Assessing Human Error Against a Benchmark of Perfection

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An increasing number of domains are providing us with detailed trace data on human decisions in settings where we can evaluate the quality of these decisions via an algorithm. Motivated by this development, an emerging line of work has begun to consider whether we can characterize and predict the kinds of decisions where people are likely to make errors.

To investigate what a general framework for human error prediction might look like, we focus on a model system with a rich history in the behavioral sciences: the decisions made by chess players as they select moves in a game. We carry out our analysis at a large scale, employing datasets with several million recorded games, and using *chess tablebases* to acquire a form of ground truth for a subset of chess positions that have been completely solved by computers but remain challenging for even the best players in the world.

We organize our analysis around three categories of features that we argue are present in most settings where the analysis of human error is applicable: the skill of the decision-maker, the time available to make the decision, and the inherent difficulty of the decision. We identify rich structure in all three of these categories of features, and find strong evidence that in our domain, features describing the inherent difficulty of an instance are significantly more powerful than features based on skill or time.

CCS Concepts: • **Applied computing → Psychology;**

Additional Key Words and Phrases: Blunder prediction, human decision-making

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1. INTRODUCTION

Several rich strands of work in the behavioral sciences have been concerned with characterizing the nature and sources of human error. These include the broad notion of *bounded rationality* [Simon 1957] and the subsequent research beginning with Kahneman and Tversky on heuristics and biases [Tversky and Kahneman 1975]. With the growing availability of large datasets containing millions of human decisions on fixed, well-defined, real-world tasks, there is an increasing need to add a new style of inquiry to this research—given a large stream of decisions, with rich information about the context of each decision, can we algorithmically characterize and predict the instances on which people are likely to make errors?

This genre of question—analyzing human errors from large traces of decisions on a fixed task—also has an interesting relation to the canonical set-up in machine learning

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applications. Typically, using instances of decision problems together with "ground truth" labels showing the correct decision, an algorithm is trained to produce the correct decision in a large fraction of instances. The analysis of human error, on the other hand, represents a twist on this formulation: given instances of a task in which we have both the correct decision *and* a human's decision, an algorithm is trained to recognize future instances on which a human is likely to make a mistake. Predicting human error from this type of trace data has a history in human factors research [Kirwan 1993; Salvendy 2012], and a nascent line of work has begun to apply current machine-learning methods to the question [Lakkaraju et al. 2015, 2016].

Model systems for studying human error. As the investigation of human error using large datasets grows increasingly feasible, it becomes useful to understand which styles of analysis will be most effective. For this purpose, as in other settings, there is enormous value in focusing on model systems where one has exactly the data necessary to ask the basic questions in their most natural formulations.

What might we want from such a model system?

(i) It should consist of a task for which the context of the human decisions has been measured as thoroughly as possible, and in a very large number of instances, to provide the training data for an algorithm to analyze errors.
(ii) So that the task is non-trivial, it should be challenging even for highly skilled human decision-makers.
(iii) Notwithstanding the previous point (ii), the "ground truth"—the correctness of each candidate decision—should be feasibly computable by an algorithm.

Guided by these desiderata, we focus in this article on chess as a model system for our analysis. In doing so, we are proceeding by analogy with a long line of work in behavioral science using chess as a model for human decision-making [Charness 1992; Chase and Simon 1973; De Groot 1978]. Chess is a natural domain for such investigations, since it presents a human player with a sequence of concrete decisions—which move to play next—with the property that some choices are better than others. Indeed, because chess provides data on hard decision problems in such a pure fashion, it has been described as the "drosophila of psychology" [Simon and Chase 1988]. (It is worth noting our focus here on *human* decisions in chess, rather than on designing algorithms to play chess. This latter problem has also, of course, generated a rich literature, along with a closely related tag-line of chess as the "drosophila of artificial intelligence" [McCarthy 1990].)

Chess as a model system for human error. Despite the clean formulation of the decisions made by human chess players, we still must resolve a set of conceptual challenges if our goal is to assemble a large corpus of chess moves with ground-truth labels that classify certain moves as errors. Let us consider three initial ideas for how we might go about this, each of which is lacking in some crucial respect for our purposes.

First, for most of the history of human decision-making research on chess, the emphasis has been on focused laboratory studies at small scales in which the correct decision could be controlled by design [Charness 1992]. In our list of desiderata, this means that point (iii), the availability of ground truth, is well under control, but a significant aspect of point (i)—the availability of a vast number of instances—is problematic due to the necessarily small scales of the studies.

A second alternative would be to make use of two important computational developments in chess—the availability of databases with millions of recorded chess games by strong players; and the fact that the strongest chess programs—generally referred to as *chess engines*—now greatly outperform even the best human players in the world. This makes it possible to analyze the moves of strong human players, in a large-scale fashion, by comparing their choices to those of an engine. This has been pursued very effectively in the last several years by Biswas and Regan [2015a, 2015b] and Regan and Biswas [2013]; they have used the approach to derive interesting insights including proposals for how to estimate the depth at which human players are analyzing a position.

For the current purpose of assembling a corpus with ground-truth error labels, however, engines present a set of challenges. The basic difficulty is that even current chess engines are far from being able to provide guarantees regarding the best move(s) in a given position. In particular, an engine may prefer move m to m' in a given position, supplementing this preference with a heuristic numerical evaluation, but m' may ultimately lead to the same result in the game, both under best play and under typical play. In these cases, it is hard to say that choosing m' should be labeled an error. More broadly, it is difficult to find a clear-cut rule mapping an engine's evaluations to a determination of human error, and efforts to label errors this way would represent a complex mixture of the human player's mistakes and the nature of the engine's evaluations.

Finally, a third possibility is to go back to the definition of chess as a deterministic game with two players (White and Black) who engage in alternating moves, and with a game outcome that is either (a) a win for White, (b) a win for Black, or (c) a draw. This means that from any position, there is a well-defined notion of the outcome with respect to optimal play by both sides—in game-theoretic terms, this is the *minimax value* of the position. In each position, it is the case that White wins with best play, or Black wins with best play, or it is a draw with best play, and these are the three possible minimax values for the position.

This perspective provide us with a clean option for formulating the notion of an error, namely the direct game-theoretic definition: a player has committed an error if their move worsens the minimax value from their perspective. That is, the player had a forced win before making their move but now they do not; or the player had a forced draw before making their move but now they do not. But there is an obvious difficulty with this route, and it is a computational one: for most chess positions, determining the minimax value is hopelessly beyond the power of both human players and chess engines alike.

We now discuss the approach we take here.

Assessing errors using tablebases. In our work, we use minimax values by leveraging a further development in computer chess—the fact that chess has been solved for all positions with at most k pieces on the board, for small values of k [Bellman 1965; Makhnychev and Zakharov 2012; Nalimov 2005; Kopec 1990]. (We will refer to such positions as $\leq k$-piece positions.) Solving these positions has been accomplished not by forward construction of the chess game tree, but instead by simply working backward from terminal positions with a concrete outcome present on the board and filling in all other minimax values by dynamic programming until all possible $\leq k$-piece positions have been enumerated. The resulting solution for all $\leq k$-piece positions is compiled into an object called a *k-piece tablebase*, which lists the game outcome with best play for each of these positions. The construction of tablebases has been a topic of interest since the early days of computer chess [Kopec 1990], but only with recent developments in computing and storage have truly large tablebases been feasible. Proprietary tablebases with $k = 7$ have been built, requiring in excess of a hundred terabytes of storage [Makhnychev and Zakharov 2012]; tablebases for $k = 6$ are much more manageable, though still very large [Nalimov 2005], and we focus on the case of $k = 6$ in what follows.[1]

[1]There are some intricacies in how tablebases interact with certain rules for draws in chess, particularly *threefold-repetition* and *the 50-move rule*, but since these have essentially negligible impact on our use of tablebases in the present work, we do not go into further details here.

Tablebases and traditional chess engines are thus very different objects. Chess engines produce strong moves for arbitrary positions, but with no absolute guarantees on move quality in most cases; tablebases, on the other hand, play perfectly with respect to the game tree—indeed, effortlessly, via table lookup—for the subset of chess containing at most k pieces on the board.

Thus, for arbitrary $\leq k$-piece positions, we can determine minimax values, and so we can obtain a large corpus of chess moves with ground-truth error labels: Starting with a large database of recorded chess games, we first restrict to the subset of $\leq k$-piece positions, and then we label a move as an error if and only it worsens the minimax value from the perspective of the player making the move. Adapting chess terminology to the current setting, we will refer to such an instance as a *blunder*.

This is our model system for analyzing human error; let us now check how it lines up with desiderata (i)–(iii) for a model system listed above. Chess positions with at most $k = 6$ pieces arise relatively frequently in real games, so we are left with many instances even after filtering a database of games to restrict to only these positions (point (i)). Crucially, despite their simple structure, they can induce high error rates by amateurs and non-trivial error rates even by the best players in the world; in recognition of the inherent challenge they contain, textbook-level treatments of chess devote a significant fraction of their attention to these positions [Fine 1941] (point (ii)). And they can be evaluated perfectly by tablebases (point (iii)).

