Polynomial Shape from Shading

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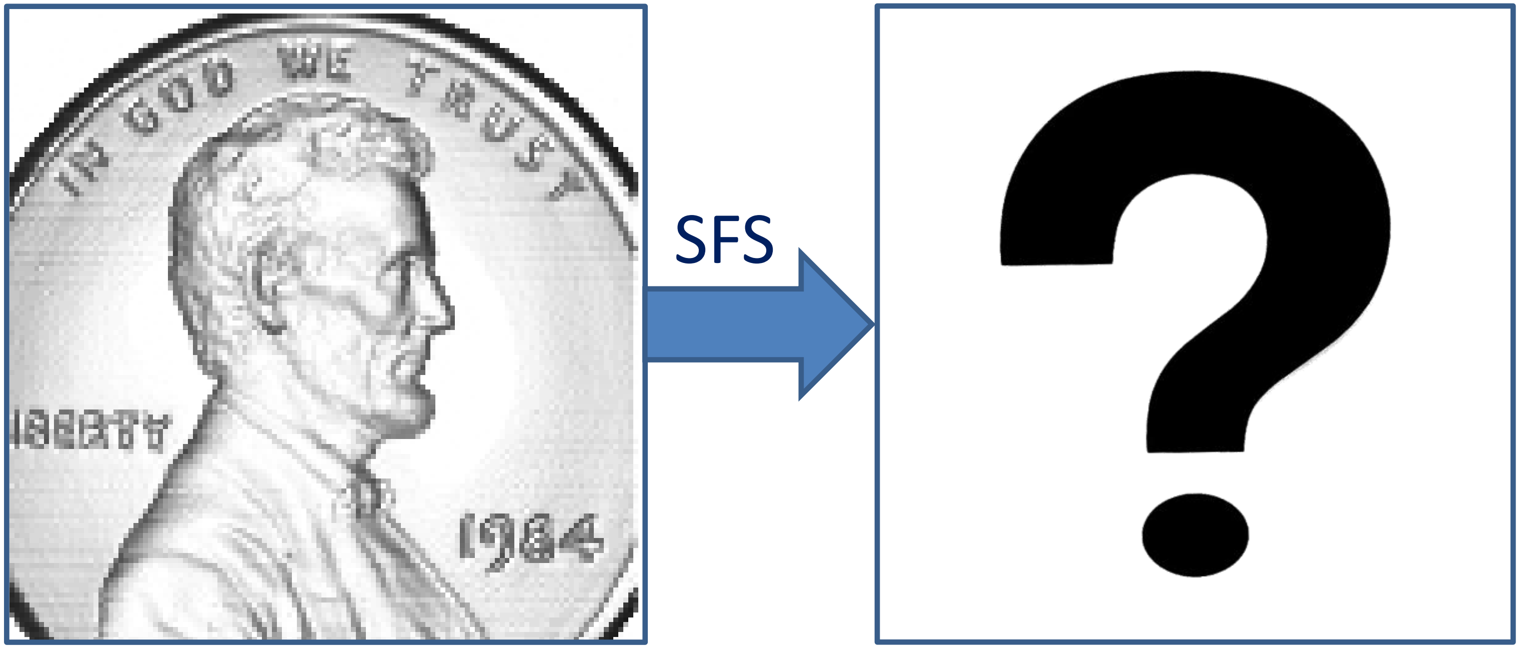
Shape from shading (SFS)



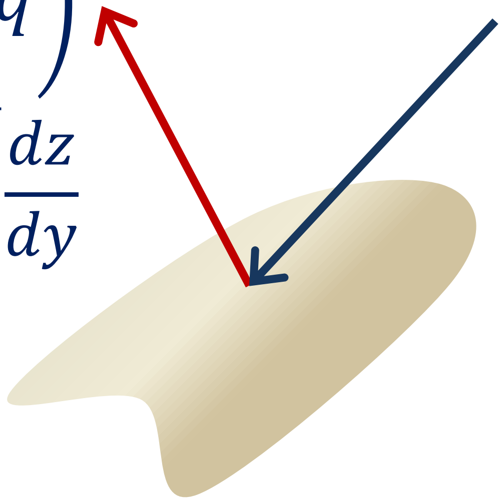
Single 2D image

3D surface

Standard Lambertian SFS

$$\mathbf{N} = \begin{pmatrix} -p \\ -q \\ 1 \end{pmatrix}$$

$$p = \frac{dz}{dx}, q = \frac{dz}{dy}$$

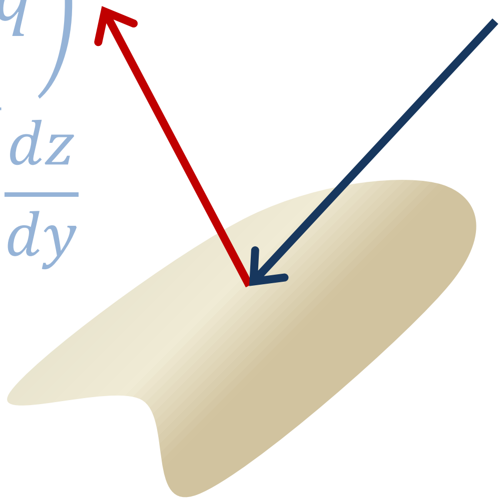


$$\mathbf{L} = (a, b, c)$$

$$\|\mathbf{L}\| = 1$$

$$I = \frac{\mathbf{L} \cdot \mathbf{N}}{\|\mathbf{N}\|} = \frac{-ap - bq + c}{\sqrt{1 + p^2 + q^2}}$$

