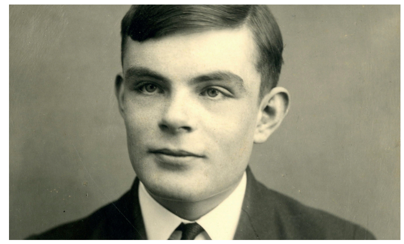
Undecidability

There are problems that cannot be solved by computer programs (i.e. algorithms) even assuming unlimited time and space.



Proved by Alan Turing in 1936

What is a computer program/algorithm?

- At the time, there were no electronic computers. A computer referred to a person who computes.
- Turing's idea of a "computer program" was a list of instructions that a person could follow.
- For us, an algorithm could refer to any of the following:
    - Racket, C, and C++ programs
    - Turing machines
    - High-level pseudo-code

What does it mean for an algorithm to solve a problem?

- The algorithm must produce the correct output for ______ input.

We focus on decision problems.
A decision problem ________________________

A decision problem is
- Decidable iff ______________________________________.
- Undecidable iff ____________________________________
  ___________________________________.

Examples of decision problems:

1. Given a propositional formula, is it satisfiable?

2. Given a predicate formula, is it valid?

3. Given a positive integer, is it prime?

4. Given a program and a Hoare triple, does the program satisfy the Hoare triple under partial correctness?

5. Given a program and a Hoare triple, does the program satisfy the Hoare triple under total correctness?

6. Given two programs, do the two programs produce the same output for every input?

7. Given a program and an input, does the program terminate on the input?

The Halting Problem:
Given a program P and an input I, will P halt on I?

- "Halts" means "terminates" or "does not get stuck".
- One of the first known undecidable problems

The Halting Theorem: There does not exist an algorithm H which solves the halting problem for every program P and input I.

Proof by contradiction:

Assume that there exists an algorithm H(P,I), which solves the halting problem for every program P and input I.

We need to derive a contradiction, which shows that H does not exist.

Our approach:
We will construct an algorithm X(P), which takes a program P as input. We will show that H always gives the wrong answer when predicting whether the program X halts on the input X. That is,

- If H(X,X) returns yes, then X does not halt on X.
- If H(X,X) returns no, then X halts on X.

The algorithm X(P) does the following three things:

(1)

(2)

(3)

Let's compare the result of X(X) and the output of H(X,X).

Therefore, our assumption must be wrong, and H does not exist.

QED

Proving Undecidability via Reduction

Now that we know the halting problem is undecidable.
How do we prove that another problem is undecidable?

- We could prove it from scratch, or…
- We could prove that this problem is as hard as the halting problem; hence it is undecidable.

Problem A is reducible to problem B.

- An algorithm for solving B could be used as a subroutine for solving A.
- If there is an algorithm to solve B, then there is an algorithm to solve A.
- If A is undecidable, then B is undecidable.

A picture to illustrate this:

Halting-no-input Problem: Given a program P (that reads no input), does P halt?

Theorem: The Halting-no-input problem is undecidable.

Proof by contradiction:

Suppose that we have an algorithm A to solve the Halting-no-input Problem. We can use it to solve the Halting problem.

QED

A proof by picture:

Total Correctness Problem: Given a Hoare triple {P} C {Q}, does C satisfy the triple under total correctness?

Theorem: The total correctness problem is undecidable.

Proof by contradiction:

Suppose that we have an algorithm A to solve the Total Correctness Problem. We can use it to solve the Halting-no-input Problem.

QED

A proof by picture:

Partial Correctness: Given a Hoare triple {P} C {Q}, does C satisfy the triple under partial correctness?

Theorem: The partial correctness problem is undecidable.

Proof by contradiction:

Suppose that we have an algorithm A to solve the Partial Correctness Problem. We can use it to solve the Halting-no-input Problem.

QED

A proof by picture: