The basic rules of natural deduction:

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\;\wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\;\wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\;\wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\;\vee\mathrm{i}_1 \qquad \dfrac{\psi}{\phi \vee \psi}\;\vee\mathrm{i}_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\;\vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\;\rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\;\rightarrow\mathrm{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\;\neg\mathrm{i}$	$\dfrac{\phi \quad \neg\phi}{\bot}\;\neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\;\bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\;\neg\neg\mathrm{e}$

Some useful derived rules:

$$\frac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\;\mathrm{MT} \qquad \frac{\phi}{\neg\neg\phi}\;\neg\neg\mathrm{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\;\mathrm{PBC} \qquad \frac{}{\phi \vee \neg\phi}\;\mathrm{LEM}$$

Figure 1.2. Natural deduction rules for propositional logic.

Natural deduction rules

① conjunction rules

(premises). formulas that we assume to be true.

$$\frac{x \quad y}{x \wedge y}\ \wedge i \qquad \text{inference line} \rightarrow \frac{\overbrace{x \wedge y}}{\underbrace{x}} \qquad \frac{x \wedge y}{y}\ \wedge e$$

conclusion. what we can derive to be true

② implication rules

$$\frac{\boxed{\begin{array}{c} x \\ \vdots \\ y \end{array}}}{x \rightarrow y}\ \rightarrow i$$

subproof → ; → assumption ; → conclusion.

$$\frac{\begin{array}{c} x \\ x \rightarrow y \end{array}}{y}\ \rightarrow e$$

(modus ponens)

③ disjunction rules

$$\frac{x}{x \vee y} \qquad \frac{y}{x \vee y}\ \vee i$$

$$\frac{x \vee y \quad \boxed{\begin{array}{c} x \\ \vdots \\ z \end{array}} \quad \boxed{\begin{array}{c} y \\ \vdots \\ z \end{array}}}{z}\ \vee e$$

$x \rightarrow z$ $y \rightarrow z$

$$\frac{\begin{array}{c} x \vee y \\ x \rightarrow z \\ y \rightarrow z \end{array}}{z}$$

(proof by cases)

④ negation rules

$\perp$ also called "bottom"

$$\frac{\boxed{\begin{array}{c} x \\ \vdots \\ \perp \end{array}}}{(\neg x)}\ \neg i$$

contradiction F (false) → $\perp$ → $\perp$

(proof by contradiction)

$$\frac{\begin{array}{c} x \\ (\neg x) \end{array}}{\perp}\ \neg e(\perp i)$$

two names.

⑤ contradiction rules

$$\frac{\begin{array}{c} x \\ (\neg x) \end{array}}{\perp}\ \perp i(\neg e) \qquad \frac{\perp}{x}\ \perp e$$

→ False. (We can ~~derive~~ infer anything from false/contradiction.

⑥ double negation rules (this is also a logical identity).

$$\frac{x}{\neg\neg x}\ \neg\neg i \qquad \frac{\neg\neg x}{x}\ \neg\neg e$$

Natural deduction rules.

A few notes about subproofs.
- It's a small proof inside a big proof.
- Inside a subproof,
  - We can start a subproof with any formula we want. (called an assumption). (The assumption doesn't need to have appeared earlier in the proof.)
  - We can use any formula that has appeared before, inside and/or outside the subproof.
- Outside a subproof:
  - No singe formula can escape a subproof. We cannot take a formula inside a subproof and use it later.
  - We have to use the subproof as a whole.

An example of or-elimination $\vee e$:

I eat an apple or I eat an orange.	$a \vee o$
If I eat an apple, then I am happy.	$a \to h$
If I eat an orange, then I am happy.	$o \to h$
Therefore, I am happy.	h

This inference is valid. You can't be sure whether I ate an apple or an orange, but either way, I am happy.