Polynomial form



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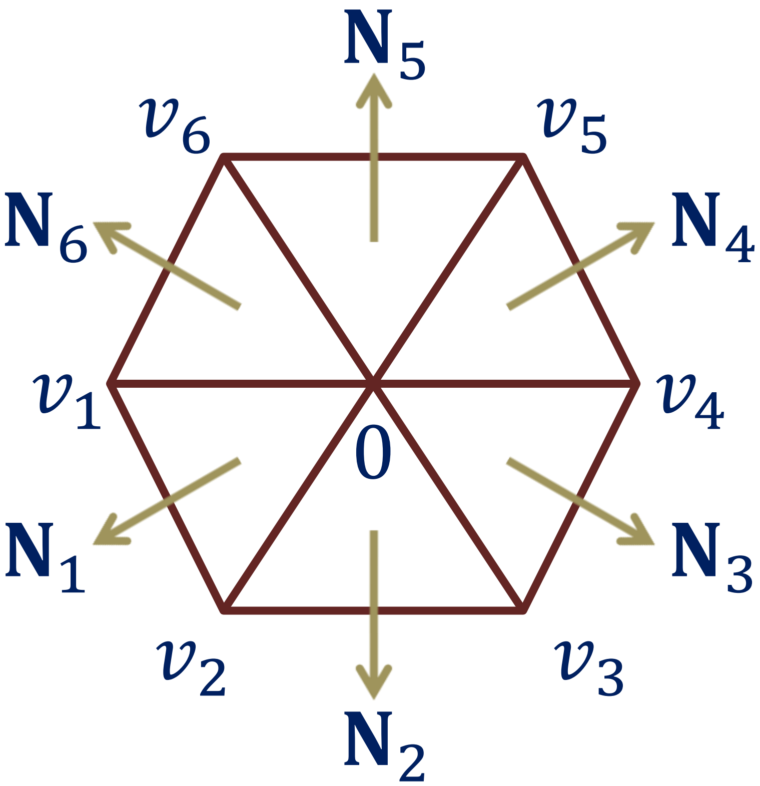
$$(1 + p^2 + q^2)I^2 - (-ap - bq + c)^2 = 0$$

$$-ap - bq + c \geq 0$$

Outline

- Advantages of the polynomial form
  - Small systems are solvable
  - Exact line search
  - Semidefinite programming (SDP) relaxation
- Shading ambiguities
  - Visualization of ambiguous solutions

SFS of a polyhedron



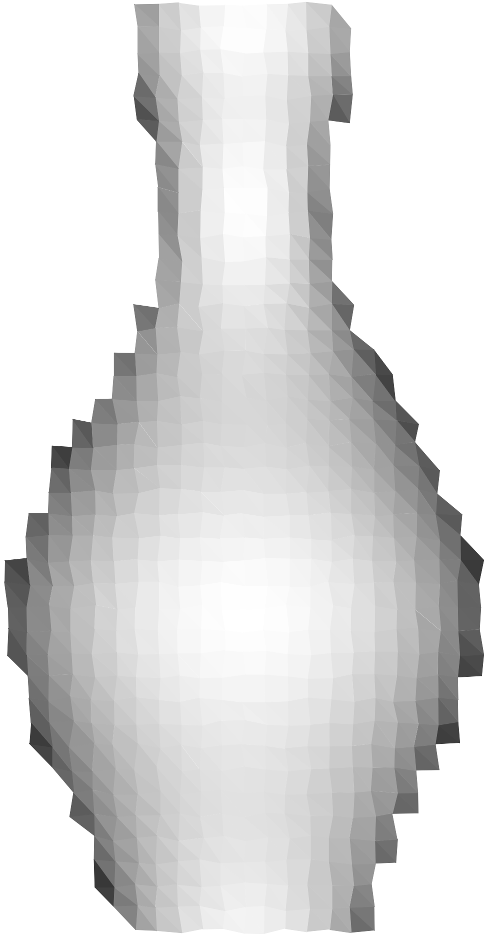
$$-p_i = \frac{y_i z_{i+1} - y_{i+1} z_i}{x_i y_{i+1} - x_{i+1} y_i}$$

$$-q_i = \frac{x_{i+1} z_i - x_i z_{i+1}}{x_i y_{i+1} - x_{i+1} y_i}$$

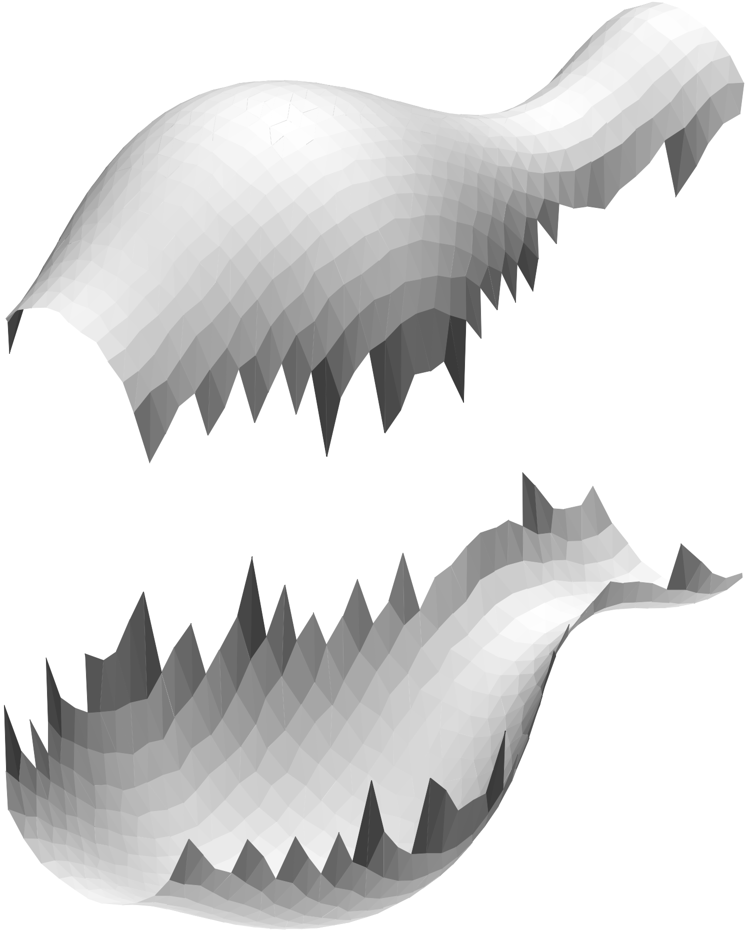
- A small polynomial system in the unknowns $z_1, \dots, z_6$ around a vertex
- All solutions to generic systems can be found by polynomial solvers (e.g. HOM4PS-2.0)

