:

```
if (  i or (not u) ) {
    if (   not (u and q) ) {
        P1
    }
    else if (  u and (not q) ) {
        P2
    }
    else {
        P3
    }
}
else {
    P4
}
```

Fragment 2:

```
if (  (i and u) and q ) {
    P3
}
else if (  (not i) and u ) {
    P4
}
else {
    P1
}
```

Simplifying Code

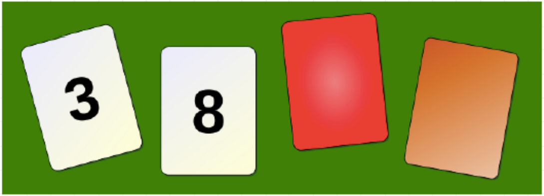
To prove that the two fragments are equivalent, show that each block of code P_1, P_2, P_3, and P_4 is executed under equivalent conditions.

Block	Fragment 1	Fragment 2
P_1	$\big(i \vee (\neg u)\big) \wedge \big(\neg(u \wedge q)\big)$	$\big(\neg(i \wedge u \wedge q)\big) \wedge \big(\neg((\neg i) \wedge u)\big)$
P_2	$\big(i \vee (\neg u)\big) \wedge \big(\neg(\neg(u \wedge q))\big)$ $\wedge \big(u \wedge (\neg q)\big)$	`F`
P_3	$\big(i \vee (\neg u)\big) \wedge \big(\neg(\neg(u \wedge q))\big)$ $\wedge \big(\neg(u \wedge (\neg q))\big)$	$(i \wedge u \wedge q)$
P_4	$\big(\neg(i \vee (\neg u))\big)$	$\big(\neg(i \wedge u \wedge q)\big) \wedge \big((\neg i) \wedge u\big)$

The solution to the last logic puzzle

Each of the four cards has a number on one side and a color on the other side. How many cards do you have to turn over to test whether this statement is true: “if a card has an even number on one side, then its opposite side is red”?

Solution: You need to turn over 2 cards. If a card has an even number on one side, then you need to check that its opposite side is red. Also, if a card is NOT red, you need to check that its opposite side has an ODD number (this is the contrapositive of the given statement). Thus, you need to turn over the second card from the left and the first card from the right.



Another logic puzzle

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie.

You meet three inhabitants: Alice, Rex and Bob.

Alice says, “Rex is a knave.”

Rex says, “it’s false that Bob is a knave.”

Bob claims, “I am a knight or Alice is a knight.”

Can you determine who is a knight and who is a knave?

Adequate sets of connectives

Learning goals

- Prove that a set of connectives is an adequate set for propositional logic by using truth tables and logical identities.
- Prove that a set of connectives is not an adequate set for propositional logic.

Some questions first

- We started propositional logic by learning these connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.
- Why did we learn these connectives?
- Using these connectives, can we express every propositional logic formula that we ever want to write?
- Are there any connectives in this set that are not necessary?
- Are there other connectives that we could define and use? Is there another set of connectives that we should have studied instead?

Some answers

Is every connective we learned necessary?

Nope!

Recall that $x \to y \equiv (\neg x) \vee y$. We don't need $\to$ at all. (We say that $\to$ is definable in terms of $\neg$ and $\vee$.)

Are there other connectives that we could define and use?

Yep! Let's take a look.

Adequate Sets of Connectives

Which set of connectives is sufficient to express every possible propositional formula?

This is called an adequate set of connectives. Any other connective connective is definable in terms of the ones in such a set.

Theorem 1. $\{\wedge, \vee, \neg\}$ is an adequate set of connectives.

Theorem 2. Each of the sets $\{\wedge, \neg\}$, $\{\vee, \neg\}$, and $\{\rightarrow, \neg\}$ is adequate.

Theorem 3. The set $\{\wedge, \vee\}$ is *not* an adequate set of connectives.

An adequate set to start with

Theorem 1. $\{\wedge, \vee, \neg\}$ is an adequate set of connectives.

Hint: use truth tables.

How many people know a way to prove this theorem?

A reduction problem

Theorem 1. $\{\wedge, \vee, \neg\}$ is an adequate set of connectives.

Now we can assume that theorem 1 holds.

Theorem 2. Each of the sets $\{\wedge, \neg\}$, $\{\vee, \neg\}$, and $\{\rightarrow, \neg\}$ is adequate.

By Theorem 1, the set $\{\wedge, \vee, \neg\}$ is adequate.

To prove that $\{\wedge, \neg\}$ is adequate, we need to show that $\vee$ is definable in terms of $\wedge$ and $\neg$.

To prove that $\{\vee, \neg\}$ is adequate, we need to show that $\wedge$ is definable in terms of $\vee$ and $\neg$.

To prove that $\{\rightarrow, \neg\}$ is adequate, we need to show that each of $\vee$ and $\wedge$ is definable in terms of $\rightarrow$ and $\neg$.

A non-adequate set

Theorem 3. The set $\{\wedge, \vee\}$ is *not* an adequate set of connectives.

Consider any formula which uses only $\wedge$ and $\vee$ as connectives. Assume that every variable in the formula is true. What is the truth value of the formula?

a. Always true
b. Always false
c. Sometimes true and sometimes false
d. Not enough information to tell

A non-adequate set

Theorem 3. The set $\{\wedge, \vee\}$ is *not* an adequate set of connectives.

Lemma: For any formula which uses only $\wedge$ and $\vee$ as connectives, if every variable in the formula is true, then the formula is true.

This lemma means that it is impossible to negate a formula using only $\wedge$ and $\vee$.

We can prove the lemma using structural induction.