Evaluating a predicate formula.

Oct 27

Finally, we are ready to evaluate formulas with free and bound variables.

How do we evaluate a formula $(\forall x\, \alpha)$ or $(\exists x\, \alpha)$?

- For $(\forall x\, \alpha)$, we need to verify that α is true for every possible value of x in the domain.
- For $(\exists x\, \alpha)$, we need to verify that α is true for at least one value of x in the domain.

Formally, the values of $(\forall x\, \alpha)$ and $(\exists x\, \alpha)$ are.

- $(\forall x\, \alpha)^{(I,E)} = T$ if $\alpha^{(I, E[x \mapsto d])} = T$ for every $d \in dom(I)$. and it is F otherwise.
- $(\exists x\, \alpha)^{(I,E)} = T$ if $\alpha^{(I, E[x \mapsto d])} = T$ for at least one $d \in dom(I)$. and it is F otherwise.

$E[x \mapsto d]$ is a new environment created by making a change to E.

$E[x \mapsto d]$ means that "keep all the mappings in E intact EXCEPT re-assign x to $d \in dom(I)$".

Example: Consider E defined below.

$E(x) = 3$, $E(y) = 3$, $E(z) = 1$. $dom(I) = \{1, 2, 3\}$.

① $E[x \mapsto 2](x) = 2$ $E[x \mapsto 2](y) = 3$ $E[x \mapsto 2](z) = 1$.

② $E[x \mapsto 2][y \mapsto 2](x) = 2$ $E[x \mapsto 2][y \mapsto 2](y) = 2$

$E[x \mapsto 2][y \mapsto 2](z) = 1$

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Our language: Variable symbols: x, y.
Predicate symbols: $Q^{(2)}$.

Interpretation I: $dom(I) = \{1, 2\}$. $Q^I = \emptyset$
Environment E: $E(x) = 1$ $E(y) = 2$.

Examples:

① Give an interpretation I_1 and an environment E_1 such that $I_1 \models_{E_1} \alpha$ where $\alpha = (\exists x\ Q(x, y))$.

- y is free in α, so its value is given by E_1.
- To make α true, we need there to be at least one tuple in Q^{I_1} and the second value in the tuple must be 2 because $E(y) = 2$.

Let $I_1 = I$ and $E_1 = E$ except that $Q^{I_1} = \{\langle 1, 2\rangle\}$.

$Q(x, y)^{(I_1, E_1[x \mapsto 1])} = T$ because $\langle 1, 2\rangle \in Q^{I_1}$.

Hence, $(\exists x\ Q(x, y))^{(I_1, E_1)} = T$ and $I_1 \models_{E_1} \alpha$.

② Give I_2 and E_2 such that $I_2 \models_{E_2} \alpha$ where $\alpha = (\forall x\ Q(x, y))$.

- y is free in α, so its value is given by E_2.
- To make α true, we need 2 tuples in Q^{I_2}, each starting with a possible value for x and ending with 2 because $E(y) = 2$.

Let $I_2 = I$ and $E_2 = E$ except that $Q^{I_2} = \{\langle 1, 2\rangle, \langle 2, 2\rangle\}$

Case $[x \mapsto 1]$ $Q(x, y)^{(I_2, E_2[x \mapsto 1])} = T$ because $\langle 1, 2\rangle \in Q^{I_2}$.

Case $[x \mapsto 2]$ $Q(x, y)^{(I_2, E_2[x \mapsto 2])} = T$ because $\langle 2, 2\rangle \in Q^{I_2}$.

Therefore, $(\forall x\ Q(x, y))^{(I_2, E_2)} = T$ and $I_2 \models_{E_2} \alpha$.

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Our language: variable symbols: x, y predicate symbols: $Q^{(2)}$

Interpretation I: $dom(I) = \{1, 2\}$ $Q^I = \phi$

Environment E: $E(x) = 1$ $E(y) = 2$.

Examples:

③ Give I_3 and E_3 such that $I_3 \vDash_{E_3} \alpha$ where $\alpha = (\exists x (\forall y\, Q(x,y)))$.

Start with $I_3 = I$ and $E_3 = E$.

(To make α true, we need at least two tuples in Q^{I_3}. Both tuples need to start with the same value for x. The second values of the tuples need to enumerate all possible domain values for y.)

Let $Q^{I_3} = \{\langle 1, 1\rangle, \langle 1, 2\rangle\}$.

To show that $\alpha^{(I_3, E_3)} = T$, it is sufficient to show that

$$(\forall y\, Q(x,y))^{(I_3, E_3[x \mapsto 1])} = T.$$

Consider all possible values for y:

Case $[y \mapsto 1]$: $Q(x, y)^{(I_3, E_3[x \mapsto 1][y \mapsto 1])} = T$.

Case $[y \mapsto 2]$: $Q(x, y)^{(I_3, E_3[x \mapsto 1][y \mapsto 2])} = T$.

Therefore, $(\forall y\, Q(x,y))^{(I_3, E_3[x \mapsto 1])} = T$ and $I_3 \vDash_{E_3} \alpha$.

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Our language: variable symbols: x, y predicate symbols: $Q^{(2)}$.

Interpretation I: $dom(I) = \{1, 2\}$ $Q^I = \emptyset$.
Environment E: $E(x) = 1$, $E(y) = 2$.

Examples:

④ Show that $I \not\models_E \alpha$ where $\alpha = (\exists x (\forall y \, Q(x, y)))$.

We need to show that $(\exists x (\forall y \, Q(x, y)))^{(I,E)} = F$, which is equivalent to

$$(\neg(\exists x (\forall y \, Q(x, y))))^{(I,E)} = T.$$

or $$(\forall x (\exists y (\neg Q(x, y))))^{(I,E)} = T.$$

Consider all possible values for x:

Case $[x \mapsto 1]$: $Q(x, y)^{(I, E[x \mapsto 1][y \mapsto 1])} = F$ because $\langle 1, 1 \rangle \notin Q^I$.

Thus, $(\exists y (\neg Q(x, y)))^{(I, E[x \mapsto 1])} = T$.

Case $[x \mapsto 2]$: $Q(x, y)^{(I, E[x \mapsto 2][y \mapsto 1])} = F$ because $\langle 2, 1 \rangle \notin Q^I$.

Thus $(\exists y (\neg Q(x, y)))^{(I, E[x \mapsto 2])} = T$.

Therefore, $(\forall x (\exists y (\neg Q(x, y))))^{(I,E)} = T$ and $I \not\models_E \alpha$.