SFS of a polyhedron

Synthetic input



Computed solutions



- Unfortunately, the method is sensitive to noise

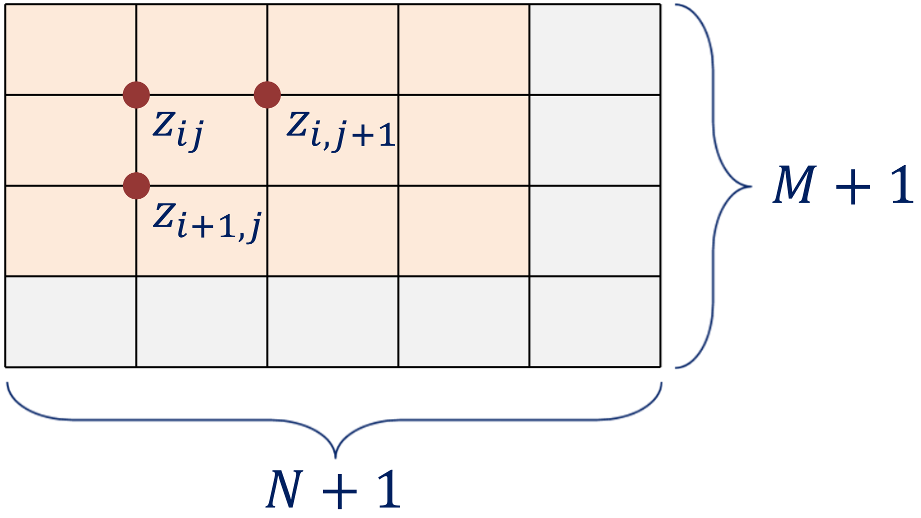
SFS on a grid

$$p_{ij} = z_{i+1,j} - z_{ij}, \;\; q_{ij} = z_{i,j+1} - z_{ij}$$

$$r_{ij} = (1 + p^2 + q^2)I^2 - (-ap - bq + c)^2$$

$$= \mathbf{z}^T \mathbf{A}_{ij} \mathbf{z} + \mathbf{e}_{ij}^T \mathbf{z} + h_{ij}$$

$$minimize \; \|\mathbf{r}\|^2 + \lambda(smoothness \; term)$$



Conjugate gradient with exact line search

$r_{ij} = \mathbf{z}^T \mathbf{A}_{ij} \mathbf{z} + \mathbf{e}_{ij}^T \mathbf{z} + h_{ij}$

$minimize\ \|\mathbf{r}\|^2 + \lambda(smoothness\ term)$

- Use line search on $\mathbf{z} = \mathbf{z}_0 + \alpha \mathbf{d}$
- Results in a quartic minimization in α
- Can be solved in closed form

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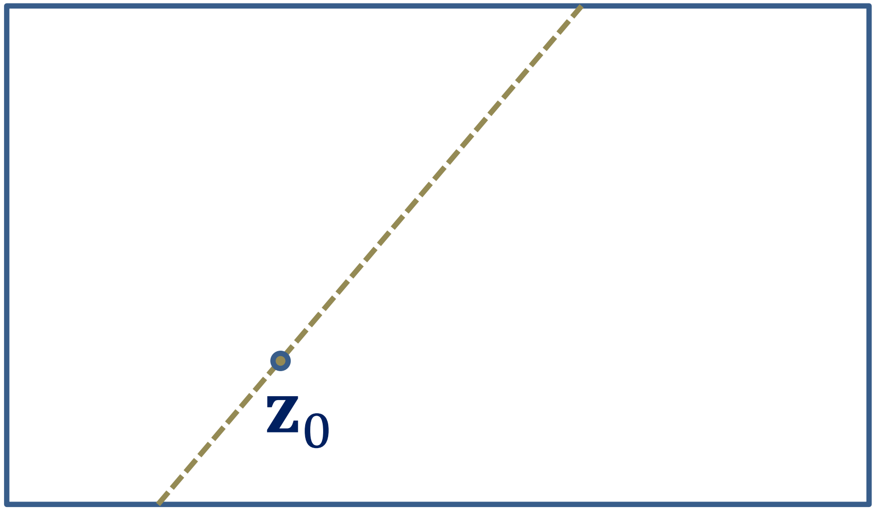

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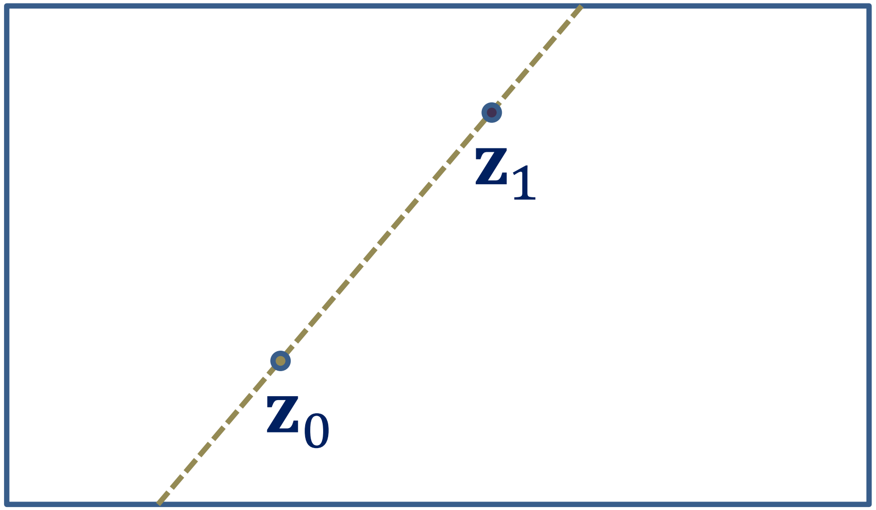