Focusing on $\leq k$-piece positions has an additional benefit, made possible by a combination of tablebases and the recent availability of databases with millions of recorded chess games. The most frequently occurring of these positions arise in our data thousands of times. As we will see, this means that for some of our analyses, we can control for the exact position on the board and still have enough instances to observe meaningful variation. Controlling for the exact position is not generally feasible with arbitrary positions arising in the middle of a chess game, but it becomes possible with the scale of data we now have, and we will see that in this case it yields interesting and in some cases surprising insights.

2. SETTING UP THE ANALYSIS

In formulating our analysis, we begin from the premise that for analyzing error in human decisions, three crucial types of features are the following:

(a) the skill of the decision-maker;
(b) the time available to make the decision; and
(c) the inherent difficulty of the decision.

Any instance of the problem will implicitly or explicitly contain features of all three types: an individual of a particular level of skill is confronting a decision of a particular difficulty, with a given amount of time available to make the decision.

In our current domain, as in any other setting where the question of human error is relevant, there are a number of basic genres of question that we would like to ask. These include the following.

—For predicting whether an error will be committed in a given instance, which types of features (skill, time, or difficulty) yield the most predictive power?
—In which kinds of instances does greater skill confer the largest relative benefit? Is it for more difficult decisions (where skill is perhaps most essential) or for easier ones (where there is the greatest room to realize the benefit)? Are there particular kinds of instances where skill does not in fact confer an appreciable benefit?
—An analogous set of questions for time in place of skill: In which kinds of instances does greater time for the decision confer the largest benefit? Is additional time more

beneficial for hard decisions or easy ones? And are there instances where additional time does not reduce the error rate?

—Finally, there are natural questions about the interaction of skill and time: Is it higher skill or lower skill decision-makers who benefit more from additional time?

These questions motivate our analyses in the subsequent sections. We begin by discussing how features of all three types (skill, time, and difficulty) are well-represented in our domain.

Our data comes from two large databases of recorded chess games. The first is a corpus of approximately 200 million games from the Free Internet Chess Server (FICS), where amateurs play each other on-line.[2] The second is a corpus of approximately 1 million games played in international tournaments by the strongest players in the world. We will refer to the first of these as the FICS dataset, and the second as the GM dataset. (GM for "grandmaster," the highest title a chess player can hold.) For each corpus, we extract all occurrences of $\leq$6-piece positions from all of the games; we record the move made in the game from each occurrence of each position, and use a tablebase to evaluate all possible moves from the position (including the move that was made). This forms a single instance for our analysis. Since we are interested in studying errors, we exclude all instances in which the player cannot possibly blunder, since there are no legal moves that would change the minimax value of the position. This means we exclude all instances in which the player to move is in a theoretically losing position—where the opponent has a direct path to checkmate—because there are no blunders in losing positions (the minimax value of the position is already as bad as possible for the player to move). Finally, there are times when players will arrive at the same position more than once in the same game; we restrict our analysis to only the first time an instance has been seen in a game. There are 24.6 million such instances in the FICS dataset, and 880,000 in the GM dataset.

We now consider how feature types (a), (b), and (c) are associated with each instance. First, for skill, each chess player in the data has a numerical rating, termed the *Elo rating*, based on their performance in the games they have played [Elo 1978; Herbrich et al. 2006]. Higher numbers indicate stronger players, and to get a rough sense of the range: most amateurs have ratings in the range 1000–2000, with extremely strong amateurs getting up to 2200–2400; players above 2500–2600 belong to a rarefied group of the world's best; and at any given time there are generally around five players in the world above 2800. If we think of a game outcome in terms of points, with 1 point for a win and 0.5 points for a draw, then the Elo rating system has the property that when a player is paired with someone 400 Elo points lower, their expected game outcome is approximately 0.91 points—an enormous advantage.[3]

For our purposes, an important feature of Elo ratings is the fact that a single number has empirically proven so powerful at predicting performance in chess games. While ratings clearly cannot contain all the information about players' strengths and weaknesses, their effectiveness in practice argues that we can reasonably use a player's rating as a single numerical feature that approximately represents their skill.

With respect to temporal information, chess games are generally played under time limits of the form, "play x moves in y minutes" or "play the whole game in y minutes." Players can choose how they use this time, so on each move they face a genuine decision about how much of their remaining allotted time to spend. The FICS dataset contains the amount of time remaining in the game when each move was played (and hence the

[2]This data is publicly available at ficsgames.org.
[3]In general, the system is designed so that when the rating difference is $400d$, the expected score for the higher ranked player under the Elo system is $1/(1 + 10^{-d})$.

amount of time spent on each move as well); most of the games in the FICS dataset are played under extremely rapid time limits, with a large fraction of them requiring that the whole game be played in 3 minutes for each player. To avoid variation arising from the game duration, we focus on this large subset of the FICS data consisting exclusively of games with 3 minutes allocated to each side.

Our final set of features will be designed to quantify the difficulty of the position on the board—i.e., the extent to which it is hard to avoid selecting a move that constitutes a blunder. There are many ways in which one could do this, and we are guided in part by the goal of developing features that are less domain-specific and more applicable to decision tasks in general. We begin with perhaps the two most basic parameters, analogues of which would be present in any setting with discrete choices and a discrete notion of error—these are the number of legal moves in the position, and the number of these moves that constitute blunders. Later, we will also consider a general family of parameters that involve looking more deeply into the search tree, at moves beyond the immediate move the player is facing.

To summarize, in a single instance in our data, a player of a given rating, with a given amount of time remaining in the game, faces a specific position on the board, and we ask whether the move they select is a blunder. We now explore how our different types of features provide information about this question, before turning to the general problem of prediction.

3. FUNDAMENTAL DIMENSIONS

3.1. Difficulty

We begin by considering a set of basic features that help quantify the difficulty inherent in a position. There are many features we could imagine employing that are highly domain-specific to chess, but our primary interest is in whether a set of relatively generic features can provide non-trivial predictive value.

Above we noted that in any setting with discrete choices, one can always consider the total number of available choices, and partition these into the number that constitute blunders and the number that do not constitute blunders. In particular, let's say that in a given chess position P, there are $n(P)$ legal moves available—these are the possible choices—and of these, $b(P)$ are blunders, in that they lead to a position with a strictly worse minimax value (and thus the other $n(P) - b(P)$ moves are non-blunders that preserve the minimax value of the position). Note that it is possible to have $b(P) = 0$, but we exclude these positions because it is impossible to blunder. Also, by the definition of the minimax value, we must have $b(P) \leq n(P) - 1$; that is, there is always at least one move that preserves the minimax value.

A global check of the data reveals an interesting bimodality in both the FICS and GM datasets: positions with small values of $b(P)$ (principally $b(P) \leq 3$) and positions with $b(P) = n(P) - 1$ are both heavily represented (see Figure 1). The former correspond to positions in which there are very few possible blunders, and the latter correspond to positions in which there is a unique correct move to preserve the minimax value. Our results will cover the full range of $(n(P), b(P))$ values, but it is useful to know that both of these extremes are well represented.

Now, let us ask what the empirical blunder rate looks like as a bivariate function of this pair of variables $(n(P), b(P))$. Over all instances in which the underlying position P satisfies $n(P) = n$ and $b(P) = b$, we define $r(n, b)$ to be the fraction of those instances in which the player blunders. How does the empirical blunder rate vary in $n(P)$ and $b(P)$? It seems natural to suppose that for fixed $n(P)$, it should generally increase in $b(P)$, since there are more possible blunders to make. On the other hand, instances with $b(P) = n(P) - 1$ often correspond to chess positions in which the only non-blunder

Fig. 1. The number of instances as a function of the two variables $(n(P), b(P))$, for the FICS dataset.