Natural deduction proof questions

1. $\{((p \land q) \to r)\} \vdash (p \to (q \to r))$

2. $\{(p \to (q \to r))\} \vdash ((p \land q) \to r)$

3. $\{(p \land (q \lor r))\} \vdash ((p \land q) \lor (p \land r))$

4. $\{(p \to q)\} \vdash ((r \lor p) \to (r \lor q))$

5. $\{(\alpha \to (\neg\alpha))\} \vdash (\neg\alpha)$

6. $\{(p \to q), (\neg q)\} \vdash (\neg p)$ (modus tollens)

7. $\{(p \to (q \to r)), p, (\neg r)\} \vdash (\neg q)$.

8. $\{((p \land (\neg q)) \to r), (\neg r), p\} \vdash q$.

9. $\{((\neg\alpha) \land (\neg\beta))\} \vdash (\neg(\alpha \lor \beta))$.

① Show that $\{((p \wedge q) \rightarrow r)\} \vdash (p \rightarrow (q \rightarrow r))$

Proof :

1	$((p \wedge q) \rightarrow r)$	premise
2	p	assumption
3	q	assumption
4	$(p \wedge q)$	$\wedge i$: 2,3
5	r	$\rightarrow e$: 1,4
6	$(q \rightarrow r)$	$\rightarrow i$: 3–5
7	$(p \rightarrow (q \rightarrow r))$	$\rightarrow i$: 2–6

QED

Fill in the missing justifications in the proof.
For each justification, indicate the name of the rule (e.g. $\wedge i$) and the numbers for the lines the rule applied to.

spare

② Show that $(p \to (q \to r)) \vdash ((p \wedge q) \to r)$

Proof:

1	$(p \to (q \to r))$	premise.
2	$(p \wedge q)$	assumption.
3.	p	$\wedge e: 2$
4	$(q \to r)$	$\to e: 1, 3$
5	q	$\wedge e: 2.$
6.	r	$\to e: 4, 5$
7.	$((p \wedge q) \to r)$	$\to i: 2\text{–}6$

QED.

Write down the premises and the conclusion.
Can I apply an elimination rule to a premise?
Can I apply an introduction rule to get to the conclusion?

③ Show that $\{(p \land (q \lor r))\} \vdash ((p \land q) \lor (p \land r))$.

Proof:

1	$(p \land (q \lor r))$	premise
2	p	$\land e$: 1
3	$(q \lor r)$	$\land e$: 1.
4	q	assumption
5	$(p \land q)$	$\land i$: 2,4
6	$((p \land q) \lor (p \land r))$	$\lor i$: 5
7	r	assumption
8	$(p \land r)$	$\land i$: 2,7.
9	$((p \land q) \lor (p \land r))$	$\lor i$: 8
10	$((p \land q) \lor (p \land r))$	$\lor e$: 3, 4-6, 7-9.

QED.

Fill in the missing justifications in the proof.
For each justification, indicate the name of the rule (e.g. $\land i$) and the numbers for the lines the rule applied to.

spare

④ Show that $\{(p \to q)\} \vdash (r \vee p) \to (r \vee q)$

Proof:

1	$(p \to q)$	premise.
2	$(r \vee p)$	assumption
3	r	assumption
4	$(r \vee q)$	$\vee i$: 3
5	p	assumption.
6	q	$\to e$: 1,5
7	$(r \vee q)$	$\vee i$: 6.
8	$(r \vee q)$	$\vee e$: 2,3-4,5-7
9	$((r \vee p) \to (r \vee q))$	$\to i$: 2-8.

QED

① Show that $\{(\alpha \to (\neg\alpha))\} \vdash (\neg\alpha)$.

Proof:

1.	$(\alpha \to (\neg\alpha))$	premise.
2	α	assumption.
3	$(\neg\alpha)$	$\to e$: 1, 2
4.	$\perp$	$\perp i$: 2, 3
5.	$(\neg\alpha)$	$\neg i$: 2–4

QED.

Fill in the missing justifications in the proof.

② Show that $\{(p \to q), (\neg q)\} \vdash (\neg p)$ (modus tollens).

Proof:

1.	$p \to q$	premise
2.	$(\neg q)$	premise
3.	p	assumption
4.	q	$\to e$: 1, 3
5.	$\perp$	$\perp i$: 2, 4
6.	$(\neg p)$	$\neg i$: 3–5

QED

③ Show that $\{(p \to (q \to r)), p, (\neg r)\} \vdash (\neg q)$.

