Tutorial 3: Decidability and Undecidability

CSC 463

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1. Recall that Rice's theorem states that if P is a non-trivial set of Turing machine encodings $\langle M \rangle$ satisfying the condition that if M_1, M_2 are Turing machines with $L(M_1) = L(M_2)$, then $\langle M_1 \rangle \in P$ if and only if $\langle M_2 \rangle \in P$ [1], we can conclude that P is an undecidable set.

   Prove Rice's Theorem using the following steps.

   (a) Explain why we can assume that P does not contain any Turing machine M with $L(M) = \emptyset$ and there is a Turing machine $M' \in P$ with $L(M') \neq \emptyset$. (Hint: Recall that if A is decidable, so is the complement $\overline{A}$.)

   (b) Construct a reduction $A_{TM} \leq_m P$ so that reduction maps instances $\langle M, w \rangle$ to Turing machines $\langle N \rangle \in P$, using M' from part (a), if and only if M accepts w.

   (c) Conclude Rice's Theorem from the reduction in part (b).

   (d) Let L be the set of Turing machine encodings $\langle M \rangle$ with less than 200 states. Show that L is decidable. Why does this **not** contradict Rice's theorem?

2. Let $T = \{\langle M \rangle : M \text{ is a Turing machine with } |L(M)| = 3\}$. Prove that T is not semidecidable.

3. Let G_1 and G_2 be context-free grammars. Show that the problem of testing whether or not $L(G_1) \subset L(G_2)$ is undecidable. You may assume that testing whether or not $L(G) = \Sigma^*$ for a context-free grammar G is undecidable.

4. Assume that $\Gamma = \{0, 1, \sqcup\}$ is the tape alphabet for all Turing machines in this problem. The **busy beaver function** $BB : \mathbb{N} \mapsto \mathbb{N}$ is defined as follows. Let M_k be the set of k-state Turing machines that halt when started on a blank tape. Define $BB(k)$ to be the maximum number of ones remaining on the tape when a machine $M \in M_k$ started on a blank tape halts. [2]

   Show that BB is not a computable function (i.e. there is no Turing machine M which on input k in unary, outputs $BB(k)$ in unary and halts). You may assume that BB is a strictly increasing function. [3]

[1] In other words, whether or not $\langle M \rangle \in P$ depends on its language $L(M)$ only

[2] Since M_k is a finite set, $BB(k)$ exists.

[3] Blog post about BB for those interested: `https://www.scottaaronson.com/blog/?p=2725`. If we know BB for large enough k, we can solve major open problems in mathematics!