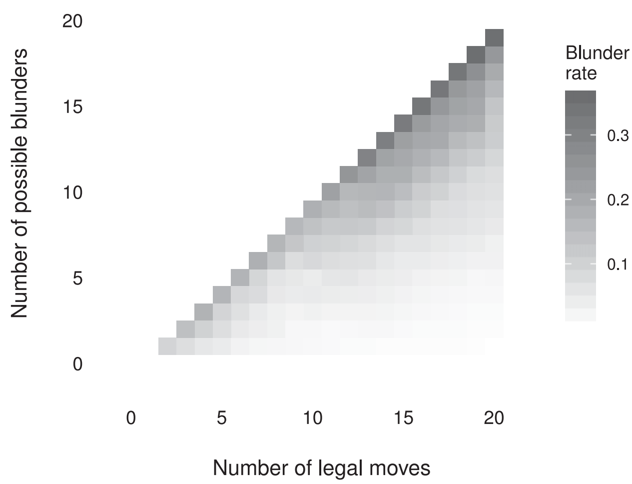
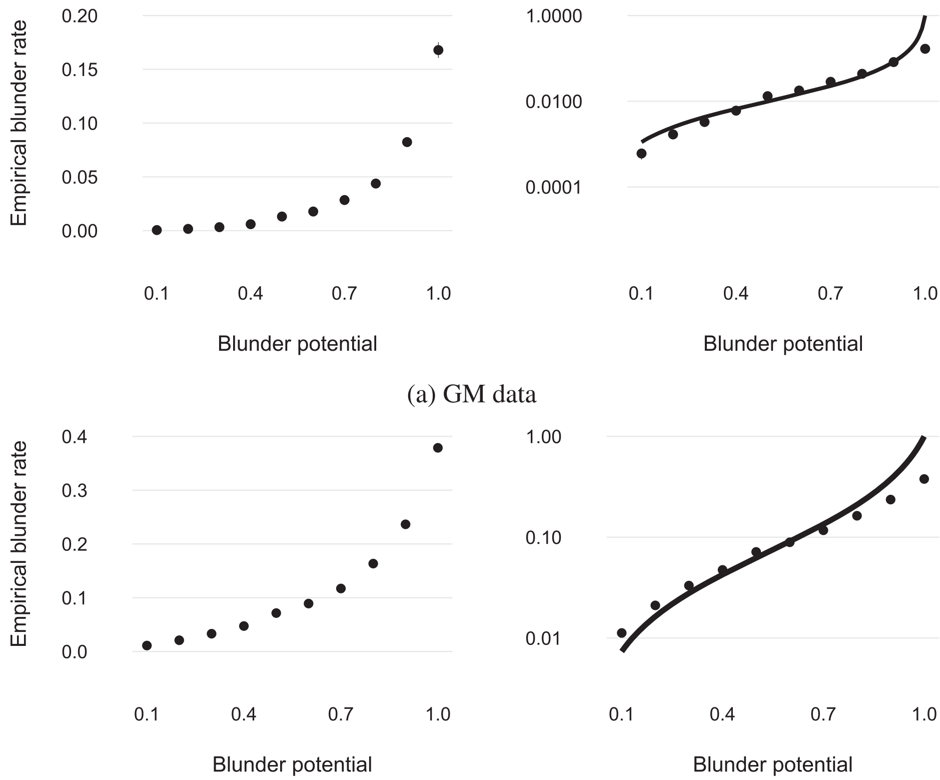


Fig. 2. A heat map showing the empirical blunder rate as a function of the two variables $(n(P), b(P))$, for the FICS dataset.

is "obvious" (for example, if there is only one way to recapture a piece), and so one might conjecture that the empirical blunder rate will be lower for this case.

In fact, the empirical blunder rate is generally monotone in $b(P)$, as shown by the heatmap representation of $r(n, b)$ in Figure 2. (We show the function for the FICS data; the function for the GM data is similar.) Moreover, if we look at the heavily-populated line $b(P) = n(P) - 1$, the blunder rate is increasing in $n(P)$; as there are more blunders to compete with the unique non-blunder, it becomes correspondingly harder to make the right choice.

Blunder Potential. Given the monotonicity we observe, there is an informative way to combine $n(P)$ and $b(P)$: by simply taking their ratio $b(P)/n(P)$. This quantity, which we term the *blunder potential* of a position P and denote $\beta(P)$, is the answer to the question, "If the player selects a move uniformly at random, what is the probability



(a) GM data

(b) FICS data

Fig. 3. The empirical blunder rate as a function of the blunder potential, shown for both the GM and the FICS data. On the left are standard axes, on the right are logarithmic y-axes. The plots on the right also show an approximate fit to the γ-value defined in Section 3.1.

that they will blunder?." This definition will prove useful in many of the analyses to follow. Intuitively, we can think of it as a direct measure of the *danger* inherent in a position, since it captures the relative abundance of ways to go wrong.

In Figure 3, we plot the function $y = r(x)$, the proportion of blunders in instances with $\beta(P) = x$, for both our GM and FICS datasets on linear as well as logarithmic y-axes. The striking regularity of the $r(x)$ curves shows how strongly the availability of potential mistakes translates into actual errors. One natural starting point for interpreting this relationship is to note that if players were truly selecting their moves uniformly at random, then these curves would lie along the line $y = x$. The fact that they lie below this line indicates that in aggregate players are preferentially selecting non-blunders, as one would expect. And the fact that the curve for the GM data lies much further below $y = x$ is a reflection of the much greater skill of the players in this dataset, a point that we will return to shortly.

The γ-value. We find that a surprisingly simple model qualitatively captures the shapes of the curves in Figure 3 quite well. Suppose that instead of selecting a move uniformly at random, a player selected from a biased distribution in which they were preferentially c times more likely to select a non-blunder than a blunder, for a parameter $c > 1$.

If this were the true process for move selection, then the empirical blunder rate of a position P would be

$$\gamma_c(P) = \frac{b(P)}{c(n(P) - b(P)) + b(P)}.$$

We will refer to this as the γ-value of the position P, with parameter c. Using the definition of the blunder potential $\beta(P)$ to write $b(P) = \beta(P)n(P)$, we can express the γ-value directly as a function of the blunder potential:

$$\gamma_c(P) = \frac{\beta(P)n(P)}{c(n(P) - \beta(P)n(P)) + \beta(P)n(P)} = \frac{\beta(P)}{c - (c-1)\beta(P)}.$$

We can now find the value of c for which $\gamma_c(P)$ best approximates the empirical curves in Figure 3. The best-fit values of c are $c \approx 15$ for the FICS data and $c \approx 100$ for the GM data, again reflecting the skill difference between the two domains. These curves are shown superimposed on the empirical plot in the figure (on the right, with logarithmic y-axes).

We note that in game-theoretic terms the γ-value can be viewed as a kind of *quantal response* [McKelvey and Palfrey 1998], in which players in a game select among alternatives with a probability that decreases according to a particular function of the alternative's payoff. Since the minimax value of the position corresponds to the game-theoretic payoff of the game in our case, a selection rule that probabilistically favors non-blunders over blunders can be viewed as following this principle. (We note that our functional form cannot be directly mapped onto standard quantal response formulations. The standard formulations are strictly monotonically decreasing in payoff, whereas we have cases where two different blunders can move the minimax value by different amounts—in particular, when a win changes to a draw versus a win changes to a loss—and we treat these the same in our simple formulation of the γ-value.)

3.2. Skill

A key focus in the previous subsection was to understand how the empirical blunder rate varies as a function of parameters of the instance. Here we continue this line of inquiry, with respect to the skill of the player in addition to the difficulty of the position.

Recall that a player's *Elo rating* is a function of the outcomes of the games they have played, and is effective in practice for predicting the outcomes of a game between two rated players [Elo 1978]. It is for this reason that we use a player's rating as a proxy for their skill. However, given that ratings are determined by which games a player wins, draws, or loses, rather than by the extent to which they blunder in $\leq$6-piece positions, a first question is whether the empirical blunder rate in our data shows a clean dependence on rating.

In fact it does. Figure 4 shows the empirical blunder rate $f(x)$ averaged over all instances in which the player has rating x. The blunder rate declines smoothly with rating for both the GM and FICS data, with a flattening of the curve at higher ratings.[4]

The Skill Gradient. We can think of the downward slope in Figure 4, as a kind of *skill gradient*, showing the reduction in blunder rate as skill increases. The steeper this reduction is in a given setting, the higher the empirical benefit of skill in reducing error.

It is therefore natural to ask how the skill gradient varies across different conditions in our data. As a first way to address this, we take each possible value of the blunder potential β (rounded to the nearest multiple of 0.1), and define the function $f_\beta(x)$ to

[4]FICS uses a slight modification of the Elo rating system called the Glicko rating system, but the general principles of the two systems are the same.

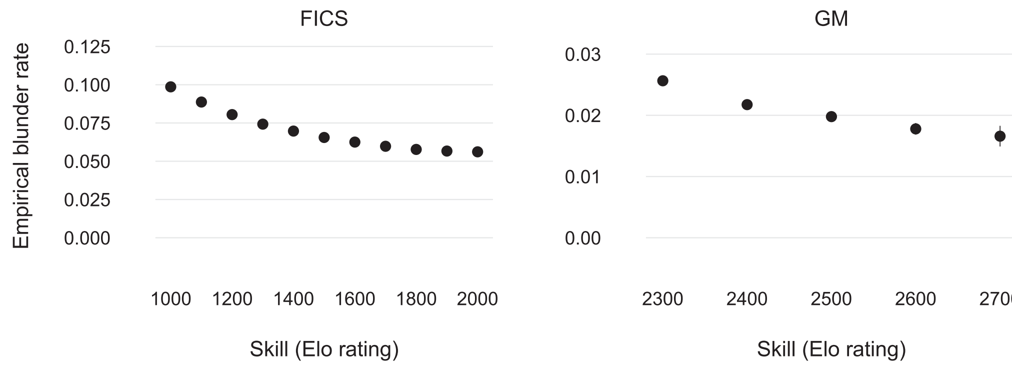


Fig. 4. The empirical blunder rate as a function of player rating, shown for both the FICS and GM datasets.