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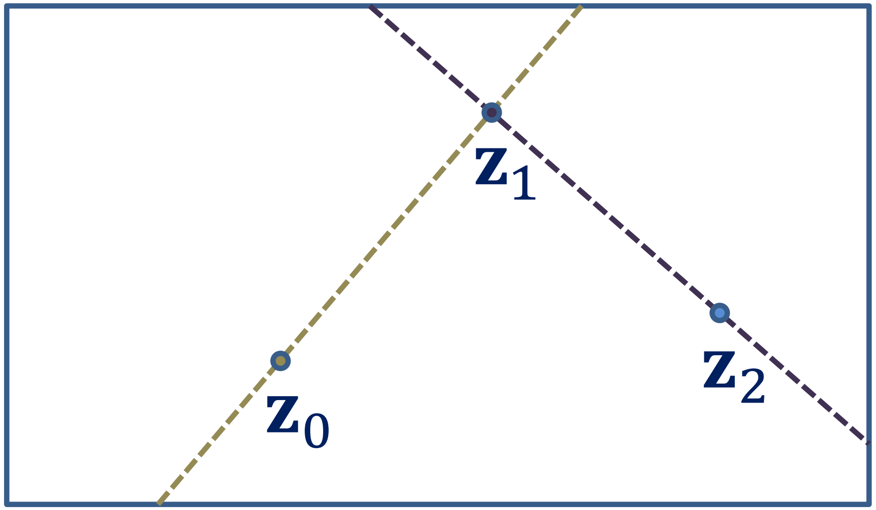

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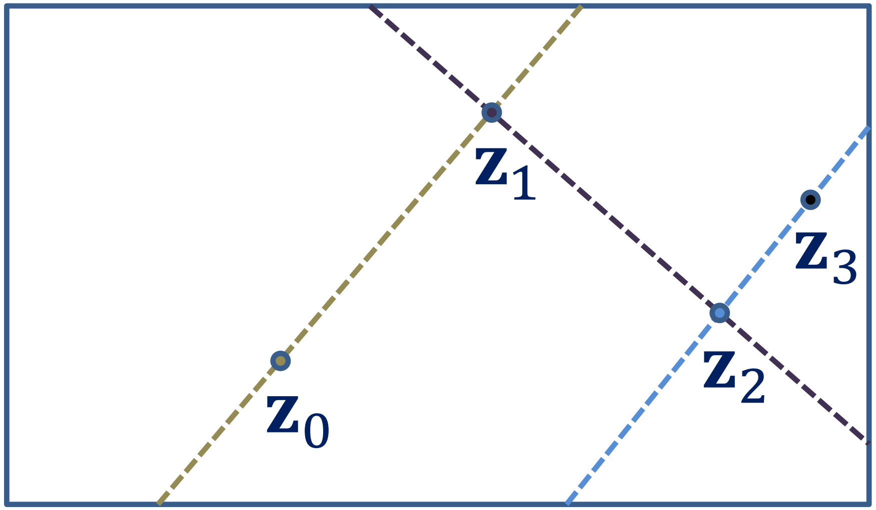
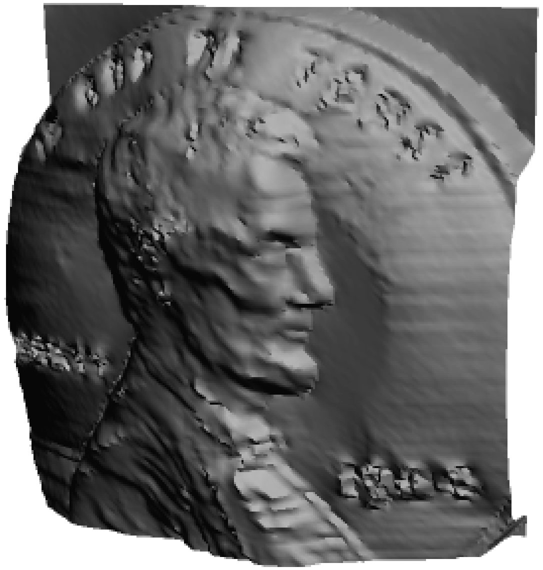


