Semantic Entailment.

Oct 23.

Predicate Logic.	Propositional Logic.
Σ semantically entails C, denoted $\Sigma \models C$, if and only if	
$I \models_E \Sigma$ implies $I \models_E C$ (or $C^{(I,E)} = T$)	$\Sigma^t = T$ implies $C^t = T$
for any interpretation I and any environment E.	for any truth valuation t.

A few notes for $\Sigma \models C$ in predicate logic:

① $\Sigma \models C$ means if a pair of I and E makes every formula in Σ true, then I and E also make C true.

② $\Sigma \models C$ means $\phi \models ((P_1 \wedge \cdots \wedge P_n) \to C)$ (if $\Sigma = \{P_1, P_2, \ldots, P_n\}$.)

③ $\Sigma \models C$ means $((P_1 \wedge \cdots \wedge P_n) \to C)$ is valid.

④ If $\phi \models C$, then C is valid because $I \models_E \phi$ for any I and E.

Prove that $\Sigma \models C$ holds.

① Proof: Assume that there is an I and an E such that $I \models_E \Sigma$.
We need to show that $I \models_E C$.

② Proof by contradiction:
Assume that there is an I and an E such that
$I \models_E \Sigma$ and $I \not\models_E C$.
We need to derive a contradiction.

Prove that $\Sigma \not\models C$.

Give an I and an E such that $I \models_E \Sigma$ and $I \not\models_E C$.

Proving Semantic Entailment. Oct 24.

Q: Show that $(\exists y (\forall x\ P(x,y))) \models (\forall x (\exists y\ P(x,y)))$.

Proof by contradiction:

- Assume that there is an interpretation I such that

$$(\exists y (\forall x\ P(x,y)))^I = T \quad \text{and} \quad (\forall x (\exists y\ P(x,y)))^I = F.$$

We need to derive a contradiction.

- Since $(\exists y (\forall x\ P(x,y)))^I = T$, there is an element $a \in dom(I)$ such that $(\forall x\ P(x,y))^{(I, E[y \mapsto a])} = T$.

If $(\forall x\ P(x,y))^{(I, E[y \mapsto a])} = T$, then

$$P(x,y)^{(I, E[x \mapsto d][y \mapsto a])} = T \text{ for every } d \in dom(I). \quad ①$$

- Since $(\forall x (\exists y\ P(x,y)))^I = F$, we know that

$$(\neg(\forall x (\exists y\ P(x,y)))^I = T \quad \text{or} \quad (\exists x (\forall y (\neg P(x,y))))^I = T.$$

This means that there is an element $b \in dom(I)$ such that $(\forall y (\neg P(x,y)))^{(I, E[x \mapsto b])} = T$.

If $(\forall y (\neg P(x,y)))^{(I, E[x \mapsto b])} = T$, then

$$(\neg P(x,y))^{(I, E[x \mapsto b][y \mapsto d])} = T \text{ for every } d \in dom(I). \quad ②$$

- By equation ①, we know that $P(x,y)^{(I, E[x \mapsto b][y \mapsto a])} = T$ ③

- By equation ②, we know that $(\neg P(x,y))^{(I, E[x \mapsto b][y \mapsto a])} = T$ ④

- ③ and ④ mean that $P(x,y)$ is T and F under I and $E[x \mapsto b][y \mapsto a]$. This is a contradiction.

QED

Proving Semantic Entailment.

Show that $\emptyset \models ((\forall x\,(\alpha \to \beta)) \to ((\forall x\,\alpha) \to (\forall x\,\beta)))$

Proof: We have no premise. So we need to show that $((\forall x\,(\alpha \to \beta)) \to ((\forall x\,\alpha) \to (\forall x\,\beta)))$ is valid.

Assume that there is an interpretation I and an environment E such that $I \models_E (\forall x\,(\alpha \to \beta))$. We need to show that $I \models_E ((\forall x\,\alpha) \to (\forall x\,\beta))$. That means, we assume that $I \models_E (\forall x\,\alpha)$ and we need to show that $I \models_E (\forall x\,\beta)$.

By definition of $\forall$, $I \models_E (\forall x\,(\alpha \to \beta))$ means that for every $a \in dom(I)$, $I \models_{E[x \mapsto a]} (\alpha \to \beta)$.

By definition of $\forall$, $I \models_E (\forall x\,\alpha)$ means that for every $a \in dom(I)$, $I \models_{E[x \mapsto a]} \alpha$.

Thus, by definition of $\to$, $I \models_{E[x \mapsto a]} \beta$ for every $a \in dom(I)$, which means $I \models_E (\forall \beta)$.

QED

Proof by contradiction: Assume that there is an interpretation I and an environment E such that

$$I \not\models_E ((\forall x\,(\alpha \to \beta)) \to ((\forall x\,\alpha) \to (\forall x\,\beta)))$$

We need to derive a contradiction.

Our assumption means that $I \models_E (\forall x\,(\alpha \to \beta))$ and $I \not\models_E ((\forall x\,\alpha) \to (\forall x\,\beta))$. The latter means that $I \models_E (\forall x\,\alpha)$ and $I \not\models_E (\forall x\,\beta)$.

By definition of $\forall$, $I \models_E (\forall x\,(\alpha \to \beta))$ means that for every $a \in dom(I)$, $I \models_{E[x \mapsto a]} (\alpha \to \beta)$.

By definition of $\forall$, $I \models_E (\forall x\,\alpha)$ means that for every $a \in dom(I)$, $I \models_{E[x \mapsto a]} \alpha$.

By definition of $\to$, we have that $I \models_{E[x \mapsto a]} \beta$ for every $a \in dom(I)$. This contradicts our assumption that $I \not\models_E (\forall x\,\beta)$.