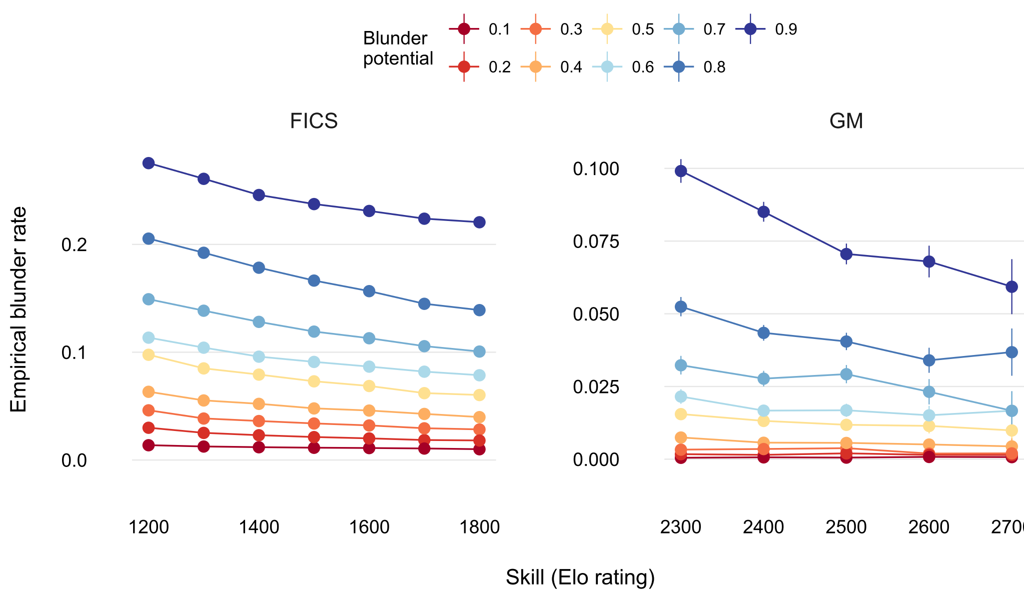


Fig. 5. The empirical blunder rate as a function of Elo rating in the FICS and GM datasets, for positions with fixed values of the blunder potential.

be the empirical error rate of players of rating x in positions of blunder potential β. Figure 5 shows plots of these curves for β equal to each multiple of 0.1, for both the GM and FICS datasets.

We observe two properties of these curves. First, there is remarkably little variation among the curves. When viewed on a logarithmic y-axis the curves are almost completely parallel, indicating the same rate of proportional decrease across all blunder potentials.

A second, arguably more striking, property is how little the curves overlap in their ranges of y-values. In effect, the curves form a kind of "ladder" based on blunder potential: for every value of the discretized blunder potential, every rating in the 1200–1800 range on FICS has a lower empirical blunder rate at blunder potential β than the best of these ratings at blunder potential $\beta + 0.2$. In effect, each additional 0.2 increment in blunder potential contributes more, averaging over all instances, to the aggregate empirical blunder rate than an additional 600 rating points, despite the

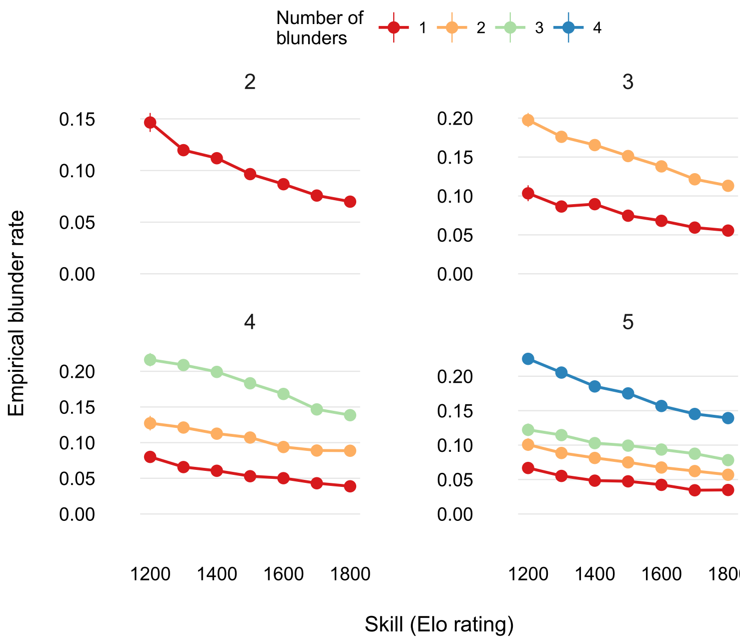


Fig. 6. The empirical blunder rate as a function of Elo rating in the FICS data, for positions with fixed values of $(n(P), b(P))$.

fact that 600 rating points represent a vast difference in chess performance. We see a similar effect for the GM data, where small increases in blunder potential have a greater effect on blunder rate than the enormous difference between a rating of 2300 and a rating of 2700. (Indeed, players rated 2700 are making errors at a greater rate in positions of blunder potential 0.9 than players rated 1200 are making in positions of blunder potential 0.3.) And we see the same effects when we separately fix the numerator and denominator that constitute the blunder potential, $b(P)$ and $n(P)$, as shown in Figure 6.

To the extent that this finding runs counter to our intuition, it bears an interesting relation to the *fundamental attribution error*—the tendency to attribute differences in people's performance to differences in their individual attributes, rather than to differences in the situations they face [Jones and Harris 1967]. What we are uncovering here is that a basic measure of the situation—the blunder potential, which as we noted above corresponds to a measure of the *danger* inherent in the underlying chess position—is arguably playing a larger role than the players' skill. This finding also relates to work of Abelson on quantitative measures in a different competitive domain, baseball, where he found that a player's batting average accounts for very little of the variance in their performance in any single at-bat [Abelson 1985].

We should emphasize, however, that despite the strong effect of blunder potential, skill does play a fundamental role in our domain, as the analysis of this section has shown. And in general it is important to take multiple types of features into account in any analysis of decision-making, since only certain features may be under our control in any given application. For example, we may be able to control the quality of the people we recruit to a decision, even if we cannot control the difficulty of the decision itself.

The Skill Gradient for Fixed Positions. Grouping positions together by common $(n(P), b(P))$ values gives us a rough sense for how the skill gradient behaves in positions

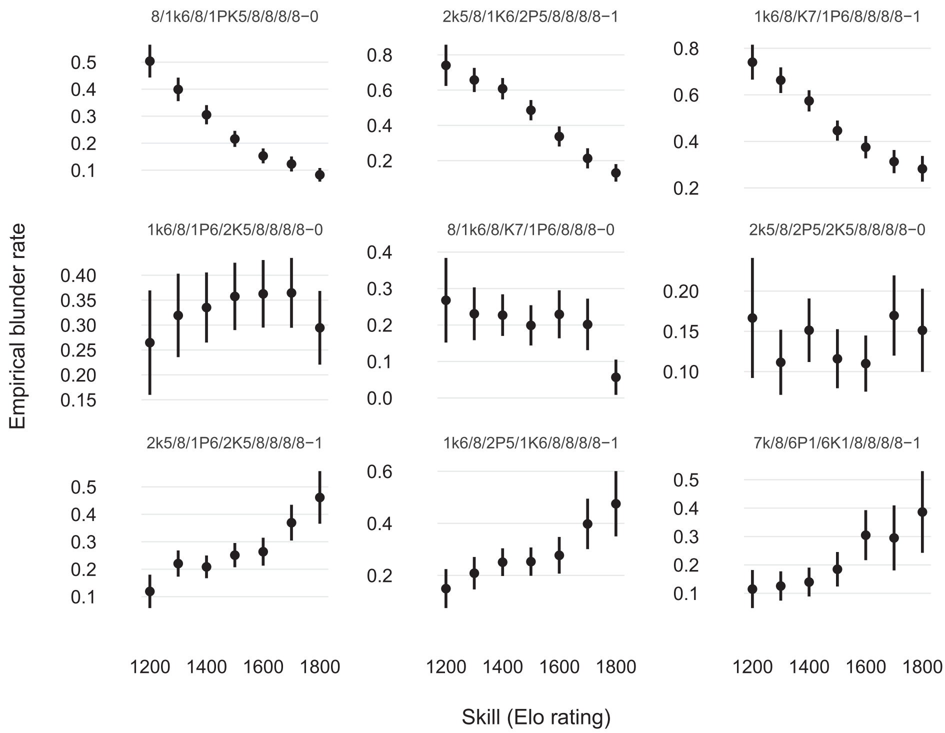


Fig. 7. The empirical blunder rate as a function of Elo rating, for a set of frequently occurring positions.

of varying difficulty. But this analysis still aggregates together a large number of different positions, each with their own particular properties. Here, we ask: how does the empirical blunder rate vary with Elo rating *when we fix the exact position on the board?*

The fact that we are able to meaningfully ask this question is based on a fact noted in Section 1, that many non-trivial ≤6-piece positions recur in the FICS data, exactly, several thousand times.[5] For each such position P, we have enough instances to plot the function $f_P(x)$, the rate of blunders committed by players of rating x in position P.