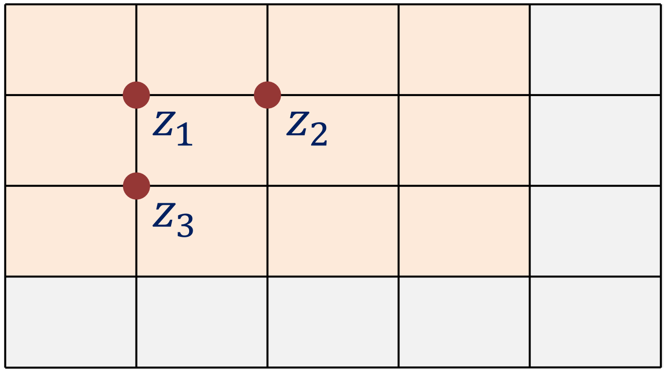
Result

input
image
and
surface

output
image
and
surface



SDP relaxation

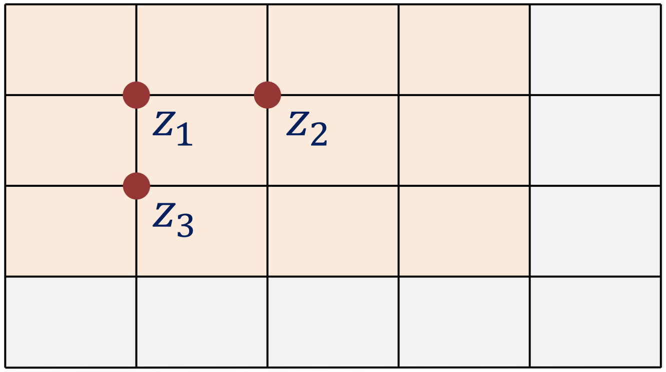


$p = z_3 - z_1,\ \ q = z_2 - z_1$

$r = (1 + p^2 + q^2)I^2 - (-ap - bq + c)^2$

ideally $r = 0$

SDP relaxation



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Each element of $\mathbf{M}$ represents a product between the corresponding monimials

$\mathbf{M}_{kl}$ are the new variables

$$\begin{array}{c|cccccccccc}
 & 1 & z_1 & z_2 & z_3 & z_1^2 & z_1z_2 & z_1z_3 & z_2^2 & z_2z_3 & z_3^2 \\
\hline
1 & & & & & & & & & & \\
z_1 & & & & & & & & & & \\
z_2 & & & & & & & & & & \\
z_3 & & & & & & & & & & \\
z_1^2 & & & & & & & & & & \\
z_1z_2 & & & & & & & \mathbf{M}_{kl} & & & \\
z_1z_3 & & & & & & & & & & \\
z_2^2 & & & & & & & & & & \\
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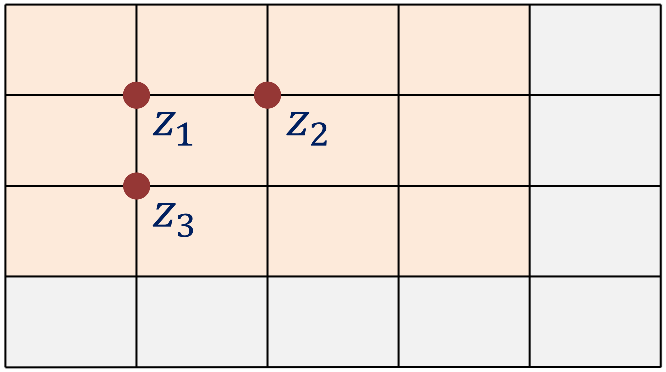
\end{array}$$

SDP relaxation



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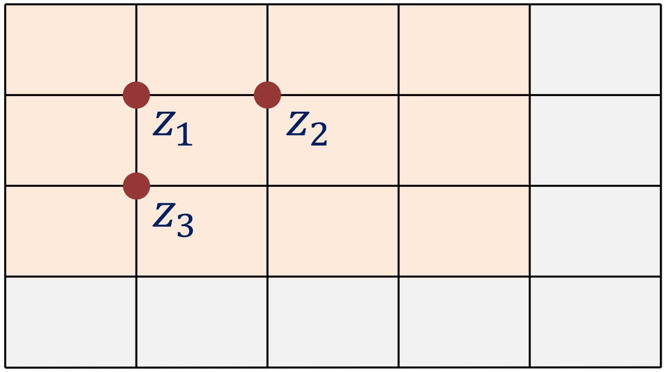
Each element of $\mathbf{M}$ represents a product between the corresponding monimials

Some entries are equal

$$\begin{array}{c|cccccccccc}
 & 1 & z_1 & z_2 & z_3 & z_1^2 & z_1z_2 & z_1z_3 & z_2^2 & z_2z_3 & z_3^2 \\
\hline
1 & & & & & & & & & & \\
z_1 & & & & & & \bigcirc & & & & \\
z_2 & & & & & & & & & & \\
z_3 & & & & & & & & & & \\
z_1^2 & & & \bigcirc & & & & & & & \\
z_1z_2 & & & & & & & \mathbf{M}_{kl} & & & \\
z_1z_3 & & & & & & & & & & \\
z_2^2 & & & & & & & & & & \\
z_2z_3 & & & & & & & & & & \\
z_3^2 & & & & & & & & & &
\end{array}$$

SDP relaxation



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z_2z_3 & & & & & & & & & & \\
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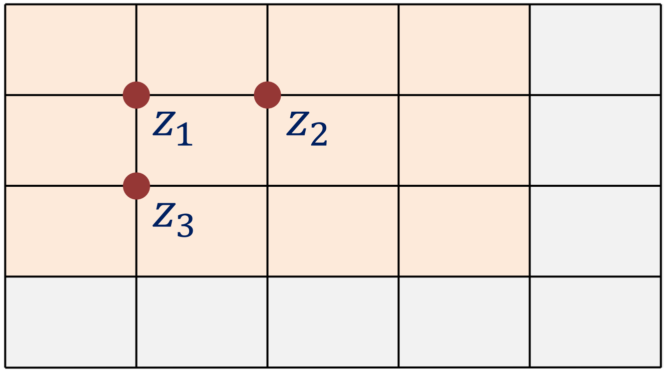
\end{array}$$

M should be symmetric positive-semidefinite

$$\mathbf{M} \succcurlyeq 0$$

SDP relaxation



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Write the quadratic constraints as linear combinations of elements of $\mathbf{M}$

$$-\varepsilon \le \mathbf{U} \bullet \mathbf{M} \le \varepsilon$$

$$\begin{array}{c|cccccccccc}
 & 1 & z_1 & z_2 & z_3 & z_1^2 & z_1 z_2 & z_1 z_3 & z_2^2 & z_2 z_3 & z_3^2 \\
\hline
1 & \textcircled{1} & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
z_1 & & & & & & & & & & \\
z_2 & & & & & & & & & & \\
z_3 & & & & & & & & & & \\
z_1^2 & & & & & & & & & & \\
z_1 z_2 & & & & & & & & & & \\
z_1 z_3 & & & & & & & & & & \\
z_2^2 & & & & & & & & & & \\
z_2 z_3 & & & & & & & & & & \\
z_3^2 & & & & & & & & & &
\end{array}$$