Proof:

1	$(p \to (q \to r))$	premise
2	p	premise.
3	$(\neg r)$	premise
4	$(q \to r)$	$\to e$: 1, 2.
5	q	assumption
6	r	$\to e$: 4, 5
7	$\bot$	$\bot i$: 5, 6.
8	$(\neg q)$	$\neg i$: 4–7

QED.

Fill in the missing justifications in the proof.

④ Show that $\{((p \land (\neg q)) \to r), (\neg r), p\} \vdash q$.

Proof:

1.	$((p \land (\neg q)) \to r)$	premise
2.	$(\neg r)$	premise
3.	p	premise
4	$\neg q$	assumption
5	$(p \land (\neg q))$	$\land i$: 3, 4
6	r	$\to e$: 1, 5
7	$\bot$	$\bot i$: 2, 6.
8	$\neg\neg q$	$\neg i$: 4–7
9	q	$\neg\neg e$: 8

QED

spare

⑤ Show that $\{((\neg\alpha)\wedge(\neg\beta))\} \vdash (\neg(\alpha\vee\beta))$
(This is one part of the De Morgan's law.)
Hints: You may want to use proof by contradiction ($\neg i$).
and proof by cases ($\vee e$).

Proof:

1.	$((\neg\alpha)\wedge(\neg\beta))$	premise
2.	$(\alpha\vee\beta)$	assumption
3.	α	assumption
4.	$(\neg\alpha)$	$\wedge e$: 1
5		
6	$\perp$	$\perp i$
7	β	assumption
8	$(\neg\beta)$	$\wedge e$: 1.
9		
10	$\perp$	$\perp i$
11	$\perp$	$\vee e$: 2,3-6,7-10
12	$(\neg(\alpha\vee\beta))$	$\neg i$: 2-11.

spare

Derived rules for natural deduction.

1. Modus Tollens (MT) $\{(a \to b), (\neg b)\} \vdash (\neg a)$

Proof:

1.	$(a \to b)$	premise
2.	$(\neg b)$	premise
3.	a	assumption
4.	b	$\to e$: 1,3
5.	$\bot$	$\bot i$: 2,4
6	$(\neg a)$	$\neg i$: 3–5.

QED.

2. Double negation introduction $(\neg\neg i)$. $\{a\} \vdash (\neg(\neg a))$

Proof:

1.	a	premise
2.	$(\neg a)$	assumption
3.	$\bot$	$\neg e$: 1,2
4.	$(\neg(\neg a))$	$\neg i$: 2,3.

3. Proof by contradiction (PBC)

$$\dfrac{\boxed{\begin{array}{c}\neg a\\ \vdots\\ \bot\end{array}}}{a}\ \text{PBC}$$

Proof:

1.	$(\neg a) \to \bot$	given
2	$(\neg a)$	assumption
3	$\bot$	$\to e$: 1,2.
4	$(\neg(\neg a))$	$\neg i$: 2,3
5.	a	$\neg\neg e$: 4.

Derived rules for natural deduction

4. Law of excluded middle (LEM) $\emptyset \vdash (a \vee (\neg a))$

Proof:

1.	$(\neg(a \vee (\neg a)))$	assumption
2	a	assumption
3	$(a \vee (\neg a))$	$\vee i$: 2
4	$\perp$	$\neg e$: 1, 3
5	$(\neg a)$	$\neg i$: 2-4
6	$(a \vee (\neg a))$	$\vee i$: 5
7	$\perp$	$\neg e$: 1, 6
8	$(\neg(\neg(a \vee (\neg a))))$	$\neg i$: 1-7
9	$(a \vee (\neg a))$	$\neg\neg e$: 8.

QED

Natural deduction examples

① $\{p, (q \to r)\} \vdash ((p \wedge q) \to r)$

Proof:

1.	p	premise
2.	$q \to r$	premise
3.	$p \wedge q$	assumption
4	q	$\wedge e$: 3
5	r	$\to e$: 2,4
6	$((p \wedge q) \to r)$	$\to i$: 3–5.