QED

Disproving Semantic Entailment

Oct 24

Q: Show that $(\forall x\, (\exists y\ P(x, y))) \not\models (\exists y\, (\forall x\ P(x, y)))$.

Proof: We need to give an interpretation I such that

$$(\forall x\, (\exists y\ P(x, y)))^I = T \quad \text{and} \quad (\exists y\, (\forall x\ P(x, y)))^I = F.$$

Define I:

Domain $dom(I) = \{1, 2, 3\}$.

Predicate: $P^I = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 3,1\rangle\}$.

Let's verify that the premise is true and the conclusion is false under I. To do this, we need to evaluate $P(x, y)$ under I and some environment E.

Let E be an arbitrary environment.

To verify that $(\forall x\, (\exists y\ P(x, y)))^I$ is true, we need to verify that $(\exists y\ P(x, y))^{(I, E[x \mapsto d])}$ is true for every $d \in dom(I)$

Case $[x \mapsto 1]$, $P(x, y)^{(I, E[x\mapsto 1][y \mapsto 1])} = T$ because $\langle 1,1\rangle \in P^I$.

Thus $(\exists y\ P(x, y))^{(I, E[x \mapsto 1])} = T$.

Case $[x \mapsto 2]$ $P(x, y)^{(I, E[x\mapsto 2][y \mapsto 3])} = T$ because $\langle 2,3\rangle \in P^I$.

Thus, $(\exists y\ P(x, y))^{(I, E[x \mapsto 2])} = T$.

Case $[x \mapsto 3]$ $P(x, y)^{(I, E[x\mapsto 3][y \mapsto 1])} = T$ because $\langle 3,1\rangle \in P^I$.

Thus $(\exists y\ P(x, y))^{(I, E[x \mapsto 3])} = T$.

Therefore, $(\forall x\, (\exists y\ P(x, y)))^I = T$.

continued on the next page...

Disproving Semantic Entailment. Oct 24.

Q: Show that $(\forall x (\exists y\ P(x,y))) \not\models (\exists y (\forall x\ P(x,y)))$.

Proof continued: We need to verify that $(\exists y (\forall x\ P(x,y)))^{I} = F$.

This is equivalent to verifying $(\neg(\exists y (\forall x\ P(x,y))))^{I} = T$.

or $(\forall y (\exists x (\neg P(x,y))))^{I} = T$.

We need to verify that

$$(\exists x (\neg P(x,y)))^{(I, E[y \mapsto d])} = T \text{ for every } d \in dom(I).$$

Case $[y \mapsto 1]$ $\quad P(x,y)^{(I, E[x \mapsto 2][y \mapsto 1])} = F$ because $\langle 2,1 \rangle \notin P^{I}$.

Thus, $(\exists x (\neg P(x,y)))^{(I, E[y \mapsto 1])} = T$.

Case $[y \mapsto 2]$ $\quad P(x,y)^{(I, E[x \mapsto 1][y \mapsto 2])} = F$ because $\langle 1,2 \rangle \notin P^{I}$.

Thus $(\exists x (\neg P(x,y)))^{(I, E[y \mapsto 2])} = T$.

Case $[y \mapsto 3]$ $\quad P(x,y)^{(I, E[x \mapsto 1][y \mapsto 3])} = F$ because $\langle 1,3 \rangle \notin P^{I}$.

Thus $(\exists x (\neg P(x,y)))^{(I, E[y \mapsto 3])} = T$.

Therefore, $(\exists y (\forall x\ P(x,y)))^{I} = F$.

QED

Disproving Semantic Entailment

Show that $((\forall x\, \alpha) \to (\forall x\, \beta)) \not\models (\forall x\, (\alpha \to \beta))$

(Thoughts: How can we come up with an interpretation I, an environment E and formulas for α and β such that
$I \models_E ((\forall x\, \alpha) \to (\forall x\, \beta))$ and $I \not\models_E (\forall x\, (\alpha \to \beta))$?

The easiest way to make $((\forall x\, \alpha) \to (\forall x\, \beta))$ true is to make $(\forall x\, \alpha)$ false, which means we need to make sure α is false for at least one element of the domain. Let α be $P(x)$ and let
$P(x)^{(I, E[x \mapsto a])} = F$ for $a \in dom(I)$.

To make $(\forall x\, (\alpha \to \beta))$, the easiest way is to make sure there exists $b \in dom(I)$ such that $\alpha^{(I, E[x \mapsto b])} = T$ and $\beta^{(I, E[x \mapsto b])} = F$.
We could let β be $(\neg P(x))$, let $dom(I) = \{a, b\}$ and let $P^I = \{b\}$.
)

Proof: Let α be $P(x)$ and β be $(\neg P(x))$.
Let an interpretation I be $dom(I) = \{a, b\}$ and $P^I = \{b\}$
Consider any environment E.

We will show that ① $I \models_E ((\forall x\, \alpha) \to (\forall x\, \beta))$ and ② $I \not\models_E (\forall x\, (\alpha \to \beta))$.

① $a \notin P^I$, so $\alpha^{(I, E[x \mapsto a])} = F$. Thus $(\forall x\, \alpha)^{(I, E)} = F$.
By definition of $\to$, $((\forall x\, \alpha) \to (\forall x\, \beta))^{(I, E)} = T$.

② $b \in P^I$, so $\alpha^{(I, E[x \mapsto b])} = T$ and $\beta^{(I, E[x \mapsto b])} = F$.
So $(\alpha \to \beta)^{(I, E[x \mapsto b])} = F$, or $(\forall x\, (\alpha \to \beta))^{(I, E)} = F$.

QED