SDP relaxation

$$\mathit{minimize}\ \sum \mathrm{trace}\left(\mathbf{M}_{ij}\right) + G \cdot \varepsilon \quad \text{s.t.}$$
$$\mathbf{M}_{ij} \succcurlyeq 0\ + \mathit{linear\ equality\ and}$$
$$\mathit{inequality\ constraints\ on}\ \mathbf{M}_{ij}$$

SDP relaxation

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Solution extraction:

$$\begin{array}{c|cccccccccc}
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\hline
1 & & \bigcirc & \bigcirc & \bigcirc & & & & & & \\
z_1 & & & & & & & & & & \\
z_2 & & & & & & & & & & \\
z_3 & & & & & & & & & & \\
z_1^2 & & & & & & & & & & \\
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z_1 z_3 & & & & & & & & & & \\
z_2^2 & & & & & & & & & & \\
z_2 z_3 & & & & & & & & & & \\
z_3^2 & & & & & & & & & &
\end{array}$$

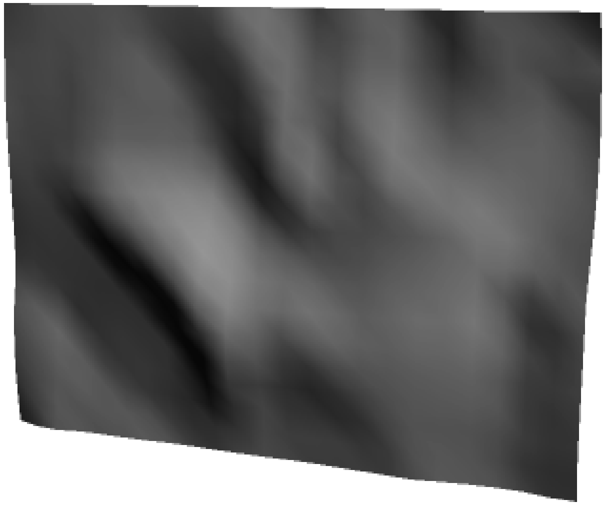
Advantages of SDP relaxations

- Don’t rely on an initial guess,
  boundary conditions or singular points
- Convex optimization
- Easy to integrate different types of constraints

Results

iterative
(48×64)

SDP
(18×24)



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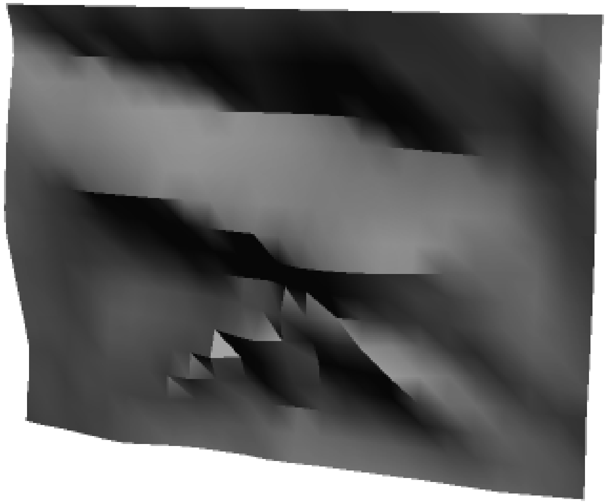

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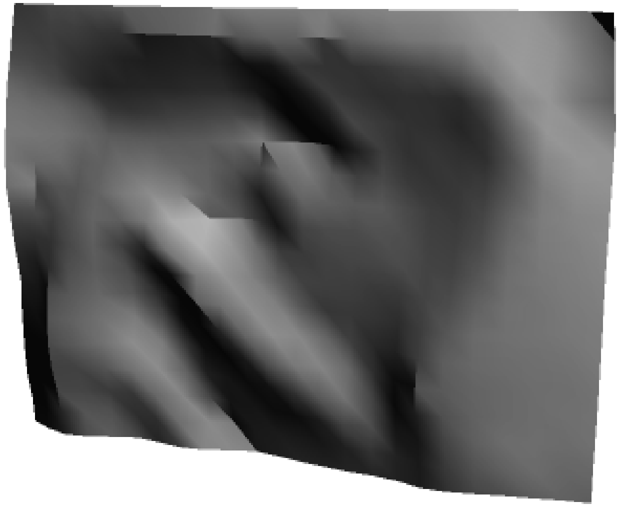

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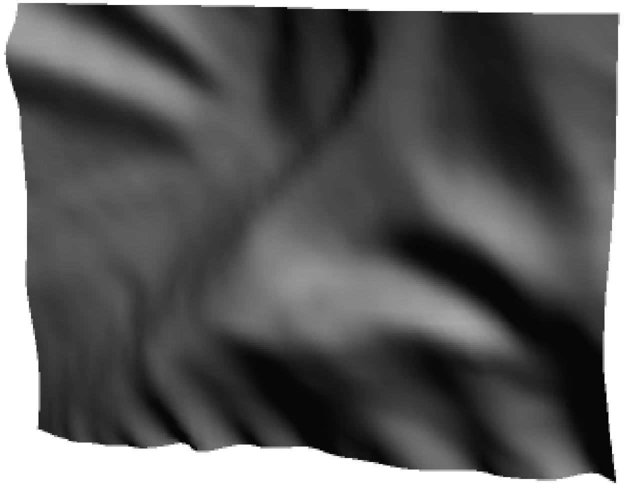