Let us say that the function $f_P(x)$ is *skill-monotone* if it is decreasing in x—that is, if players of higher rating have a lower blunder rate in position P. A natural conjecture would be that every position P is skill-monotone, but in fact this is not the case. Among the most frequent positions, we find several that we term *skill-neutral*, with $f_P(x)$ remaining approximately constant in x, as well as several that we term *skill-anomalous*, with $f_P(x)$ increasing in x. Figure 7 shows a subset of the most frequently occurring positions in the FICS data that contains examples of each of these three types: skill-monotone, skill-neutral, and skill-anomalous.[6]

The existence of skill-anomalous positions is surprising, since there is a no *a priori* reason to believe that chess as a domain should contain common situations in which

[5]To increase the amount of data we have on each position, we group together positions that are equivalent by symmetry: we can apply a left-to-right reflection of the board, or we can apply a top-bottom reflection of the board (also reversing the colors of the pieces and the side to move), or we can do both. Each of the four resulting positions is equivalent under the rules of chess.

[6]For readers interested in looking at the exact positions in question, each position in Figure 7 is described in Forsyth–Edwards notation above the panel in which its plot appears. The final digit after the dash indicates the player to move: 0 means it's Black to move, and 1 means it's White to move.

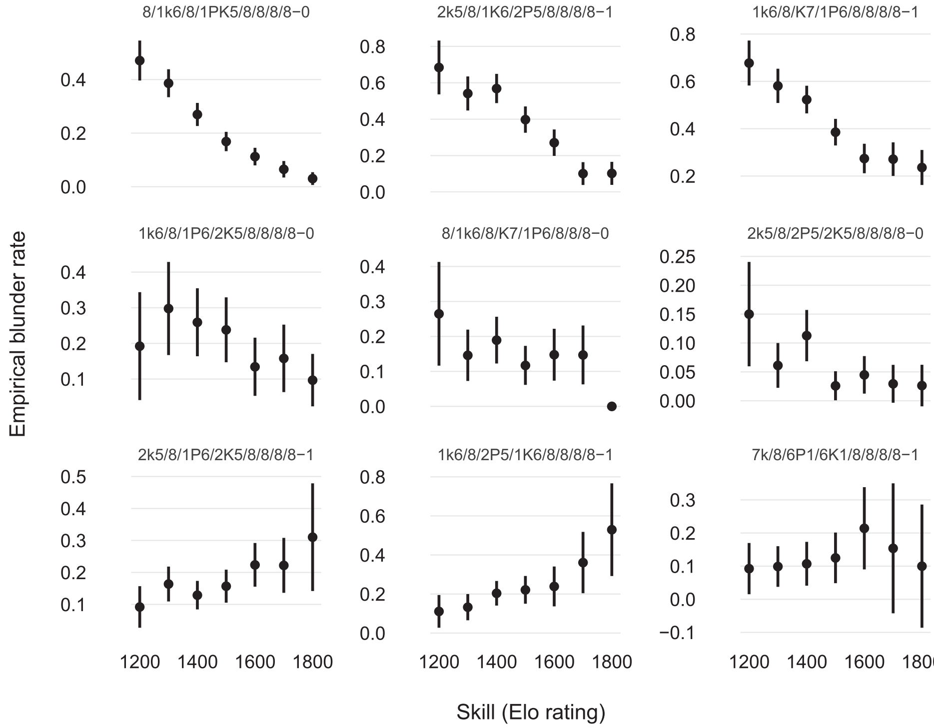


Fig. 8. The empirical blunder rate as a function of Elo rating, for the same set of frequently occurring positions, restricted to instances in which the player to move has at least 10 seconds remaining and spends more than the global median value of 0.735 seconds thinking.

stronger players make more errors than weaker players. Moreover, the behavior of players in these particular positions does not seem explainable by a strategy in which they are deliberately making a one-move blunder for the sake of the overall game outcome. In each of the skill-anomalous examples in Figure 7, the player to move has a forced win, and the position is reduced enough that the worst possible game outcome for them is a draw under any sequence of moves, so there is no long-term value in blundering away the win on their present move.

We can also perform additional checks on the result as follows. Since we have grouped together all instances with the same position in our analyses, it could be the case that in certain positions there is a correlation between a player's rating and the time they spend on a move, and that confounds our analysis. To address this potential issue, in Figure 8 we re-plot the functions $f_P(x)$ for the same positions P, restricting to the subset of instances in which the players use more than the median amount of time spent across all of these nine positions (the median is 0.735 seconds), and in which they have at least 10 seconds remaining for the game (which, as we will see in Figure 9, is where the blunder rate starts leveling off as a function of the time remaining). The data is noisier since we have restricted to a subset of instances per position, but the categorization of positions into skill-monotone, skill-neutral, and skill-anomalous positions is unchanged. These restrictions suggest that it is not because stronger players are making their moves extremely quickly, or under more severe time pressure, that we are seeing skill-neutral and skill-anomalous positions.

Skill-anomalous positions are reminiscent of, and possibly related to, the phenomenon of *U-shaped development* in psychology, where people become worse at a

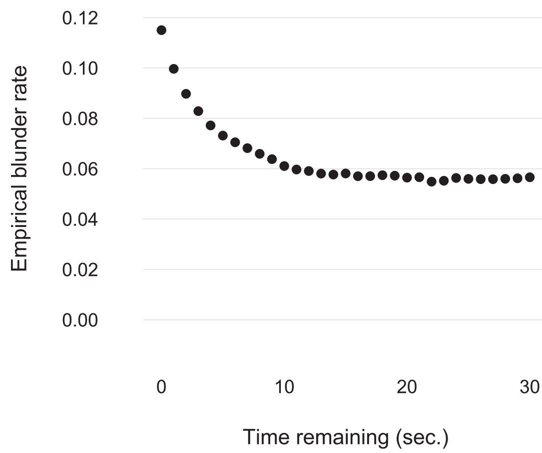


Fig. 9. The empirical blunder rate as a function of time remaining.

task before they start to improve again [Taatgen and Anderson 2002; Namy et al. 2004]. The canonical example is first language acquisition, where, for example, children learning English will first use the word "spoke," then use the incorrect "speaked" as they learn the rule for appending "-ed" to form the past tense, then revert back to using "spoke" as they learn the exceptions to the general rule [Brown 1973; Rumelhart and McClelland 1985]. There are also examples in domains closer to chess; for example, 9-year-olds consistently underperform both 7-year-olds and 11-year-olds on a class of simple algebra problems [McNeil 2007]. Our skill-anomalous $f_P(x)$ curves do not form a full U-shape, but this is presumably due to a lack of data for very high-rated players in these positions—we expect that at some point in chess development, performance on these skill-anomalous positions improves again.

3.3. Time

Finally, we consider our third category of features, the time that players have available to make their moves, and the time they spend in making individual moves. Recall that players have to make their own decisions about how to allocate a fixed budget of time across an entire game. The FICS data has information about the time remaining associated with each move in each game, so we focus our analysis on FICS in this subsection. Specifically, as noted in Section 2, FICS games are generally played under extremely rapid conditions, and for uniformity in the analysis we focus on the most commonly occurring FICS time constraint—the large subset of games in which each player is allocated 3 minutes for the whole game.

As a first object of study, let's define the function $g(t)$ to be the empirical blunder rate in positions where the player begins considering their move with t seconds left in the game. Figure 9 shows a plot of $g(t)$; it is natural that the blunder rate increases sharply as t approaches 0, though it is notable how flat the value of $g(t)$ becomes once t exceeds roughly 10 seconds.

The Time Gradient. The plot in Figure 9 can be viewed as a basic kind of *time gradient*, analogous to the skill gradient, showing the overall improvement in empirical blunder rate that corresponds with having extra time available. Here too we can look at the time gradient restricted to positions with fixed blunder potential, rating, or fixed blunder potential and player rating.

First, since the curve in Figure 9 aggregates data from players at all skill levels, it could potentially obfuscate how blunder rates vary with time remaining for players of

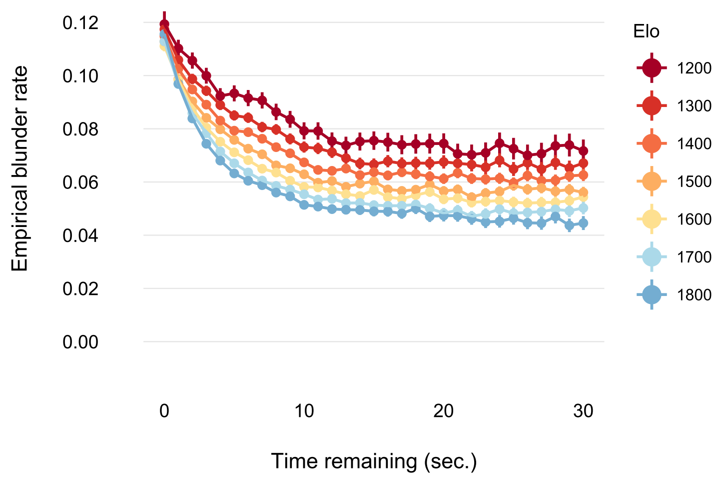


Fig. 10. The empirical blunder rate as a function of time remaining for several skill levels.

similar skill. Thus, we define the function $g_r(t)$ to be the same as $g(t)$, but restricted to players whose rating (rounded to the nearest 100 points) is r. We show $g_r(t)$ for multiple choices of r in Figure 10. Reassuringly, $g_r(t)$ has the same shape as $g(t)$ for every r: the blunder rates start to rapidly increase once players have fewer than 10 seconds for the rest of the game, but above 10 seconds they stabilize at a rating-specific value. Furthermore, the $g_r(t)$ curves are roughly the same distance apart for all t—with one important exception. As t approaches 0, the blunder rates of different skill levels start to converge. For example, when players have just 10 seconds left for the whole game, their blunder rates vary from approximately 5% to 7.5%, depending on their skill. But under extreme time pressure, blunder rates are all close to 11% regardless of skill level. This is perhaps surprising, since one might have expected that the more extreme the time pressure, the more beneficial it is to have higher skill. Our results show the opposite: the marginal benefit of being high-skill is greater when time pressure is less extreme.

