Semantics of Predicate Logic

Oct 19

The question: Given a well-formed predicate logic formula, is it T or F in some context?

In propositional logic, a truth valuation is enough to assign a meaning to a formula.
In predicate logic, we need a lot more.

Properties of formulas

① A formula α is valid if $I \models_E \alpha$ for every interpretation I and environment E.
- $I \models_E \alpha$ means "I and E make α true or satisfy α."
- I and E together make up the context.
- "valid" is analogous to "tautology" in prop logic.

② A formula α is satisfiable if $I \models_E \alpha$ for some interpretation I and environment E.

③ A formula α is unsatisfiable if $I \not\models_E \alpha$ for every interpretation I and environment E.
- "unsatisfiable" is analogous to "contradiction".

Most predicate formulas are satisfiable but not valid because we have a great deal of freedom to choose I and E.

Evaluating a predicate logic formula

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Let's define a set of symbols first.

Constant symbols : a, b, c
Variable symbols : x, y, z.
Function symbols : $f^{(1)}, h^{(2)}$
Predicate symbols : $P^{(1)}, Q^{(2)}$

These symbols do not have intrinsic meanings. They only get meanings through interpretations and environments.

Recall that there are two kinds of expressions:
terms and formulas.
Let's divide them into 3 categories.

Category 1 : terms and formulas without variables.
Examples: terms : $f(h(f(a), f(c)))$, $f(h(b, f(a)))$
formulas : $Q(f(c), a)$, $P(h(f(a), f(c)))$

We only need an interpretation to give these expressions meaning

Category 2 : terms and formulas with free variables only.
Examples: terms : $h(f(a), z)$ $f(h(y, c))$
formulas : $P(h(f(a), z))$ $Q(y, h(a, b))$.

We need an interpretation and an environment to give these expression meaning.

Category 3 : formulas with free and bound variables
Examples : $(\exists x (\forall y\ R(x, y, h(z, c))))$

Evaluating a Predicate Logic Formula.

An interpretation I consists of a domain and the meanings for all of the constant, function and predicate symbols.

Example: Interpretation I_1:

Domain $D = \{1, 2, 3\}$

Constants: $a^{I_1} = 1$, $b^{I_1} = 2$, $c^{I_1} = 3$

Functions: f^{I_1}: $f(1) = 2$, $f(2) = 3$, $f(3) = 1$.

h^{I_1}: $h(x, y) = \min(x, y)$, $\forall x, y \in D$.

Predicates: $P^{I_1} = \{1, 3\}$

$$\underbrace{Q^{I_1} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}}$$

This notation means $Q(a,b)^{I_1} = T$, $Q(c,c)^{I_1} = T$, $Q(c,a)^{I_1} = T$ and Q is false for any other 2-tuples in D^2.

An interpretation gives a meaning to every symbol even if our terms or formulas do not use the symbol.

Let's interpret the expressions in category 1:

$f(h(f(a), f(c)))^{I_1} = 2$

$f(h(b, f(a)))^{I_1} = 3$

$Q(f(c), a)^{I_1} = F$

$P(h(f(a), f(c)))^{I_1} = T$

Evaluating a Predicate Logic Formula

Questions:

① We saw that $Q(f(c),a)^{I_1} = F$, so $I_1 \not\models Q(f(c),a)$.

Is there an interpretation I_2 such that $I_2 \models Q(f(c),a)$?

Let's start with $I_2 = I_1$ and make a small change to I_2

Note that

$$f(c)^{I_2} = 1 \text{ and } a^{I_2} = 1$$

To make Q true, we only need to make sure that $\langle 1,1\rangle \in Q^{I_2}$. So let $Q^{I_2} = \{\langle 1,1\rangle\}$.

$$Q(f(c),a)^{I_2} = T, \text{ and so } I_2 \models Q(f(c),a)$$

② We saw that $P(h(f(a),f(c)))^{I_1} = T$.

Is there an interpretation I_3 such that $I_3 \not\models P(h(f(a),f(c)))$?

Let's start with $I_3 = I_1$.

Note that

$$h(f(a),f(c))^{I_3} = h(b,a)^{I_3} = 1.$$

To make this false, we only need to make sure that $1 \notin P^{I_3}$. Let's define $P^{I_3} = \emptyset$ (the empty set, this predicate P is always false.)

So $P(h(f(a),f(c)))^{I_3} = F$ and $I_3 \not\models P(h(f(a),f(c)))$.

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A note about functions in an interpretation.

A function symbol $f^{(k)}$ must be interpreted as a function f^I that is total on D.

$$f^{(k)}: \underbrace{D \times \cdots \times D}_{k \text{ copies.}} \to D$$

① Any k-tuple in D^k can be an input to $f^{(k)}$.

② The output of f^I must be in D.

Examples of non-total functions:

① $f(x, y) = x - y$ Domain $D = \mathbb{N}$ natural numbers.

$f(1,2) = 1-2 = -1 \notin \mathbb{N}$.

② $f(x) = \sqrt{x}$ Domain $D = \mathbb{Z}$ Integers.

(1) $-1 \in D$ cannot be an input to f.

(2) $f(2) = \sqrt{2} \notin D$

Examples of total functions

① $D = \{1, 2, 3\}$

$f(1) = 1$, $f(2) = 1$, $f(3) = 1$

② $D = \mathbb{N}$ natural numbers

$f(x) = x+1 \quad \forall x \in \mathbb{N}$.

Evaluating a Predicate Logic Formula

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To evaluate terms and formulas with variables, we need an interpretation and an environment.

An environment maps every variable symbol to an element of the domain.

- An environment is only used to interpret free variables.
- Bound variables get their meanings through the corresponding quantifiers.

Example: Environment E_1:

$E_1(x) = 3$, $E_1(y) = 3$, $E_1(z) = 1$

Consider expressions in category 2:

$h(f(a), z)^{(I_1, E_1)} = 1$ $\quad$ $P(h(f(a), z))^{(I_1, E_1)} = T$

$f(h(y, c))^{(I_1, E_1)} = 1$ $\quad$ $Q(y, h(a, b))^{(I_1, E_1)} = T$

Questions:

① Give an I_4 and E_4 such that $I_4 \not\models_{E_4} P(h(f(a), z))$.
- Let's start with $I_4 = I_1$ and $E_4 = E_1$.
- The easiest way to make P false is to let $P^{I_4} = \phi$. (This means P^{I_4} is always false. Thus, $I_4 \not\models_{E_4} P(h(f(a), z))$ if $P^{I_4} = \phi$.

② Give an I_5 and E_5 such that $I_5 \not\models_{E_5} Q(y, h(a, b))$.
Using the same idea as above, let $I_5 = I_1$, $Q^{I_5} = \phi$ and $E_5 = E_1$.
We have $I_5 \not\models_{E_5} Q(y, h(a, b))$.