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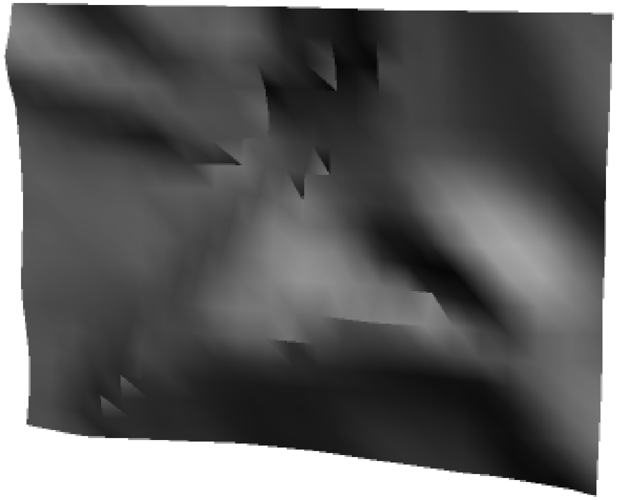


SDP – room for improvements

- Relaxation is not tight
- Slow, applicable to very small images
- Regularization terms
- Solution extraction scheme

Shading ambiguities

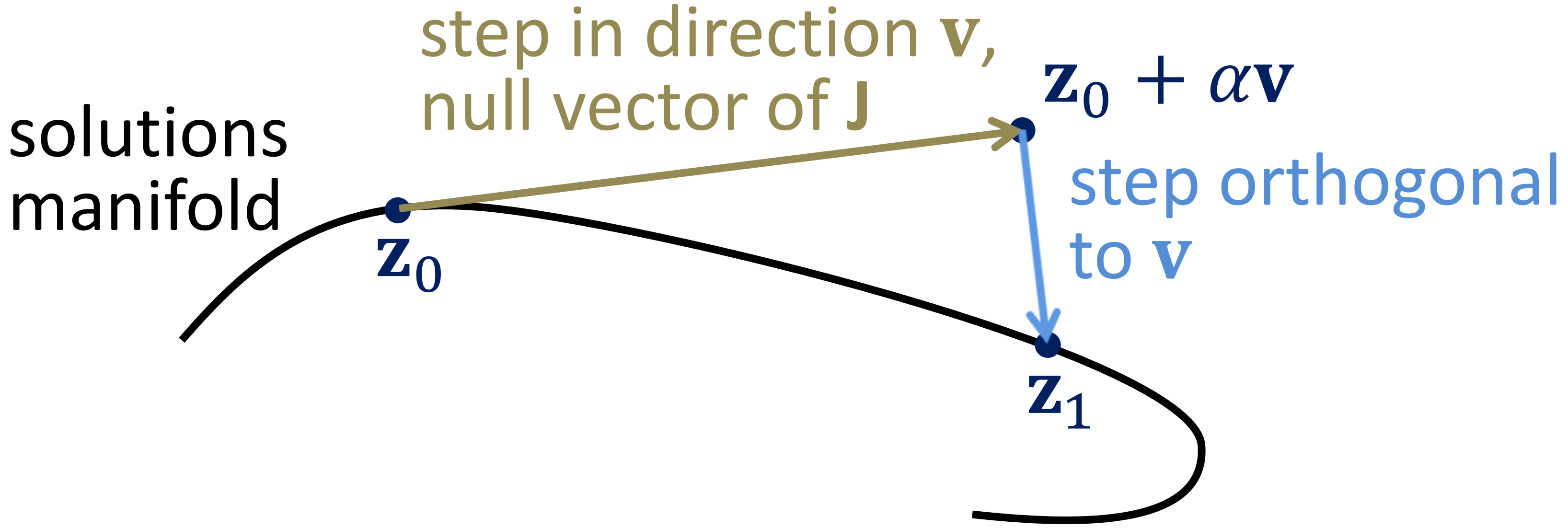
- Known to exist in the continuous case
- For the discrete case, we show that the implicit function theorem implies the existence of a manifold of solutions (subject to conditions)

Visualizing SFS ambiguities

$$\|\mathbf{r}(\mathbf{z}_0+\mathbf{v})\|^2 \approx \|\underbrace{\mathbf{r}(\mathbf{z}_0)}_{0}\|^2 + 2\underbrace{\mathbf{r}^T\mathbf{J}}_{0}\mathbf{v} + \mathbf{v}^T\mathbf{J}^T\mathbf{J}\mathbf{v}$$

$\mathbf{J} = \left[\frac{d\mathbf{r}}{dz}(\mathbf{z}_0)\right]$ is the Jacobian of $\mathbf{r}$ at $\mathbf{z}_0$

On the extended grid **J** is $MN \times (M+1)\,(N+1)$, with at least $M+N+1$ null vectors