Next, we examine Figure 11, which shows $g_\beta(t)$, the blunder rate for players within a narrow skill range (1500–1599 Elo) with t seconds remaining in positions with blunder potential β. In this sense, it is a close analogue of Figure 5, which plotted $f_\beta(x)$; and for values of t above 8 seconds, it shows a very similar “ladder” structure in which the role of blunder potential is dominant. Specifically, for every β, players are blundering at a lower rate with 8 to 12 seconds remaining at blunder potential β than they are with over a minute remaining at blunder potential $\beta + 0.2$. A small increase in blunder potential has a more extensive effect on blunder rate than a large increase in available time.

We can separate the instances further both by blunder potential and by the rating of the player, via the function $g_{\beta,r}(t)$ which gives the empirical blunder rate with t seconds remaining when restricted to players of rating r in positions of blunder potential β. Figure 12 plots these functions, with a fixed value of β in each panel. We can compare curves for players of different rating, observing that for higher ratings the curves are steeper: extra time confers a greater relative empirical benefit on higher rated players. Across panels, we see that for higher blunder potential the curves become somewhat shallower: more time provides less relative improvement as the density of possible blunders proliferates. But equally or more striking is the fact that all curves retain a roughly constant shape, even as the empirical blunder rate climbs by an order of magnitude from the low ranges of blunder potential to the highest.

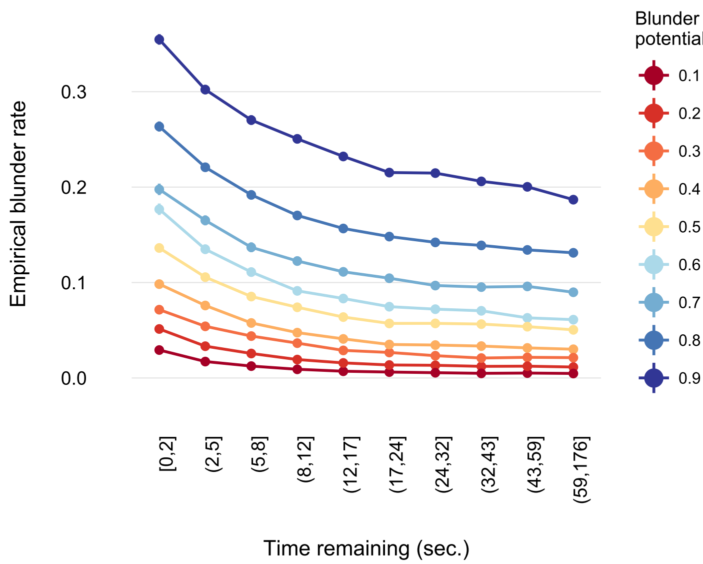


Fig. 11. The empirical blunder rate as a function of the time remaining, for positions with fixed blunder potential values.

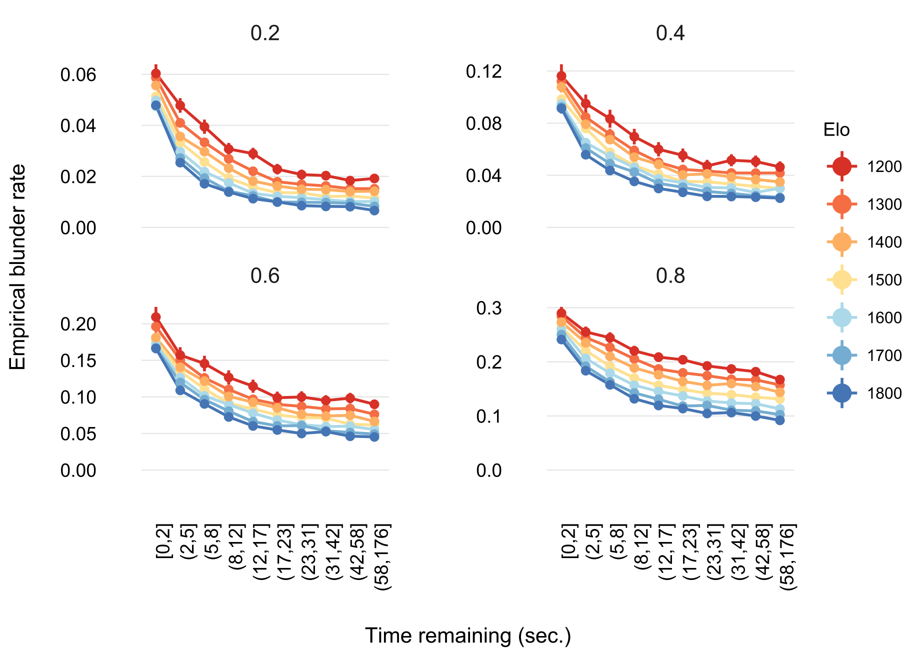


Fig. 12. The empirical blunder rate as a function of time remaining, fixing the blunder potential and player rating.

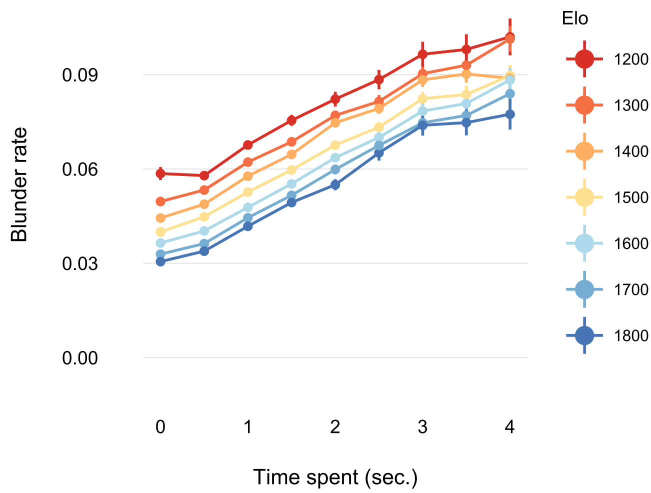


Fig. 13. The empirical blunder rate as a function of time spent on the move separated by rating bucket. The data is restricted to positions where the player to move have at least 20 seconds remaining.

Comparing across points in different panels helps drive home the role of blunder potential even when considering skill and time simultaneously. Consider for example (a) instances in which players rated 1200 (at the low end of the FICS data) with 5–8 seconds remaining face a position of blunder potential 0.4, contrasted with (b) instances in which players rated 1800 (at the high end of the FICS data) with 42–58 seconds remaining face a position of blunder potential 0.8. As the figure shows, the empirical blunder rate is lower in instances of type (a)—a weak player in extreme time pressure is making blunders at a lower rate because they are dealing with positions that contain less danger.

Time Spent on a Move. Thus far, we have looked at how the empirical blunder rate depends on the amount of time remaining in the game. But we can also ask how the probability of a blunder varies with the amount of time the player actually spends considering their move before playing it. When a player spends more time on a move, should we predict they are less likely to blunder (because they gave the move more consideration) or more likely to blunder (because the extra time suggests they did not know what to do)?

The data turns out to be strongly consistent with the latter view. As shown in Figure 13 (where we restrict to instances in which players have at least 20 seconds remaining), the empirical blunder rate is higher for players who spend more time playing a move. The increase is large: in aggregate, nearly-instantly played moves are blunders only 4% of the time, whereas when players take more than 3 seconds on a move they blunder more than 8% of the time—more than twice as often. Of course, it is unlikely that spending longer on a particular move decreases decision quality; a more natural hypothesis is that these curves are produced by an effect where players are aware of when they are unsure what to do, and spend more time on more difficult decisions. The extra time they spend does not entirely make up for their uncertainty, however, since blunder rates still rise as time spent increases. Furthermore, the slopes of the curves are virtually parallel, meaning that every additional second of time spent corresponds to a constant increase in the blunder rate regardless of rating. Even though blunder rates are higher for moves with more time spent, the spread of blunder rates among players of different skill levels stays roughly the same.

4. PREDICTION

We have now seen how the empirical blunder rate depends on our three fundamental dimensions: difficulty, the skill of the player, and the time available to them. We now turn to a set of tasks that allow us to further study the predictive power of these dimensions.

4.1. Greater Tree Depth

In order to formulate our prediction methods for blunders, we first extend the set of features available for studying the difficulty of a position. Once we have these additional features, we will be prepared to develop the predictions themselves.

