CS 245 – Examples of structural induction proofs September 14, 2017
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Version 1: I describe the property in English in the proof.

Lemma 2: Every well-formed formula φ has an equal number of opening and closing brackets.

Define $P(\varphi)$ to be φ has an equal number of opening and closing brackets.

Proof by structural induction:

<u>Base case</u>: φ is a propositional variable.
A propositional variable has zero opening bracket and zero closing bracket. Thus, the number of opening and closing brackets in φ are equal.

<u>Induction step</u>:

Let $op(x)$ and $cl(x)$ denote the number of opening and closing brackets in x respectively.

Case 1: φ is a well-formed formula of the form $(\neg x)$ where x is a well-formed formula.

Induction hypothesis: Assume that x has an equal number of opening and closing brackets.
We need to prove that $(\neg x)$ has an equal number of opening and closing brackets.

$op(\ (\neg x)\ )$
$= 1 + op(x)$ by inspection of $(\neg x)$
$= 1 + cl(x)$ by the induction hypothesis
$= cl(\ (\neg x)\ )$ by inspection of $(\neg x)$
Thus, $(\neg x)$ has an equal number of opening and closing brackets.

Case 2: φ is a well-formed formula of the form $(x * y)$ where x and y are well-formed formulas and * is one of the four binary connectives ($\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$).

Induction hypothesis: Assume that each of x and y has an equal number of opening and closing brackets.
We need to prove that $(x * y)$ has an equal number of opening and closing brackets.

$op((x * y))$
$= 1 + op(x) + op(y)$ by inspection of $(x * y)$
$= 1 + cl(x) + cl(y)$ by the induction hypothesis
$= cl((x * y))$ by inspection of $(x * y)$
Thus, $(x * y)$ has an equal number of opening and closing brackets.

By the principle of structural induction, every well-formed formula φ has an equal number of opening and closing brackets. QED

Version 2: I describe the property using P(.) in the proof.

Lemma 2: Every well-formed formula φ has an equal number of opening and closing brackets.

Define $P(\varphi)$ to be φ has an equal number of opening and closing brackets.

Proof by structural induction:

<u>Base case</u>: φ is a propositional variable. We need to prove that $P(\varphi)$ holds.

A propositional variable has zero opening bracket and zero closing bracket. Thus, $P(\varphi)$ holds.

<u>Induction step</u>:

Let $op(x)$ and $cl(x)$ denote the number of opening and closing brackets in x respectively.

Case 1: φ is a well-formed formula of the form $(\neg x)$ where x is a well-formed formula.

Induction hypothesis: Assume that $P(x)$ holds.
We need to prove that $P((\neg x))$ holds.

$op((\neg x))$
$= 1 + op(x)$ by inspection of $(\neg x)$
$= 1 + cl(x)$ by the induction hypothesis
$= cl((\neg x))$ by inspection of $(\neg x)$

Thus, $P((\neg x))$ holds.

Case 2: φ is a well-formed formula of the form $(x * y)$ where x and y are well-formed formulas and * is one of the four binary connectives ($\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$).

Induction hypothesis: Assume that $P(x)$ and $P(y)$ hold.
We need to prove that $P((x * y))$ holds.

$op((x * y))$
$= 1 + op(x) + op(y)$ by inspection of $(x * y)$
$= 1 + cl(x) + cl(y)$ by the induction hypothesis
$= cl((x * y))$ by inspection of $(x * y)$

Thus, $P((x * y))$ holds.

By the principle of structural induction, $P(\varphi)$ holds for every well-formed formula φ. QED