step in direction $\mathbf{v}$, null vector of **J**

$\mathbf{z}_0 + \alpha\mathbf{v}$

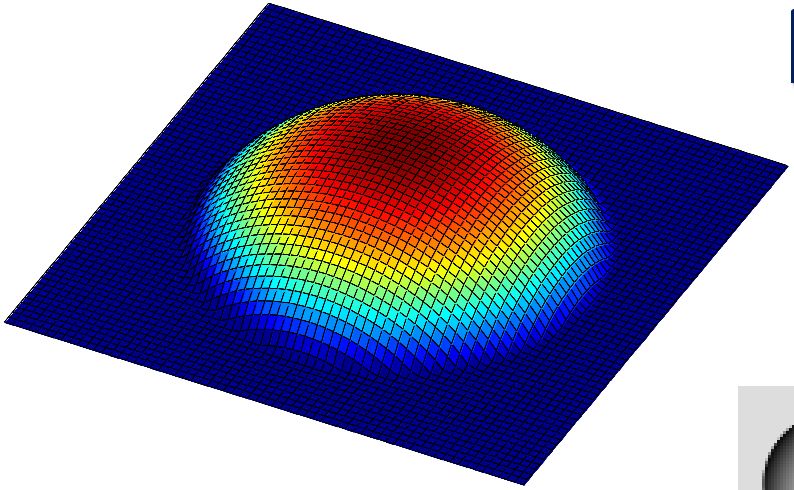
solutions manifold

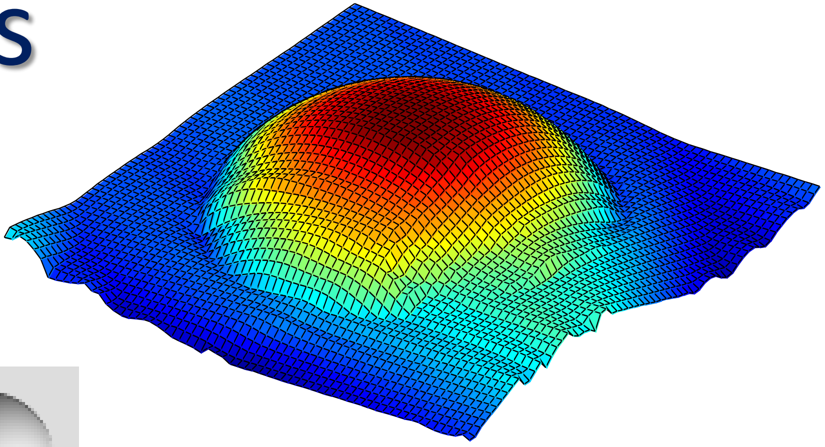
step orthogonal to $\mathbf{v}$

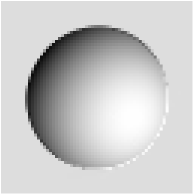
$\mathbf{z}_0$

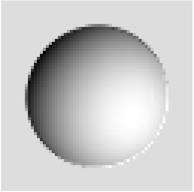
$\mathbf{z}_1$

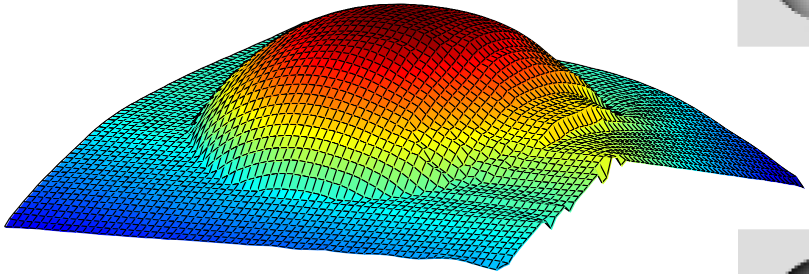
Results

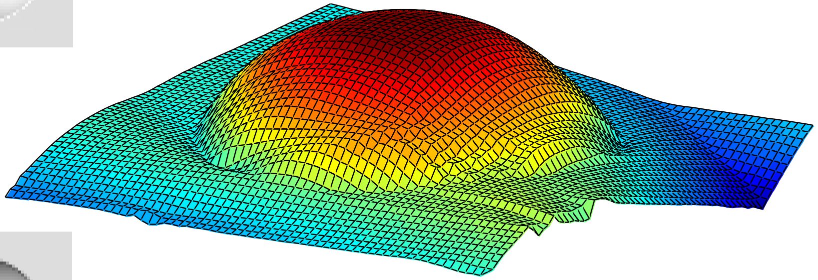


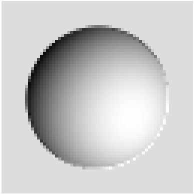


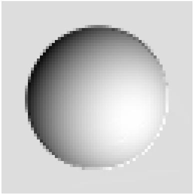


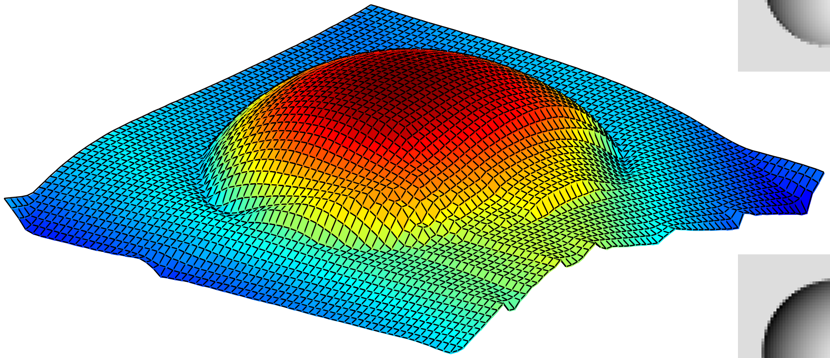


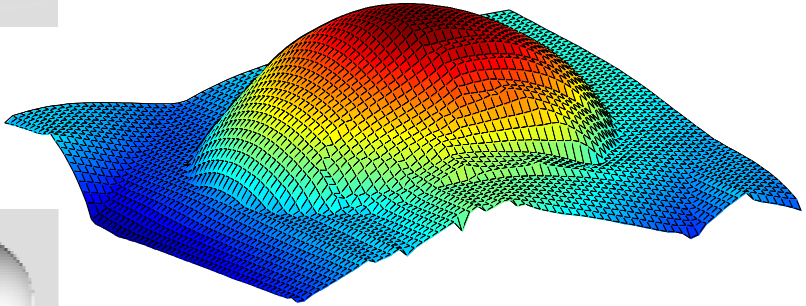


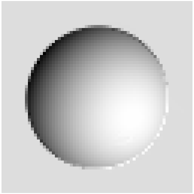


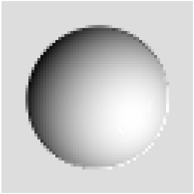












What shape is this?

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