Thus far, when we have considered a position's difficulty, we have used information about the player's immediate moves, and then invoked a tablebase to determine the outcome after these immediate moves. We now ask whether it is useful for our task to consider longer sequences of moves beginning at the current position. Specifically, if we consider all d-move sequences beginning at the current position, we can organize these into a *game tree* of depth d with the current position P as the root, and nodes representing the states of the game after each possible sequence of $j \leq d$ moves. Chess engines use this type of tree as their central structure in determining which moves to make; it is less obvious, however, how to make use of these trees in analyzing blunders by human players, given players' imperfect selection of moves even at depth 1.

Let us introduce some notation to describe how we use this information. Suppose our instance consists of position P, with n legal moves, of which b are blunders. We will denote the moves by $m_1, m_2, \ldots, m_n$, leading to positions $P_1, P_2, \ldots, P_n$, respectively, and we will suppose they are indexed so that $m_1, m_2, \ldots, m_{n-b}$ are the non-blunders, and $m_{n-b+1}, \ldots, m_n$ are the blunders. We write T_0 for the indices of the non-blunders $\{1, 2, \ldots, n-b\}$ and T_1 for the indices of the blunders $\{n-b+1, \ldots, n\}$. Finally, from each position P_i, there are n_i legal moves, of which b_i are blunders.

The set of all pairs (n_i, b_i) for $i = 1, 2, \ldots, n$ constitutes a potentially useful source of information in the depth-2 game tree from the current position. What might it tell us?

First, suppose that position P_i, for $i \in T_1$, is a position reachable via a blunder m_i. Then if the blunder potential $\beta(P_i) = b_i/n_i$ is large, this means that it may be challenging for the opposing player to select a move that capitalizes on the blunder m_i made at the root position P; there is a reasonable chance that the opposing will instead blunder, restoring the minimax value to something larger. This, in turn, means that it may be harder for the player in the root position of our instance to see that move m_i, leading to position P_i, is in fact a blunder. The conclusion from this reasoning is that when the blunder potentials of positions P_i for $i \in T_1$ are large, it suggests a larger empirical blunder rate at P.

It is less clear what to conclude when there are large blunder potentials at positions P_i for $i \in T_0$—positions reachable by non-blunders. Again, it suggests that player at the root may have a harder time correctly evaluating the positions P_i for $i \in T_0$; if they appear better than they are, it could lead the player to favor these non-blunders. On the other hand, the fact that these positions are hard to evaluate could also suggest a general level of difficulty in evaluating P, which could elevate the empirical blunder rate.

There is also a useful aggregation of this information, as follows. If we define $b(T_1) = \sum_{i \in T_1} b_i$ and $n(T_1) = \sum_{i \in T_1} n_i$, and analogously for $b(T_0)$ and $n(T_0)$, then the ratio $\beta_1 = b(T_1)/n(T_1)$ is a kind of aggregate blunder potential for all positions reachable by blunders, and analogously for $\beta_0 = b(T_0)/n(T_0)$ with respect to positions reachable by non-blunders.

In the next subsection, we will see that the four quantities $b(T_1)$, $n(T_1)$, $b(T_0)$, $n(T_0)$ indeed contain useful information for prediction, particularly when looking at families of instances that have the same blunder potential at the root position P. We note that

Table I. Features for Blunder Prediction

Feature	Description
$\beta(P)$	Blunder potential ($b(P)/n(P)$)
$a(P)$, $b(P)$	# of non-blunders, # of blunders
$a(T_0)$, $b(T_0)$	# of non-blunders and blunders available to opponent following a non-blunder
$a(T_1)$, $b(T_1)$	# of non-blunders and blunders available to opponent following a blunder
$\beta_0(P)$	Opponent's aggregate β after a non-blunder
$\beta_1(P)$	Opponent's aggregate β after a blunder
Elo, Opp-elo	Player ratings (skill level)
t	Amount of time player has left in game

one can construct analogous information at greater depths in the game tree, by similar means, but we find in the next subsection that these do not currently provide improvements in prediction performance, so we do not discuss greater depths further here.

4.2. Prediction Results

We develop three nested prediction tasks: in the first task we make predictions about an unconstrained set of instances; in the second we fix the blunder potential at the root position; and in the third we control for the exact position.

Task 1. In our first task we formulate the basic error-prediction problem: we have a large collection of human decisions for which we know the correct answer, and we want to predict whether the decision-maker will err or not. In our context, we predict whether the player to move will blunder, given the position they face and the various features of it we have derived, how much time they have to think, and their skill level. In the process, we seek to understand the relative value of these features for prediction in our domain.

We restrict our attention to the 6.6 million instances that occurred in the 320,000 empirically most frequent positions in the FICS dataset. Since the rate of blundering is low in general, we down-sample the non-blunders so that half of our remaining instances are blunders and the other half are non-blunders. This results in a balanced dataset with 600,000 instances, and we evaluate model performance with accuracy. For ease of interpretation, we use both logistic regression and decision trees. Since the relative performance of these two classifiers is virtually identical, but decision trees perform slightly better, we only report the results using decision trees here.

Table I defines the features we use for prediction. In addition to the notation defined thus far, we define: $S = \{\text{Elo}, \text{Opp-elo}\}$ to be the skill features consisting of the rating of the player and the opponent; $a(P) = n(P) - b(P)$ for the number of non-blunders in position P; $D_1 = \{a(P), b(P), \beta(P)\}$ to be the difficulty features at depth 1; $D_2 = \{a(T_0), b(T_0), a(T_1), b(T_1), \beta_0(P), \beta_1(P)\}$ as the difficulty features at depth 2 defined in the previous subsection; and t as the time remaining.

In Table II, we show the performance of various combinations of our features. The most striking result is how dominant the difficulty features are. Using all of them together gives 0.75 accuracy on this balanced dataset, halfway between random guessing and perfect performance. In comparison, skill and time are much less informative on this task. The skill features S only give 55% accuracy, time left t yields 53% correct predictions, and neither adds predictive value once position difficulty features are in the model.

The weakness of the skill and time features is consistent with our findings in Section 3, but still striking given the large ranges over which the Elo ratings and time

Table II. Accuracy Results on Task 1

Model	Accuracy
Random guessing	0.50
$\beta(P)$	0.73
D_1	0.73
D_2	0.72
$D_1 \cup D_2$	**0.75**
S	0.55
$S \cup D_1 \cup D_2$	**0.75**
$\{t\}$	0.53
$\{t\} \cup D_1 \cup D_2$	**0.75**
$S \cup \{t\} \cup D_1 \cup D_2$	**0.75**

Statistical significances such as p-values are all approaching 0 as we are computing over hundreds of thousands of instances and have fewer than 10 features.

remaining can extend. In particular, a player rated 1800 will almost always defeat a player rated 1200, yet knowledge of rating is not providing much predictive power in determining blunders on any individual move. Similarly, a player with 10 seconds remaining in the entire game is at an enormous disadvantage compared to a player with two minutes remaining, but this too is not providing much leverage for blunder prediction at the move level. While these results only apply to our particular domain, it suggests a genre of question that can be asked by analogy in many domains. (To take one of many possible examples, one could similarly ask about the error rate of highly skilled drivers in difficult conditions versus bad drivers in safe conditions.)

Another important result is that most of the predictive power comes from depth-1 features of the tree. This tells us the immediate situation facing the player is by far the most informative feature.

Finally, we note that the prediction results for the GM data (where we do not have time information available) are closely analogous; we get a slightly higher accuracy of 0.77, and again it comes entirely from our basic set of difficulty features for the position.

Human Performance on a Version of Task 1. Given the accuracy of algorithms for Task 1, it is natural to consider how this compares to the performance of human chess players on such a task.

To investigate this question, we developed a version of Task 1 as a web app quiz and promoted it on two popular Internet chess forums. Each quiz question provided a pair of ≤6-piece instances with White to move, each showing the exact position on the board, the ratings of the two players, and the time remaining for each. The two instances were chosen from the FICS data with the property that White blundered in one of them and not the other, and the quiz question was to determine in which instance White blundered.

In this sense, the quiz is a different type of chess problem from the typical style, reflecting the focus of our work here: rather than "White to play and win," it asked "Did White blunder in this position?." Averaging over approximately 6,000 responses to the quiz from 720 participants, we find an accuracy of 0.69, non-trivially better than random guessing but also non-trivially below our model's performance of 0.79.[7]

[7]Note that the model performs slightly better here than in our basic formulation of Task 1, since there instances were not presented in pairs but simply as single instances drawn from a balanced distribution of positive and negative cases.

The relative performance of the prediction algorithm and the human forum participants forms an interesting contrast, given that the human participants were able to use domain knowledge about properties of the exact chess position while the algorithm is achieving almost its full performance from a single number—the blunder potential—that draws on a tablebase for its computation. We also investigated the extent to which the guesses made by human participants could be predicted by an algorithm; our accuracy on this was in fact lower than for the blunder-prediction task itself, with the blunder potential again serving as the most important feature for predicting human guesses on the task.