QED

② De Morgan's law (direction ①)

$\{(\neg(p \vee q))\} \vdash ((\neg p) \wedge (\neg q))$

Proof:

1.	$\neg(p \vee q)$	premise.
2	p	assumption
3	$(p \vee q)$	$\vee i$: 2
4	$\perp$	$\perp i$: 1,3.
5	$(\neg p)$	$\neg i$: 2–4
6	q	assumption
7	$(p \vee q)$	$\vee i$: 6
8	$\perp$	$\perp i$: 1,7
9	$(\neg q)$	$\neg i$: 6-8
10	$(\neg p) \wedge (\neg q)$	$\wedge i$: 5,9.

③ De Morgan's Law (direction ②)

$\{((\neg p)\wedge(\neg q))\} \vdash (\neg(p \vee q))$

Proof:

1.	$((\neg p)\wedge(\neg q))$	premise
2.	$(\neg p)$	$\wedge e: 1$
3.	$(\neg q)$	$\wedge e: 1$
4	$(p \vee q)$	assumption
5	p	assumption
6	$\perp$	$\perp i: 2,5$
7	q	assumption
8	$\perp$	$\perp i: 3,7$
9	$\perp$	$\vee e: 4, 5\text{-}6, 7\text{-}8$
10	$(\neg(p \vee q))$	$\neg i: 4-9.$

QED

④ $\{(p \to q)\} \vdash ((\neg p) \vee q)$

Proof:

1	$(p \to q)$	premise
2	$(\neg((\neg p)\vee q))$	assumption
3	$(\neg p)$	assumption
4	$((\neg p)\vee q)$	$\vee i: 3$
5	$\perp$	$\perp i: 2,4$
6	$(\neg(\neg p))$	$\neg i: 3-5$
7	p	$\neg\neg e: 6$
8	q	$\to e: 1,7$
9	$((\neg p)\vee q)$	$\vee i: 8$
10	$\perp$	$\perp i: 2,9$
11	$(\neg(\neg((\neg p)\vee q)))$	$\neg i: 2-10$
12	$((\neg p)\vee q)$	$\neg\neg e: 11$

Natural deduction examples

⑤ $\{(\alpha \vee \beta), (\neg\alpha)\} \vdash \beta$

Proof:

1.	$(\alpha \vee \beta)$	premise
2	$(\neg\alpha)$	premise
3	α	assumption
4	$\perp$	$\perp i: 2,3$
5	β	$\perp e: 4$
6	β	assumption
7	β	$\vee e: 1, 3\text{–}5, 6$

QED

⑥ $\{\} \vdash ((p \wedge q) \rightarrow p)$

Proof:

1.	$(p \wedge q)$	assumption
2	p	$\wedge e: 1$
3	$((p \wedge q) \rightarrow p)$	$\rightarrow i: 1\text{–}2$

QED

⑦ $\{\} \vdash (\alpha \vee (\neg\alpha))$ (This is a useful derived rule called the law of excluded middle.)

Proof:

1.	$(\neg(\alpha \vee (\neg\alpha)))$	assumption
2	α	assumption
3	$(\alpha \vee (\neg\alpha))$	$\vee i: 2$
4	$\perp$	$\perp i: 1,3$
5	$(\neg\alpha)$	$\neg i: 2\text{–}4$
6	$(\alpha \vee (\neg\alpha))$	$\vee i: 5$
7	$\perp$	$\perp i: 1,6$
8	$(\neg(\neg(\alpha \vee (\neg\alpha))))$	$\neg i: 1\text{–}7$
9	$(\alpha \vee (\neg\alpha))$	$\neg\neg e: 8$

QED

⑧ $\{(a \to b), (\neg b)\} \vdash (\neg a)$

(This is a useful derived rule called "modus tollens".)

Proof:

1.	$(a \to b)$	premise
2.	$(\neg b)$	premise
3	a	assumption
4	b	$\to e$: 1, 3
5	$\bot$	$\bot i$: 2, 4
6.	$(\neg a)$	$\neg i$: 3–5

QED