Problem Set 4

CSC 463

Due by April 2, 2020 at 11:59pm

You may consult with others concerning the general approach for solving problems on assignments, but you must write up all solutions entirely on your own. Copying assignments is a serious academic offense and will be dealt with accordingly. You may cite results discussed in lecture or the course textbook. You are encouraged to write precisely and concisely; it should be possible to write your solutions to the problem set within a page per question. Submit your work online by uploading a pdf file or image of your solutions onto Crowdmark. In all problems, you can assume that you are given valid encodings as input to the problem.

1. We say that two propositional Boolean formulas ϕ_1, ϕ_2 are **equivalent** if they have the same set of variables and the same truth table (i.e. $\phi_1(x_1, \ldots, x_n) = \phi_2(x_1, \ldots, x_n)$ for every possible truth assignment of $x_1, \ldots, x_n$). For instance, the CNF formulas

$$\phi_1 = (x \vee y \vee z) \wedge (x \vee y \vee \overline{z}) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee y \vee z)$$

   and

$$\phi_2 = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$$

   are equivalent since both are true if and only if at least two of x, y, z assigned true. The length of a formula is defined as the total number of variables and logical connectives appearing in the formula. We say that a formula ϕ is **minimal** if there is **no** shorter formula ψ that is equivalent to ϕ.

   Let **MIN-FORMULA** be the problem of deciding if a Boolean formula ϕ is minimal. Show that **MIN-FORMULA** can be decided in polynomial space.

2. In the next two problems, we will study **2SAT**, the satisfiability problem for CNF Boolean formulas where there are exactly two different variables per clause. The **implication graph** G_ϕ of a 2CNF formula ϕ containing n variables is a **directed** graph with $2n$ vertices V, one for every possible literal in ϕ, and edges $(\overline{l_1}, l_2)$ and $(\overline{l_2}, l_1)$ for every clause $(l_1 \vee l_2)$ in ϕ. Prove the following properties of implication graphs:

   (a) Suppose there is a directed path between literals l_1 and l_2 in G_ϕ. Then show that there is also a directed path between $\overline{l_2}$ and $\overline{l_1}$. Furthermore, show that if τ is a truth assignment satisfying ϕ where $\tau(l_1)$ is true, then $\tau(l_2)$ is true as well.

   (b) Using the observations in the previous part, show that ϕ is unsatisfiable if and only if there is some directed cycle in G_ϕ containing a variable x and its complement $\overline{x}$.

3. Show that **2SAT** is **NL**$-$Complete by using the observations in the previous parts. You may use the fact that **2SAT** is **NL**-Complete if and only if the complement $\overline{\textbf{2SAT}}$ is also **NL**-Complete, and the **NL**$-$Completeness of **PATH**.