Task 2. Given how powerful the depth-1 features are, we now control for $b(P)$ and $n(P)$ and investigate the predictive performance of our features once blunder potential has been fixed. Our strategy on this task is very similar to before: we compare different groups of features on a binary classification task and use accuracy as our measure. These groups of features are D_2, S, $S \cup D_2$, $\{t\}$, $\{t\} \cup D_2$, and the full set $S \cup \{t\} \cup D_2$. For each of these models, we have an accuracy score for every $(b(P), n(P))$ pair. The relative performances of the models are qualitatively similar across all $(b(P), n(P))$ pairs: again, position difficulty dominates time and rating, this time at depth 2 instead of depth 1. In all cases, the performance of the full feature set is best (the mean accuracy is 0.71), but D_2 alone achieves 0.70 accuracy on average. This further underscores the importance of position difficulty.

Additionally, inspecting the decision tree models reveals a very interesting dependence of the blunder rate on the depth-1 structure of the game tree. First, recall that positions with $b(P) = 1$ and positions with $b(P) = n(P) - 1$ both occur frequently in our datasets. In so-called “only-move” situations, where there is only one move that is not a blunder, the dependence of blunder rate on D_2 is as one would expect: the higher the $b(T_1)$ ratio, the more likely the player is to blunder. But for positions with only one blunder, the dependence reverses: blunders are *less* likely with higher $b(T_1)$ ratios. Understanding this latter effect is an interesting open question.

Task 3. Our final prediction question is about the degree to which time and skill are informative once the position has been fully controlled for. In other words, once we understand everything we can about a position’s difficulty, what can we learn from the other dimensions? To answer this question, we set up a final task where we fix the position completely, create a balanced dataset of blunders and non-blunders, and consider how well time and skill predict whether a player will blunder in the position or not. We do this for all 25 instances of positions for which there are over 500 blunders in our data. On average, knowing the rating of the player alone results in an accuracy of 0.62, knowing the times available to the player and his opponent yields 0.54, and together they give 0.63. Thus once difficulty has been completely controlled for, there is still substantial predictive power in skill and time, consistent with the notion that all three dimensions are predictive of blunders.

5. FURTHER DIRECTION: THE ROLE OF RECENT HISTORY

There are other dimensions in addition to skill, time, and the difficulty of the instance that one could imagine using as information in estimating the probability of error in a given decision, and in this final section we suggest one such source of information that leads to a set of intriguing open questions.

This source of information is the recent history of actions leading up to the current instance. In the case of chess as a domain, this would most naturally correspond to using information not just about the current position on the board, but also the position immediately preceding it in the game, and the move by the opponent that brought the game to the current position. (One could also extend this to consider the most recent

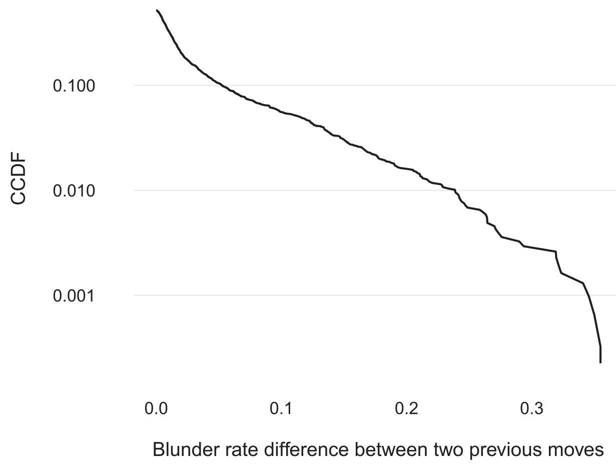


Fig. 14. Complementary cumulative distribution of the blunder rate difference in frequently occurring positions depending on which of the two most common immediately preceding moves was played by the opponent.

few moves by both players and the positions they were played in, but for simplicity we can ask this question just about the single immediately preceding position and move.)

In the case of the single immediately preceding position and move, it is natural to ask whether the move that led to this position should be useful at all in predicting errors, given that the rules of chess (and most other tasks like chess) imply that decision-making is almost always *memoryless*—all that matters is the current position on the board.[8] A basic way to make this question precise is to define $s_m(P)$ to be the empirical blunder rate in position P when the preceding move by the opponent was m. How often does it happen that for two preceding moves ℓ and m we have significantly different values for $s_\ell(P)$ and $s_m(P)$?

In fact we find that this happens surprisingly often in our data; the empirical blunder rate in a given position can vary considerably depending on the previous move by the opponent. If we fix a non-trivial threshold k and consider all positions that occur at least k times in our data, we can consider the difference $s_\ell(P)$ and $s_m(P)$ for the two most common preceding moves ℓ and m. In more than 10% of these instances, $|s_\ell(P) - s_m(P)|$ is at least 0.05, and in more than 1% of them it is at least 0.20. As one sees from our earlier analyses, these are large differences in qualitative terms.

Figure 14 gives the full complementary cumulative distribution for the quantity $|s_\ell(P) - s_m(P)|$ over all frequently occurring positions, where for P the moves ℓ and m are the two most common immediately preceding moves by the opponent, and both occurred at least 50 times.

Given the fact that the task we are studying depends only on the current position, it is interesting to ask why one finds such non-trivial differences in blunder rate as a function of the immediately preceding move. We pose this primarily as an open question, but we observe that there are two distinct families of reasons that could be operating. The first is a behavioral one, explored in psychological studies of chess by Bilalić et al. [2008]: the immediately preceding course of the game might have caused the player to focus on certain aspects of the position rather than others, and

[8]This is not always true; the *threefold-repetition rule* and the *the 50-move rule* for draws in chess imply that there are situations in which a player has to think about the sequence of events leading to the current position. But these cases are very rare, and as noted earlier they do not play a significant role in any of our analyses here.

so when they come to the present position P, the aspects that are psychologically salient to them may be different depending on whether the game arrived at P via ℓ or m. This in turn could have an effect on whether they notice that certain moves from position P are blunders. A second category of reasons relates more directly to the notion of skill: it appears from the instances where $s_\ell(P)$ and $s_m(P)$ differ significantly that the immediately preceding position conveys information about whether the player selecting the move, or the player's opponent, or both players, are displaying a good or bad sense about how to handle the course of the particular game. Certain preceding positions suggest that they are following a good plan, while others suggest that they are following a misguided plan. This suggests that the recent history of preceding positions is providing us with information about the player's understanding of the current position—a kind of skill specific to the instance at hand, rather than their overall skill as reflected through Elo rating.

There are natural analogies for both of these categories of reasons to other domains: in any situation where someone is making a decision in the present, knowledge of what has happened in the recent past can both suggest something about what they view as salient, and also suggest something about the skill that they are displaying in handling the current situation. These considerations, and what we see in our data here, suggest that including these types of information about the recent past can be useful as features in assessing error; we leave the deeper exploration of such categories of features as a direction for further work.

6. CONCLUSION

We have used chess as a model system to investigate the types of features that help in analyzing and predicting error in human decision-making. Chess provides us with a highly instrumented domain in which the time available to and skill of a decision-maker are often recorded, and, for positions with few pieces, the set of optimal decisions can be determined computationally using *tablebases*. Furthermore, online chess servers provide us with a massive amount of human play to study.

Through our analysis we have seen that the inherent difficulty of the decision, even approximated simply by the proportion of available blunders in the underlying position, can be a much more powerful source of information than the skill or time available. We have also identified a number of other phenomena, including the ways in which players of different skill levels benefit differently, in aggregate, from easier instances or more time. And we have found, surprisingly, that there exist *skill-anomalous* positions in which weaker players commit fewer errors than stronger players.

We believe there are natural opportunities to apply the article's framework of skill, time, and difficulty to a range of settings in which human experts make a sequence of decisions, some of which turn out to be in error. In doing so, we may be able to differentiate between domains in which skill, time, or difficulty emerge as the dominant source of predictive information. Many questions in this style can be asked. For a setting such as medicine, is the experience of the physician or the difficulty of the case a more important feature for predicting errors in diagnosis? Or to recall an analogy raised in the previous section, for micro-level mistakes in a human task such as driving, we think of inexperienced and distracted drivers as a major source of risk, but how do these effects compare to the presence of dangerous road conditions?

Finally, there are a number of interesting further avenues for exploring our current model domain of chess positions via tablebases. First, we note that our definition of blunders, while concrete and precisely aligned with the minimax value of the game tree, is not the only definition that could be considered even using tablebase evaluations. In particular, it would also be possible to consider "softer" notions of blunders. Suppose for example that a player is choosing between moves m and m', each leading to a position whose minimax value is a draw, but suppose that the position arising after m is more difficult for the opponent, and produces a much higher empirical probability that the opponent will make a mistake at some future point and lose. Then it can be viewed as a kind of blunder, given these empirical probabilities, to play m' rather than the more challenging m. This is sometimes termed *speculative play* [Jansen 1990], and it can be thought of primarily as a refinement of the coarser minimax value. Another direction is to more fully treat the domain as a competitive activity between two parties. For example, is there evidence in the kinds of positions we study that stronger players are not only avoiding blunders, but also steering the game toward positions that have higher blunder potential for their opponent? More generally, the interaction of competitive effects with principles of error-prone decision-making can lead to a rich collection of further questions.

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